EQUIVALENCE BETWEEN STRICT VISCOSITY SOLUTION AND VISCOSITY SOLUTION IN THE WASSERSTEIN SPACE AND REGULAR EXTENSION OF THE HAMILTONIAN IN L 2 IP

Chloé Jimenez

EQUIVALENCE BETWEEN STRICT VISCOSITY SOLUTION AND VISCOSITY SOLUTION IN THE WASSERSTEIN SPACE AND REGULAR EXTENSION OF THE HAMILTONIAN IN $L^2_{\mathbb{P}}$.

CHLOÉ JIMENEZ

To Giuseppe Buttazzo, a kind magician who makes Mathematics even more beautiful. Thank you Giuseppe and Happy Birthday!

Abstract. This article aims to build bridges between several notions of viscosity solution of first order dynamic Hamilton-Jacobi equations. The first main result states that, under assumptions, the definitions of Gangbo-Nguyen-Tudorascu and Marigonda-Quincampoix are equivalent. Secondly, to make the link with Lions' definition of solution, we build a regular extension of the Hamiltonian in $L^2_{\mathbb{P}} \times L^2_{\mathbb{P}}$. This extension allows to give an existence result of viscosity solution in the sense of Gangbo-Nguyen-Tudorascu, as a corollary of the existence result in $L^2_{\mathbb{P}} \times L^2_{\mathbb{P}}$. We also give a comparison principle for rearrangement invariant solutions of the extended equation. Finally we illustrate the interest of the extended equation by an example in Multi-Agent Control.

Keywords: Optimal Transport, Viscosity solutions, Hamilton-Jacobi equations, Multi-Agent Optimal Control.

1. Introduction

In this article, we are interested in Hamilton-Jacobi equations in the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$. Such equations arise naturally in many different fields. Up to my knowledge, the fist appearance can be dated to 2008 on the one hand in Differential Games with incomplete informations [8], on the other hand in [16] with an equation coming from Fluid Mechanics. These kind of equations also appear for instance in Mean Field Games (see [19],[6], [15], [7] among others) and Multi-Agent Control ([20], [23], [10], [3]...). In many cases, the viscosity solution of such equation is expected to be the limit of a sequence of solutions of Hamilton-Jacobi equations in finite dimension. In the setting of Multi-Agent optimal control, trajectories in the Wasserstein space aim to model trajectories of flocks, herds or crowds with a high number of individuals. The position of the agents at a time t is represented through a probability measure μ_t. A natural question is weather the value function can be obtained as a limit, as $N \to +\infty$ of a sequence of value functions, each one being the appropriate value of a control problem in $(\mathbb{R}^d)^N$ of N trajectories of N agents in $\mathbb{R}^d$. The answer is positive and was proved in [9] in a quite general case. At the level of solutions of Hamilton-Jacobi equations, this is done [15] and [7] for some equations arising in Mean Field Games.

Date: September 29, 2023.

We focuses on dynamic first order backward Hamilton-Jacobi-Bellman equations on the Wasserstein space and on a time interval $[0, T[$ with the shape:

$$\text{(HJ)} \quad \partial_t u(t,\mu) + \mathcal{H}(\mu, D_\mu u(t,\mu)) = 0 \ \forall (\mu, t) \in \mathcal{P}_2(\mathbb{R}^d) \times [0, T[$$

where $\mathcal{H} : (\mu, p) \in \mathcal{P}_2(\mathbb{R}^d) \times L^2_\mu(\mathbb{R}^d) \mapsto \mathcal{H}(\mu, p) \in \mathbb{R}$. This equation will be coupled with a final condition

$$u(T, \mu) = \mathcal{G}(\mu) \ \forall \mu \in \mathcal{P}_2(\mathbb{R}^d)$$

with $\mathcal{G} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ bounded uniformly continuous.
Several notions of viscosity solutions in $\mathcal{P}_2(\mathbb{R}^d)$ exists in the literature. We can cite among other the definitions of Cardaliaguet-Quincampoix [8], Gangbo-Nugyen-Tudorascu [16], Ambrosio-Feng [2], Lions [19] (see also [6]), Marigonda-Quincampoix [20], [23], [24], Badreddine-Frankowska [3], [4], or more recently Conforti-Kraaij-Tonon [11] and Jerhaoui-Zidani [21], [22].
We concentrate on the notions of Gangbo-Nugyen-Tudorascu, Marigonda-Quincampoix and Lions. In the continuity of [17] and [24], we aim to establish bridges between these three notions. In a first step, we will deal with the two first definitions.

The definition of subsolution of Gangbo-Nugyen-Tudorascu requires a notion of superdifferential. Namely for $u : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, $(t_0, \mu_0) \in [0, T[\times \mathcal{P}_2(\mathbb{R}^d)$, the couple $(p_t, p_\mu) \in \mathbb{R} \times L^2_{\mu_0}(\mathbb{R}^d, \mathbb{R}^d)$ belongs to $D^+ u(t_0, \mu_0)$ if:
$\forall (t, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, $\gamma \in \Pi_o(\mu_0, \nu)$:

$$u(t,\nu) - u(t_0, \mu_0) \le p_t \times (t - t_0) + \int_{\mathbb{R}^{2d}} p_\mu(x) \cdot (y - x) \, d\gamma(x,y) + o\left(\sqrt{|t - t_0|^2 + W_2^2(\mu_0, \nu)}\right),$$

and $p_\mu \in \mathcal{T}_{\mu_0}(\mathbb{R}^d)$, the tangent space to $\mathcal{P}_2(\mathbb{R}^d)$ at μ_0 (see section 2.1). Here, $\Pi_o(\mu_0, \nu)$ denotes the set of optimal transport plans between μ_0 and ν (see section 2.1). Then, classically, u is a viscosity subsolution of (HJ) if for all $(t_0, \mu_0) \in [0, T[\times \mathcal{P}_2(\mathbb{R}^d)$:

$$p_t + \mathcal{H}(\mu_0, p_\mu) \ge 0 \quad \forall (p_t, p_\mu) \in D^+ u(t_0, \mu_0).$$

Note that the equation being backward, the inequality in the definition of viscosity solution is reversed. The definition of subsolution of Marigonda-Quincampoix will be called *strict* viscosity subsolution and requires the notion of approximate superdifferential $D^+_\varepsilon u(\cdot, \cdot)$ associated to the definition above with $\varepsilon > 0$ in the spirit of [12]. Then the map u is a strict viscosity subsolution of (HJ) if it exists $C : \mathcal{P}_2(\mathbb{R}^d) \to [0, +\infty[$ (with some regularity assumption) such that for all $(t_0, \mu_0) \in [0, T[\times \mathcal{P}_2(\mathbb{R}^d)$ and all $\varepsilon > 0$:

$$p_t + \mathcal{H}(\mu_0, p_\mu) \ge -C(\mu_0)\varepsilon \quad \forall (p_t, p_\mu) \in D^+_\varepsilon u(t_0, \mu_0).$$

The first main result of this paper states that, under some assumptions on $\mathcal{H}$, both notions of strict and non-strict viscosity solutions are equivalent. As done in [12] in finite dimension, for this type of result some regularity of $\mathcal{H}$ is needed with respect to the couple (μ, p) which belongs to the domain of $\mathcal{H}$:

$$\mathcal{F}_2(\mathbb{R}^d) := \{(\mu, p) : \mu \in \mathcal{P}_2(\mathbb{R}^d), \, p \in L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)\}. \tag{1}$$

The question of a suitable convergence in $\mathcal{F}_2(\mathbb{R}^d)$ is not easy and was already addressed in [1] p 127 and in [24], section 3.4. A good regularity assumption to get equivalence between strict and non-strict solutions is the following one:

$$\lim_{n \to +\infty} W_2((Id \times p_n)\sharp \mu_n, (Id \times p)\sharp \mu) = 0 \Rightarrow \lim_{n \to +\infty} \mathcal{H}(\mu_n, p_n) = \mathcal{H}(\mu, p) \tag{2}$$

for any (μ, p), $(\mu_n, p_n)_n$ in $\mathcal{F}_2(\mathbb{R}^d)$. The other significant difficulty is to express the notion of strict solutions in terms of quite regular test functions. It was already proved in [24] that the definition of strict solution is equivalent to a definition involving test functions $v : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$. Nevertheless a test function at a point (t_0, μ_0) is only differentiable at (t_0, μ_0) with no guarantee away from this point. This lack of regularity prevents, for instance, to repeat the arguments of [12] to prove equivalence between strict and non-strict viscosity solution. We will introduce a class of more regular test functions w. Their superdifferential is non-empty everywhere, and for any $(t_n, \mu_n)_n$ converging to (t_0, μ_0), up to a subsequence, we can find $(p_{t,n}, p_{\mu,n})_n$ in $D^+v(t_n, \mu_n)$ converging to the derivative of w at (t_0, μ_0).

To introduce viscosity solutions to (HJ) in the sense of Lions, considering a complete probability space $(\Omega, B(\Omega), \mathbb{P})$, we need the lift U of u and H of $\mathcal{H}$:

$$U(t, X) := u(t, X\sharp\mathbb{P}) \ \forall (t, X) \in [0, T[\times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$$

where $X\sharp\mathbb{P}$ is the law of X,

$$H(X, p \circ X) := \mathcal{H}(\mu, p) \ \forall (\mu, p) \in \mathcal{F}_2(\mathbb{R}^d),\ X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \text{ with } X\sharp\mathbb{P} = \mu. \tag{3}$$

Moreover an extension $\overline{\mathrm{H}}$ of H to all $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ is involved. Then u is a viscosity solution of (HJ) in the sense of Lions if its lift is a viscosity solution of

$$(\mathrm{HJ}_2) \quad \partial_t U + \overline{\mathrm{H}}(X, DU(t, X)) = 0 \quad \forall (t, X) \in [0, T[\times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d).$$

This notion is relative to the choice of the extension $\overline{\mathrm{H}}$. The question is: what would be a good choice of extension $\overline{\mathrm{H}}$ of H? The answer depends on the expected properties of $\overline{\mathrm{H}}$. Hereafter we discuss several choices of extensions. In [17], to define the extension $\overline{\mathrm{H}}(X, Z)$ for some $(X, Z) \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$, the authors use the orthogonal projection $p^o_X(Z) \circ X$ of Z, on

$$\{p \circ X : p \in \mathcal{T}_{X\sharp\mathbb{P}}(\mathbb{R}^d)\} \text{ where } p^o_X(Y) \in \mathcal{T}_{X\sharp\mathbb{P}}(\mathbb{R}^d).$$

The set $\mathcal{T}_{X\sharp\mathbb{P}}(\mathbb{R}^d)$ being the tangent space to $\mathcal{P}_2(\mathbb{R}^d)$ at $X_\sharp\mathbb{P}$ (see subsection 2.1). Then, they choose $\overline{\mathrm{H}}(X, Z) = H(X, p^o_X(Z) \circ X) = \mathcal{H}(\mu, p^o_X)$, we will denote this extension H_{GT}. This choice is motivated by the following outstanding result of [17]:

$$Z \in D^+U(X) \Rightarrow p^o_X(Z) \in D^+u(X\sharp\mathbb{P}) \text{ where } U \text{ is the lift of } u.$$

This extension is of particular interest as it gives the equivalence: u is a viscosity solution of (HJ) (in the sense of Gangbo-Nguyen-Tudorascu) if and only if its lift U is a solution of (HJ_2). Another possibility, appearing in [10], is given by: $\overline{\mathrm{H}}(X, Z) = H(X, p_X(Z) \circ X) = \mathcal{H}(X\sharp\mathbb{P}, p_X(Z))$ where $p_X(Z) \circ X$ is the orthogonal projection on

$$H_X := \{p \circ X : p \in L^2_{X\sharp\mathbb{P}}(\Omega, \mathbb{R}^d)\}.$$

We will denote this extension by H_{CMQ}. Unfortunately both H_{GT} and H_{CMQ} fail to be regular on (X, Z) even if $\mathcal{H}$ is quite nice which prevents to apply classical results contained for instance in [12], [13]. Let us illustrate this lack of regularity by a simple example.

Example 1.1. *In dimension $d = 1$, we consider the following Hamiltonian on $\mathcal{F}_2(\mathbb{R})$:*

$$\mathcal{H}(\mu, p) = \|p\|_{L^2_\mu}.$$

We choose the probability space:

$$(\Omega, B(\Omega), \mathbb{P}) \textit{ with } \Omega = [-1/2; 1/2], \quad \mathbb{P} = \mathcal{L}^1 \lfloor [-1/2; 1/2]$$

where $B([-1/2, 1/2])$ is the Borelian tribe. Consider the following probabilities on $\mathcal{P}_2(\mathbb{R})$: $\mu_n = n\mathcal{L}^1 \lfloor [\frac{-1}{2n}, \frac{1}{2n}]$, $\nu = \frac{1}{2}(\delta_1 + \delta_{-1})$ and the optimal transport map from μ_n to ν: $T_n(x) = -\mathbf{1}_{[-\frac{1}{2n},0[}(x) + \mathbf{1}_{]0,\frac{1}{2n}]}(x)$. Then

$$p_n(x) = (x+1)\mathbf{1}_{[-\frac{1}{2n},0[}(x) + (x-1)\mathbf{1}_{]0,\frac{1}{2n}]}(x) \textit{ belongs to } \mathcal{T}_{\mu_n}(\mathbb{R}^d)$$

$$\textit{and } \mathcal{H}(\mu_n, p_n) = W_2^2(\mu_n, \nu) = \frac{1}{12n^2} - \frac{1}{2n} + 1.$$

The probability μ_n can be represented in $L^2_{\mathbb{P}}(\Omega, \mathbb{R})$ by $X_n(\omega) = \frac{\omega}{n}$, so we get:

$$H_{\mathrm{CMQ}}(X_n, p_n \circ X_n) = H_{\mathrm{GT}}(X_n, p_n \circ X_n) = \mathcal{H}(\mu_n, p_n) = \frac{1}{12n^2} - \frac{1}{2n} + 1.$$

Then, passing to the limit, $(X_n)_n$ converges to 0 (of law δ_0), and $(p_n \circ X_n)_n$ converges to $Z = \mathbf{1}_{[-\frac{1}{2},0[} - \mathbf{1}_{]0,\frac{1}{2}]}$ of law ν. As $\mathcal{T}_{\delta_0}(\mathbb{R}^d) = L^2_{\delta_0}(\Omega, \mathbb{R}^d)$ can be identified to $\mathbb{R}$, we easily see that $p_X(Z) = p^o_X(Z) = 0$. Finally:

$$H_{\mathrm{CMQ}}(X, Z) = H_{\mathrm{GT}}(X, Z) = \mathcal{H}(\delta_0, 0) = 0 < 1 = W_2(\delta_0, \nu) = \lim_{n \to +\infty} W_2(\mu_n, \nu)$$

$$= \lim_{n \to +\infty} H_{GT}(X_n, p_n \circ X_n) = \lim_{n \to +\infty} H_{CMQ}(X_n, p_n \circ X_n).$$

The inequality is due to the fact that the Hamiltonians H_{GT} and H_{CMQ} do not capture the fact that the optimal plan between δ_0 and ν divides masses.

In the example above, a natural choice of extension is $\overline{\mathrm{H}}(X, Z) = \|Z\|_{L^2_{\mathbb{P}}}$ which is indeed used in [19] and [6]. In the present paper we aim to build a regular extension, namely continuous. Again, the question of a good topology on $\mathcal{F}_2(\mathbb{R}^d)$ arises. We show that if H has a continuous extension $\overline{\mathrm{H}}$, necessarily, $\mathcal{H}$ satisfies (2). Conversely, assuming $\mathcal{H}$ is uniformly continuous with respect to this convergence, we can extend H to $\overline{\mathrm{H}}$ continuously on $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)^2$. This regularity allows to give an existence result for viscosity solution of (HJ) (Theorem 4.8) (in the sense of Gangbo-Nguyen-Tudorascu), using the results of Crandall and Lions [13]. We also show that this extension is meaningful and give a specific comparison principle (note that classic comparison principle will also apply).

The paper is organized as follows. In section 2, we will give the definitions and notations and recall useful results. We will also give two example of Hamiltonians which will be used all along the article. Section 3 is devoted to strict and non strict solutions. After giving the definitions in subsection 3.1, subsection 3.2 deals with test functions and regular test functions. Finally we prove the equivalence between strict and non strict solution in subsection 3.3. In Section 4 , we build a regular $\overline{\mathrm{H}}$ (subsection 4.1) as a consequence we get an existence result for (HJ). Then, in subsection 4.2, we study the properties of rearrangement invariant viscosity solutions of the extended equation and give a comparison principle. We end the section by an example (subsection 4.3) which emphasize that this extension is meaningful for Optimal Control problems. Finally the Appendix 5 contains a density result in $\mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$.

2. Preliminaries

2.1. **Optimal transport and the Wasserstein space.** We fix and integer $d \in \mathbb{N}^*$. For any $N \in \mathbb{N}^*$ $(= d,\ 2d,\ 3d...)$, we denote by $\mathcal{P}_2(\mathbb{R}^N)$ the *Wasserstein space*:

$$\mathcal{P}_2(\mathbb{R}^N) = \{\mu \in \mathcal{P}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^2 \, d\mu(x) < +\infty\}$$

where $\mathcal{P}(\mathbb{R}^N)$ is the set of probability measures on $\mathbb{R}^N$ and $|\cdot|$ is the Euclidian norm. We will write $m_2(\mu)$ for the moment of order 2 of $\mu \in \mathcal{P}_2(\mathbb{R}^d)$:

$$m_2(\mu) := \int_{\mathbb{R}^d} |x|^2 \, d\mu(x).$$

We will denote by π_1 and π_2 the first and second projection on $\mathbb{R}^d \times \mathbb{R}^d$:

$$\begin{array}{rcl} \pi_1 : \mathbb{R}^d \times \mathbb{R}^d & \to & \mathbb{R}^d, \\ (x,y) & \mapsto & x \end{array} \qquad \begin{array}{rcl} \pi_2 : \mathbb{R}^d \times \mathbb{R}^d & \to & \mathbb{R}^d. \\ (x,y) & \mapsto & y \end{array}$$

we will also use:

$$\begin{array}{rcl} \pi_{1,2} : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d & \to & \mathbb{R}^d \times \mathbb{R}^d, \\ (x,y,z) & \mapsto & (x,z) \end{array} \qquad \begin{array}{rcl} \pi_{1,3} : \mathbb{R}^d \times \mathbb{R}^d & \to & \mathbb{R}^d. \\ (x,y,z) & \mapsto & (x,z) \end{array}$$

For any $T \in L^2_\mu(\mathbb{R}^N, \mathbb{R}^{N'})$, $\mu \in \mathcal{P}_2(\mathbb{R}^N)$, *the push forward* $T\sharp\mu \in \mathcal{P}_2(\mathbb{R}^{N'})$ is defined by:

$$(T\sharp\mu)(A) = \mu(T^{-1}(A)) \text{ for all } A \text{ in the Borel Tribe } B(\mathbb{R}^{N'}).$$

The space $\mathcal{P}_2(\mathbb{R}^N)$ is a Polish space while equipped with the *Wasserstein distance* defined for all $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^N)$:

$$W_2(\mu,\nu) = \inf \left\{ \left(\int_{R^N \times \mathbb{R}^N} |x-y|^2 \, d\gamma(x,y) \right)^{1/2} : \ \gamma \in \Pi(\mu,\nu) \right\}$$

where $\Pi(\mu,\nu) = \{\gamma \in \mathcal{P}_2(\mathbb{R}^N \times \mathbb{R}^N) : \ p_1\sharp\gamma = \mu, \ p_2\sharp\gamma = \nu\}$ is called the set of *transport plans*.
The infimum is always a minimum and the set of *optimal transport plans* is denoted by $\Pi_o(\mu,\nu)$. When μ is absolutely continuous with respect to the Lebesgue measure there exists a so called *optimal transport map* $T \in L^2_\mu(\mathbb{R}^N, \mathbb{R}^N)$ such that $T\sharp\mu = \nu$ and:

$$\Pi_o(\mu,\nu) = \{(Id \times T)\sharp\mu\}.$$

We recall the following result:

Proposition 2.1. *Let $(\Omega, B(\Omega), \mathbb{P})$ any probability space with Ω a Polish space, $B(\Omega)$ the Borel tribe and $\mathbb{P}$ without atom. Let $\mu \in \mathcal{P}_2(\mathbb{R}^N)$, then it exists $T : \Omega \to \mathbb{R}^N$ that pushes forward $\mathbb{P}$ to μ: $T\sharp\mathbb{P} = \mu$.*

We will use *the tangent space* to $\mathcal{P}_2(\mathbb{R}^d)$ at $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ (see [1], section 8.4.):

$$\begin{array}{rcl} \mathcal{T}_\mu(\mathbb{R}^d) & = & \overline{\{\lambda(Id - T) : \ \lambda > 0, \ (Id \times T)\sharp\mu \in \Pi_o(\mu, T\sharp\mu)\}}^{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)} \\ & = & \overline{\{\nabla\varphi : \ \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d)\}}^{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}. \end{array}$$

We refer to [25] and [1] for more informations on $\mathcal{P}_2(\mathbb{R}^d)$.

2.2. **$L^2_{\mathbb{P}}$-representation of the Wasserstein space.** Let $(\Omega, B(\Omega), \mathbb{P})$ a fixed probability space satisfying the assumptions of Proposition 2.1. Then, for any $\mu \in \mathcal{P}_2(\mathbb{R}^N)$ it exists $X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^N)$ such that $\mu = X\sharp\mathbb{P}$. The norm and the scalar product of this space will be simply denoted $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$. It holds:

$$W_2(\mu,\nu) = \min\left\{\|X-Y\| : \ X\sharp\mathbb{P} = \mu, \ Y\sharp\mathbb{P} = \nu\right\}.$$

The following lemma is very useful while considering two random variables with the same law:

Lemma 2.2. *Let $X, Y \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ such that $X\sharp\mathbb{P} = Y\sharp\mathbb{P}$. Then, for any $\varepsilon > 0$, there exists $\tau : \Omega \to \Omega$ bijective satisfying:*

(i) *τ and τ^{-1} are measure-preserving that is $\tau\sharp\mathbb{P} = \tau^{-1}\sharp\mathbb{P} = \mathbb{P}$,*
(ii) $\|Y - X \circ \tau\|_{L^\infty_{\mathbb{P}}(\Omega,\mathbb{R}^d)} \leq \varepsilon$.

We also have:

Lemma 2.3. a) *For any X_n, X in $L^2_{\mathbb{P}}$, we have:*

$$\|X_n - X\| \to 0 \Rightarrow W_2(X_n\sharp\mathbb{P}, X\sharp\mathbb{P}) \to 0.$$

b) *For any μ_n, μ in $\mathcal{P}_2(\mathbb{R}^d)$, if $W_2(\mu_n, \mu) \to 0$ then it exists $(X_n)_n$, X in $L^2_{\mathbb{P}}$ such that:*

$$\|X_n - X\| \to 0.$$

For any $u : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, we call *lift* of u the map:

$$U : X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \mapsto U(X) = u(X\sharp\mathbb{P}) \in \mathbb{R}.$$

This map is said to be *rearrangement invariant*, that is: $U(X) = U(Y)$ for all $X, Y \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ such that $X\sharp\mathbb{P} = Y\sharp\mathbb{P}$.
Note that u is *continuous* in $\mathcal{P}_2(\mathbb{R}^d)$ iff its lift U is *continuous* in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$.
We recall the definition of the *Fréchet superdifferential* of any map $V : L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \to \mathbb{R}$:

Definition 2.4. Let $X_0, Z \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$, $p_t \in \mathbb{R}$, $t_0 \in [0, T[$. We will write that $(p_t, Z) \in D^+U(t_0, X_0)$ iff for all $(t, Y) \in \mathbb{R} \times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$:

$$V(t,Y) - V(t_0, X_0) \leq p_t(t-t_0)\langle Z, Y - X_0\rangle + o\left(\sqrt{|t-t_0|^2 + \|Y - X_0\|^2}\right).$$

The set $D^+U(t_0, X_0)$ is called the *(Fréchet) superdifferential* of V at (t_0, X_0). We have symmetric definitions for the *(Fréchet) subdifferential* of V.

If $H : L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \to \mathbb{R}$, U is a viscosity subsolution of

$$\partial_t U(t,X) + H(X, D_X U(t,X)) = 0 \, \forall t \in [0,T[, \ X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$$

if for all $(t_0, X_0) \in [0, T[\times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$, all $(p_t, Z) \in D^+U(t_0, X_0)$, it holds:

$$p_t + H(X_0, Z) \geq 0.$$

The definition is symmetric for viscosity supersolution.
For any $X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$, we denote $H_X := \{p \circ X : p \in L^2_{X\sharp\mathbb{P}}(\Omega, \mathbb{R}^d)\}$.
The *orthogonal projection* of $Y \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ on H_X (see [24], Lemma 2.3) is given by $p_X(Y) \circ X$ with

$$p_X(Y) := x \mapsto \int_{\mathbb{R}^d} y d\gamma^x(y) \text{ where } \gamma := (X \times Y)\sharp\mathbb{P}.$$

We will also consider the *orthogonal projection* $p^o_X(Y) \circ X$ of $Y \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ on $\{p \circ X : p \in \mathcal{T}_{X\sharp\mathbb{P}}(\mathbb{R}^d)\}$ for a fixed $X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$. When $\gamma = (X \times Y)\sharp\mathbb{P}$ is an optimal transport plan, we have: $p^o_X(Y) = p_X(Y)$. The following is fundamental:

Proposition and Definition 2.5. *(see* [6]*,* [17] *and* [24]*) Let* $U : L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \to \mathbb{R}$ *be rearrangement invariant and differentiable at* $X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ *of law* μ*, then it exists* $p \in \mathcal{T}_\mu(\mathbb{R}^d)$ *such that* $DU(X) = p \circ X$.
If U *is the lift of* $u : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$*, then we say that* u *is differentiable at* μ *and we denote* $D_\mu u = p$.

Example 2.6. (see [6] and [1])
(i) Let $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d)$ and $u : \mu \in \mathcal{P}_2(\mathbb{R}^d) \mapsto u(\mu) = \int_{\mathbb{R}^d} \varphi(x)\, d\mu(x)$. Then for all $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, u is differentiable at μ_0 and $D_\mu u(\mu_0) = \nabla\varphi$.
(ii) Let $\nu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ and $u : \mu \in \mathcal{P}_2(\mathbb{R}^d) \mapsto u(\mu) = W_2^2(\mu, \nu_0)$. Then u is differentiable at μ_0 if and only if $\Pi_o(\mu_0, \nu_0) = \{(Id \times T)\sharp\mu_0\}$ for some $T \in L^2_{\mu_0}(\mathbb{R}^d, \mathbb{R}^d)$ and in this case: $D_\mu u(\mu_0)(x) = 2(x - Tx)$.

2.3. **Other notations and objects.** In all the article we will say that $\omega : [0, +\infty[\to [0, +\infty[$ is a modulus of continuity if it is continuous, non-decreasing and such that $\omega(0) = 0$. Moreover we will call $(R, t) \in [0, +\infty[^2 \mapsto \omega_R(t) = \omega(R, t)$ a local modulus if $\omega_R(\cdot)$ is a modulus of continuity for any $R \geq 0$ and ω is continuous and non-decreasing in both variables.

2.4. **Example of Hamiltonians and assumptions.** We will always assume the following minimal regularity of $\mathcal{H}$:

$$\forall \mu \in \mathcal{P}_2(\mathbb{R}^d), \quad p \in L^2_\mu(\mathbb{R}^d) \mapsto \mathcal{H}(\mu, p) \text{ is continuous .} \tag{4}$$

We have two typical examples in mind for the Hamiltonian $\mathcal{H}$:

Example 2.7. We first consider a Hamiltonian appearing in [16] and [17]:

$$\mathcal{H}_1(\mu, p) := \frac{1}{2} \int_{\mathbb{R}^d} |p(x)|^2 d\mu(x) + \mathcal{V}(\mu).$$

We will assume that it exists a local modulus $(R, t) \mapsto \omega_R(t)$ such that for all $R > 0$,

$$|\mathcal{V}(\mu) - \mathcal{V}(\nu)| \leq \omega_R(W_2(\mu, \nu)) \quad \forall \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d) \text{ with } m_2(\mu), m_2(\nu) \leq R^2.$$

The Hamiltonian $\mathcal{H}_1$ satisfies the following regularity property:
for all $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, $p \in L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$, $q \in L^2_\nu(\mathbb{R}^d, \mathbb{R}^d)$ with

$$m_2(\mu) + \|p\|^2_{L^2_\mu} \leq R^2, \quad m_2(\nu) + \|q\|^2_{L^2_\nu} \leq R^2,$$

it holds:

$$|\mathcal{H}_1(\mu, p) - \mathcal{H}_1(\nu, q)| \leq R W_2(p\sharp\mu, q\sharp\nu) + \omega_R(W_2(\mu, \nu)) \tag{5}$$

$$\leq R\, W_2((Id \times p)\sharp\mu, (Id \times q)\sharp\nu) + \omega_R(W_2((Id \times p)\mu, (Id \times q)\nu)). \tag{6}$$

Indeed, by the triangular inequality applied to W_2:

$$\begin{aligned} \mathcal{H}_1(\mu, p) - \mathcal{H}_1(\nu, q) &= \frac{1}{2}\left(W_2^2(p\sharp\mu, \delta_0) - W_2^2(q\sharp\nu, \delta_0)\right) + \mathcal{V}(\mu) - \mathcal{V}(\nu) \\ &\leq \frac{1}{2}\left(W_2(p\sharp\mu, \delta_0) + W_2(p\sharp\mu, \delta_0)\right) W_2(p\sharp\mu, q\sharp\nu) + \mathcal{V}(\mu) - \mathcal{V}(\nu). \end{aligned}$$

Note that, in particular, (5) implies for all $p, q \in L^2_\mu$ with $\|p\|_{L^2_\mu}, \|q\|_{L^2_\mu} \le R$:

$$|\mathcal{H}_1(\mu,p) - \mathcal{H}_1(\mu,q)| \le R\|p-q\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}. \tag{7}$$

Remark 2.8. With our definition of viscosity solution (see definition 3.6), the Hamilton Jacobi equation considered in [16] is in fact:

$$-\partial_t u(\mu,t) - \mathcal{H}_1(\mu, D_\mu u(\mu,t)) = 0$$

together with an initial condition $u(0,\mu_0) = \mathcal{G}(\mu_0)$.

Example 2.9. We will also consider an example coming from [20] and [23]:

$$\mathcal{H}_2(\mu,p) := \inf\left\{\int_{\mathbb{R}^d} f(x,\mathbf{u}(x),\mu)\cdot p(x)\, d\mu(x) :\ \mathbf{u} : \mathbb{R}^d \to \mathbf{U} \text{ Borel}\right\}$$

where $\mathbf{U}$ is a compact, convex and separable Banach space and $f : \mathbb{R}^d \times \mathbf{U} \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ satisfies the following assumptions:

- f is affine in $\mathbf{u}$, that is for all $(x,\mu) \in \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$, $\mathbf{u},\mathbf{v} \in \mathbf{U}$ and $t \in [0,1]$:
$$f(x,(1-t)\mathbf{u} + t\mathbf{v},\mu) = (1-t)f(x,\mathbf{u},\mu) + tf(x,\mathbf{v},\nu),$$
- f is continuous and it exists $L > 0$ such that for all $(x,\mathbf{u},\mu) \in \mathbb{R}^d \times \mathbf{U} \times \mathcal{P}_2(\mathbb{R}^d)$ and $(y,\nu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$:
$$|f(x,\mathbf{u},\mu) - f(y,\mathbf{u},\nu)| \le L\left(W_2(\mu,\nu) + |x-y|\right).$$

The Hamiltonian $\mathcal{H}_2$ satisfies the following regularity properties:

Lemma 2.10. *It exists $M \ge 0$ depending only on f such that*

$$|\mathcal{H}_2(\mu,p) - \mathcal{H}_2(\nu,q)| \le (M + 4LR)W_2((Id\times p)\sharp\mu, (Id \times q)\sharp\nu))$$

for all $R > 0$ and all $\mu,\nu \in \mathcal{P}_2(\mathbb{R}^d)$, $p \in L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$, $q \in L^2_\nu(\mathbb{R}^d,\mathbb{R}^d)$ such that:

$$m_2(\mu) + \|p\|^2_{L^2_\mu} \le R^2, \quad m_2(\nu) + \|q\|^2_{L^2_\nu} \le R^2.$$

Moreover, for all $\mu, \in \mathcal{P}_2(\mathbb{R}^d)$, $p \in L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$, $q \in L^2_\nu(\mathbb{R}^d,\mathbb{R}^d)$ such that $m_2(\mu) \le R^2$, it holds:

$$|\mathcal{H}_2(\mu,p) - \mathcal{H}_2(\mu,q)| \le (M + 4LR)\|p-q\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}.$$

Proof. First, setting $M := \max_{\mathbf{u}\in\mathbf{U}} |f(0,\mathbf{u},\delta_0)|$, note that for any Borel map $x \in \mathbb{R}^d \mapsto \mathbf{u}(x) \in \mathbf{U}$, $\rho \in \mathcal{P}_2(\mathbb{R}^d)$ and $\gamma \in \mathcal{P}_2(\mathbb{R}^{2d})$, using the Lipschitz regularity of f, it holds:

$$\int_{\mathbb{R}^{2d}} |f(x_2,\mathbf{u}(x_1),\rho)|^2 d\gamma(x_1,x_2) \le \int_{\mathbb{R}^{2d}} |f(0,\mathbf{u}(x_1),\delta_0) + L(|x_2| + W_2(\rho,\delta_0))|^2 d\gamma(x_1,x_2)$$

so that if $\int |x_2|^2\, d\gamma(x_1,x_2) \le R^2$ and $m_2(\rho) \le R^2$:

$$\left(\int_{\mathbb{R}^{2d}} |f(x_2,\mathbf{u}(x_1),\rho)|^2 d\gamma(x_1,x_2)\right)^{1/2} \le M + 2LR \tag{8}$$

Set $\mu,\nu \in \mathcal{P}_2(\mathbb{R}^d)$, $p \in L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$, $q \in L^2_\nu(\mathbb{R}^d,\mathbb{R}^d)$ as in the lemma. Take $\varepsilon > 0$ and $\mathbf{u}$ ε-optimal for $\mathcal{H}_2(\mu,p)$, then choosing $m(x_1,y_1,x_2,y_2) \in \Pi_o((Id\times p)\sharp\mu, (Id\times q)\sharp\nu))$, we have:

$$\mathcal{H}_2(\mu,p) \ \ge \ \int_{\mathbb{R}^d} f(x_1,\mathbf{u}(x_1),\mu)\cdot p(x_1)\, d\mu(x_1) - \varepsilon$$

$$\begin{aligned}
&\geq \int_{\mathbb{R}^{4d}} [f(x_1,\mathbf{u}(x_1),\mu) - f(x_2,\mathbf{u}(x_1),\nu)]\cdot y_1 \; dm(x_1,y_1,x_2,y_2) \\
&\quad + \int_{\mathbb{R}^{4d}} f(x_2,\mathbf{u}(x_1),\nu)\cdot y_1 \; dm(x_1,y_1,x_2,y_2) - \varepsilon \\
&\geq \int_{\mathbb{R}^{4d}} [f(x_1,\mathbf{u}(x_1),\mu) - f(x_2,u(x_1),\nu)]\cdot y_1 \; dm(x_1,y_1,x_2,y_2) \\
&\quad + \int_{\mathbb{R}^{4d}} f(x_2,\mathbf{u}(x_1),\nu)\cdot y_2 \; dm(x_1,y_1,x_2,y_2) \\
&\quad + \int_{\mathbb{R}^{4d}} f(x_2,\mathbf{u}(x_1),\nu)\cdot (y_1-y_2) \; dm(x_1,y_1,x_2,y_2) - \varepsilon
\end{aligned}$$

then, disintegrating m as

$$m(x_1,y_1,x_2,y_2) = m^{x_2,y_2}(x_1,y_1)\delta_{q(x_2)}(y_2)d\nu(x_2),$$

by Jensen's inequality and the affine property of f, we get:

$$\begin{aligned}
\mathcal{H}_2(\mu,p) \;\geq\; & \int_{\mathbb{R}^{4d}} [f(x_1,\mathbf{u}(x_1),\mu) - f(x_2,\mathbf{u}(x_1),\nu)]\cdot y_1 \; dm(x_1,y_1,x_2,y_2) \\
& + \int_{\mathbb{R}^{d}} f\left(x_2, [\int \mathbf{u}(x_1) dm^{x_2,q(x_2)}(x_1,y_1)], \nu\right)\cdot q(x_2)\; d\nu(x_2) \\
& + \int_{\mathbb{R}^{4d}} f(x_2,\mathbf{u}(x_1),\nu)\cdot (y_1-y_2) \; dm(x_1,y_1,x_2,y_2) - \varepsilon
\end{aligned}$$

and by Cauchy-Schwarz:

$$\mathcal{H}_2(\mu,p) \geq -R\left[\int_{\mathbb{R}^{4d}} |f(x_1,\mathbf{u}(x_1),\mu) - f(x_2,\mathbf{u}(x_2),\nu)|^2 dm\right]^{1/2}$$

$$+\mathcal{H}_2(\nu,q) - \left[\int_{\mathbb{R}^{4d}} |f(x_2,\mathbf{u}(x_1),\nu)|^2 dm\right]^{1/2}\left(\int |y_1-y_2|^2 dm\right)^{1/2} - \varepsilon.$$

Finally by inequality (8), the Lipschitz property of f and the choice of m:

$$\begin{aligned}
\mathcal{H}_2(\mu,p) \geq \mathcal{H}_2(\nu,q) - LR&\left[\left(\int |x_1-x_2|^2 dm\right)^{1/2} + W_2(\mu,\nu)\right] \\
&-(M+2LR)\times W_2((Id\times p)\sharp\mu,(Id\times q)\sharp\nu)) - \varepsilon \\
&\geq \mathcal{H}_2(\nu,q) - (M+4LR)W_2((Id\times p)\sharp\mu,(Id\times q)\sharp\nu)) - \varepsilon.
\end{aligned}$$

The conclusion follows by making $\varepsilon \to 0$. $\square$

All over the article, we will use the following assumptions on $\mathcal{H}$, or similar ones:

(A_0) It exists a local modulus $(R,t)\mapsto \omega_R(t)$ such that:

$$|\mathcal{H}(\mu,p) - \mathcal{H}(\nu,q)| \leq \omega_R(W_2((Id\times p)\sharp\mu,(Id\times q)\sharp\nu))$$

for all (μ,p), (ν,q) in $\mathcal{F}_2(\mathbb{R}^d)$ such that:

$$m_2(\mu) + \|p\|^2_{L^2_\mu} \leq R^2, \quad m_2(\nu) + \|q\|^2_{L^2_\nu} \leq R^2.$$

(A_1) It exists $k : [0,+\infty[\to \mathbb{R}$ a modulus of continuity such that for all for all $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, $p\in L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$, $q\in L^2_\nu(\mathbb{R}^d,\mathbb{R}^d)$ with

$$m_2(\mu)\leq R^2, \quad \|p\|_{L^2_\mu}\leq R, \quad \|q\|_{L^2_\mu}\leq R,$$

it holds:

$$|\mathcal{H}(\mu,p)-\mathcal{H}(\mu,q)|\leq k(R)\|p-q\|_{L^2_\mu}.$$

3. Viscosity solutions and strict viscosity solutions

3.1. **Definitions of viscosity solutions.** In this section we introduce two different notions of viscosity solutions. The first one was introduced in [16] by Gangbo, Nguyen and Tudorascu, while the second one (that we call strict viscosity solution) comes from the work of Marigonda and Quincampoix ([20] and [24]).

We introduce the definitions of superdifferential ([16] and [17]) and approximate superdifferential ([20], [23], [24]) of a function $u:[0,T]\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$:

Proposition and Definition 3.1 (Superdifferential and Approximate Superdifferentials)**.**
Let $(t_0,\mu_0)\in[0,T[\times\mathcal{P}_2(\mathbb{R}^d)$ and $\varepsilon\geq 0$. The ε-superdifferential of u at (t_0,μ_0) is the set $D^+_\varepsilon u(t_0,\mu_0)$ of elements $(p_t,p_\mu)\in\mathbb{R}\times L^2_{\mu_0}(\mathbb{R}^d,\mathbb{R}^d)$ such that:

- *$p_\mu\in\mathcal{T}_{\mu_0}(\mathbb{R}^d)$ (see subsection 2.1)*
- *for all $(t,\nu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, $\gamma\in\Pi_o(\mu_0,\nu)$:*

$$u(t,\nu)-u(t_0,\mu_0)\leq p_t(t-t_0)+\int_{\mathbb{R}^{2d}}p_\mu(x)\cdot(y-x)\;d\gamma(x,y)$$
$$+\varepsilon\sqrt{|t-t_0|^2+W_2^2(\mu_0,\nu)}+o\left(\sqrt{|t-t_0|^2+W_2^2(\mu_0,\nu)}\right).$$

Or, equivalently to the last inequality:

$$u(t,\nu)-u(t_0,\mu_0)\leq p_t\times(t-t_0)+\int_{\mathbb{R}^{2d}}p_\mu(x)\cdot(y-x)\;d\gamma(x,y)$$
$$+\varepsilon\sqrt{|t-t_0|^2+\|y-x\|^2_{L^2_\gamma}}+o\left(\sqrt{|t-t_0|^2+\|y-x\|^2_{L^2_\gamma}}\right)$$

for all $(t,\nu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, $\gamma\in\Pi(\mu_0,\nu)$.

The ε-subdifferential is defined in a symmetric way: $D^-_\varepsilon u(t_0,\mu_0)=-D^+_\varepsilon(-u)(t_0,\mu_0)$.

When $\varepsilon=0$, we will talk about superdifferential and subdifferential and we will write $D^+u(t_0,\mu_0)$ and $D^-u(t_0,\mu_0)$.

In this definition we slightly abused notations by writing:

$$\|y-x\|_{L^2_\gamma}=\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}|y-x|^2\;d\gamma(x,y)\right)^{1/2}.$$

Remark 3.2.
- If $D^-u(t_0,\mu_0)\cap D^+u(t_0,\mu_0)\neq\emptyset$, then u is differentiable at (t_0,μ_0) and $D^-u(t_0,\mu_0)=D^+u(t_0,\mu_0)=\{Du(t_0,\mu_0)\}$ (see [17] for the details).
- The continuity of u implies that D^+u is non empty in a dense subset of $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$ (see [24]).

Example 3.3. If $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ is fixed, the map $u : \mu \mapsto W_2^2(\mu, \nu)$ is super-differentiable everywhere (see [1]), more precisely for every $\gamma(x, y) \in \Pi_o(\mu_0, \nu)$, the map $x \mapsto 2\left(x - \int_{\mathbb{R}^d} y d\gamma^x(y)\right)$ belongs to $D^+u(\mu_0)$.

Remark 3.4. The approximate superdifferential used in [24] is slightly smaller than in the above definition, the p_μ being constrained to be in a strictly smaller subset of $\mathcal{T}_{\mu_0}(\mathbb{R}^d)$. Thanks to Proposition 3.11 (see also Remark 3.12) below, this difference doesn't change the definition of viscosity solutions.
In [20] and [23], the superdifferential is even smaller but is not suitable to prove the comparison principle inside these articles and should be changed to the definition of [24] or to the present definition.

The superdifferential of u is linked to the superdifferential of its lift U by the following result (we refer to subsection 2.2 for the meaning of the notations p_X and p_X^o):

Proposition 3.5. *(see* [17]*, Theorem 3.17 and* [24]*, Proposition 3.14) Let* $(t_0, \mu_0) \in [0, T[\times \mathcal{P}_2(\mathbb{R}^d)$.

(i) *If* (p_t, p_μ) *belongs to* $D^+u(t_0, \mu_0)$ *and* $X\sharp\mathbb{P} = \mu_0$*, then* $(p_t, p_\mu \circ X) \in D^+U(t_0, X)$.

(ii) *Let* (p_t, ξ) *belongs to* $D^+U(t_0, X)$ *and* $X\sharp\mathbb{P} = \mu_0$*, then:*

$$(p_t, p_X(\xi) \circ X),\ (p_t, p_X^o(\xi) \circ X) \in D^+U(t_0, X),$$

(iii) *Let* (p_t, ξ) *belongs to* $D^+U(t_0, X)$ *and* $X\sharp\mathbb{P} = \mu_0$*, then*

$$(p_t, p_X^o(\xi)) \in D^+u(t_0, \mu_0).$$

The definitions of viscosity solutions and strict viscosity solutions follows:

Definition 3.6 (Viscosity Solutions [16], [17])**.** Let $w : [0, T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ continuous.

- w is a *viscosity subsolution* of (HJ) if for all $(t_0, \mu_0) \in [0, T[\times \mathcal{P}_2(\mathbb{R}^d)$:
$$p_t + \mathcal{H}(\mu_0, p_\mu) \geq 0 \quad \forall (p_t, p_\mu) \in D^+w(t_0, \mu_0).$$
- w is a *viscosity supersolution* of (HJ) if for all $(t_0, \mu_0) \in [0, T[\times \mathcal{P}_2(\mathbb{R}^d)$:
$$p_t + \mathcal{H}(\mu_0, p_\mu) \leq 0 \quad \forall (p_t, p_\mu) \in D^-w(t_0, \mu_0).$$
- w is a *viscosity solution* if it is both a supersolution and a subsolution.

Definition 3.7 (Strict viscosity solutions)**.** Let $w : [0, T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ continuous.

- w is a *strict viscosity subsolution* of (HJ) if it exists
$$C : \mathcal{P}_2(\mathbb{R}^d) \to [0, +\infty[$$
bounded on bounded sets such that for all $(t_0, \mu_0) \in [0, T[\times \mathcal{P}_2(\mathbb{R}^d)$ and $\varepsilon > 0$:
$$p_t + \mathcal{H}(\mu_0, p_\mu) \geq -C(\mu_0)\,\varepsilon \quad \forall (p_t, p_\mu) \in D_\varepsilon^+w(t_0, \mu_0).$$
- w is a *strict viscosity supersolution* of (HJ) if if it exists
$$C : \mathcal{P}_2(\mathbb{R}^d) \to [0, +\infty[$$
bounded on bounded sets such that for all $(t_0, \mu_0) \in [0, T[\times \mathcal{P}_2(\mathbb{R}^d)$ and $\varepsilon > 0$:
$$p_t + \mathcal{H}(\mu_0, p_\mu) \leq C(\mu_0)\,\varepsilon \quad \forall (p_t, p_\mu) \in D_\varepsilon^-w(t_0, \mu_0).$$

- w is a *strict viscosity solution* if it is both a strict supersolution and a stict subsolution.

Remark 3.8. In the definition of [24], $C(\cdot)$ is constant, nevertheless, all the results of [24] used here work for a non constant $C(\cdot)$.

Remark 3.9. The terminology "**strict** viscosity subsolution" is borrowed from [12] (see also [14]). Nevertheless, the notion introduced in [12] is slightly different. Adapting it to our setting gives a new notion of subsolution that we call Lions' strict viscosity subsolution:
w is a strict viscosity subsolution of (HJ) if for all $\varepsilon > 0$, $(t_0, \mu_0) \in [0, T[\times \mathcal{P}_2(\mathbb{R}^d)$:

$$p_t + \sup \Big\{ \mathcal{H}(\mu_0, p_\mu + \xi) :\ \|\xi\|_{L^2_\mu} \leq \varepsilon \Big\} \geq 0 \quad \forall (p_t, p_\mu) \in D^+_\varepsilon w(t_0, \mu_0).$$

We define in the same way Lions' strict viscosity supersolution. The following result can be easily proved:

Lemma 3.10. *Assume $\mathcal{H}$ satisfies the following assumption:*
(A_1') It exists $k : [0, +\infty[\to \mathbb{R}$ a modulus of continuity such that for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $p, q \in L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$ with $m_2(mu) \leq R^2$, it holds:

$$|\mathcal{H}(\mu, p) - \mathcal{H}(\mu, q)| \leq k(R) \|p - q\|_{L^2_\mu}.$$

Then, if u is a Lions' strict viscosity subsolution (resp. supersolution) of (HJ), it is also a strict viscosity subsolution (resp. supersolution) of (HJ) as written in Definition 3.7.

Note that the assumption (A_1') is stronger than (A_1). It is satisfied by the Hamiltonian $\mathcal{H}_2$ of Example 2.9, but not by $\mathcal{H}_1$ of Example 2.7.

The following useful property allows to restrict the set of elements of the superdifferential in the definitions of strict viscosity solutions:

Proposition 3.11. *For any $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, let $F_{\mu_0} \subset L^2_{\mu_0}(\mathbb{R}^d, \mathbb{R}^d)$ a dense subset of $\mathcal{T}_{\mu_0}(\mathbb{R}^d)$. Let $w : [0, T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ continuous and assume the regularity assumption (4) on $\mathcal{H}$ holds.*

- *The function w is a strict subsolution of* (HJ) *if and only if it exists*

$$C : \mathcal{P}_2(\mathbb{R}^d) \to [0, +\infty[$$

bounded on bounded sets such that for all $(t_0, \mu_0) \in [0, T[\times \mathcal{P}_2(\mathbb{R}^d)$ and $\varepsilon > 0$

$$p_t + \mathcal{H}(\mu_0, p_\mu) \geq -C(\mu_0)\, \varepsilon \quad \forall (p_t, p_\mu) \in D^+_\varepsilon w(t_0, \mu_0) \cap (\mathbb{R} \times F_{\mu_0}).$$

- *In particular, the function w is a strict subsolution of* (HJ) *if and only if it exists*

$$C : \mathcal{P}_2(\mathbb{R}^d) \to [0, +\infty[$$

bounded on bounded sets such that for all $(t_0, \mu_0) \in [0, T[\times \mathcal{P}_2(\mathbb{R}^d)$ and $\varepsilon > 0$

$$p_t + \mathcal{H}(\mu_0, \nabla \varphi) \geq -C(\mu_0)\, \varepsilon \quad \forall \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d),\ (p_t, \nabla \varphi) \in D^+_\varepsilon w(t_0, \mu_0).$$

Proof: Assume it exists

$$C : \mathcal{P}_2(\mathbb{R}^d) \to [0, +\infty[$$

bounded on bounded sets such that for all $(t_0,\mu_0)\in]0,T[\times\mathcal{P}_2(\mathbb{R}^d)$

$$p_t+\mathcal{H}(\mu_0,p_\mu)\geq -C(\mu_0)\,\varepsilon\quad \forall (p_t,p_\mu)\in D^+_\varepsilon w(t_0,\mu_0)\cap(\mathbb{R}\times F_{\mu_0}).$$

Le us prove w is a strict supersolution of (HJ) (the opposite implication is straightforward).

Let (t_0,μ_0) in $[0,T[\times\mathcal{P}_2(\mathbb{R}^d)$ and $(p_t,p_\mu)\in D^+_\varepsilon w(t_0,\mu_0)$, for any $\delta>0$, take $p_\delta\in F_{\mu_0}$ such that $\|p_\mu-p_\delta\|^2_{L^2_{\mu_0}}\leq\delta$. Then, using Cauchy-Schwarz inequality, we have for all $(t,\nu)\in[0,T[\times\mathcal{P}_2(\mathbb{R}^d)$ and all $\gamma\in\Pi_o(\mu_0,\nu)$:

$$\begin{aligned}
& u(t,\nu)-u(t_0,\mu_0)\leq p_t(t-t_0)+\int_{\mathbb{R}^d\times\mathbb{R}^d}p_\mu(x)\cdot(y-x)\;d\gamma(x,y)\\
& +\varepsilon\sqrt{|t-t_0|^2+W_2^2(\mu_0,\nu)}+o\left(\sqrt{|t-t_0|^2+W_2^2(\mu_0,\nu)}\right)\\
\leq\;& p_t(t-t_0)+\int p_\delta(x)\cdot(y-x)\;d\gamma+\int(p_\mu-p_\delta)(x)\cdot(y-x)\;d\gamma(x,y)\\
& +\varepsilon\sqrt{|t-t_0|^2+W_2^2(\mu_0,\nu)}+o\left(\sqrt{|t-t_0|^2+W_2^2(\mu_0,\nu)}\right)\\
\leq\;& p_t(t-t_0)+\int p_\delta(x)\cdot(y-x)\;d\gamma(x,y)\\
& +(\varepsilon+\delta)\sqrt{|t-t_0|^2+W_2^2(\mu_0,\nu)}+o\left(\sqrt{|t-t_0|^2+W_2^2(\mu_0,\nu)}\right).
\end{aligned}$$

This proves (p_t,p_δ) belongs to $D^+_{\varepsilon+\delta}w(t_0,\mu_0)$ so that:

$$p_t+\mathcal{H}(\mu_0,p_\delta)\geq -C(\mu_0)(\varepsilon+\delta).$$

The result follows by making δ tend to 0 using (4).
QED.

Remark 3.12.
- One can choose as dense subset of $\mathcal{T}_{\mu_0}(\mathbb{R}^d)$ the convex cone generated by the optimal displacement or anti-diplacements:
$$dis^-(\mu_0):=\{\lambda(T-Id):\ \lambda>0,\ (Id\times T)\sharp\mu\in\Pi_o(\mu,T\sharp\mu)\},$$
$$dis^+(\mu_0):=\{\lambda(Id-T):\ \lambda>0,\ (Id\times T)\sharp\mu\in\Pi_o(\mu,T\sharp\mu)\}.$$
In [24], the subdifferential is restricted to the set $dis^-(\mu_0)$ while the subdifferential is restricted to $dis^+(\mu_0)$ (see also [20] and [23]).
- Note that the proof of Proposition 3.11 requires to use the stronger notion of **strict** subsolutions and does not work for non-strict viscosity solutions.

3.2. **Test functions and regular tests functions.** In this section, we study how the notion of strict viscosity solution can be expressed with test functions satisfying some regularity of the sub/superdifferential.

Definition 3.13 (ε-Test functions)**.** Let $u:[0,T]\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$, $\varepsilon>0$ and $(t_0,\mu_0)\in[0,T[\times\mathcal{P}_2(\mathbb{R}^d)$.

The map $v:[0,T]\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ is an ε-*supertest* function for u at (t_0,μ_0) if:
- v is *continuous, differentiable at* (t_0,μ_0),
- $u(t_0,\mu_0)=v(t_0,\mu_0)$,

- it exists $r > 0$ such that $u(t,\nu) \le v(t,\nu) + \varepsilon\sqrt{|t-t_0|^2 + W_2^2(\mu_0,\nu)}$ for all $(t,\nu) \in [0,T[\times\mathcal{P}_2(\mathbb{R}^d)$ with $|t-t_0|^2 + W_2^2(\mu_0,\nu) < r^2$.

The map v is an ε-*subtest* function for u at (t_0,μ_0) if $-v$ is an ε-*supertest* function for $-u$ at (t_0,μ_0).

The following result can be proved by slightly adapting the proof of Theorem 3.30 in [24]:

Proposition 3.14. *Let $u : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be continuous.*
Assume (4) holds then the following assertions are equivalent:

(i) *u is a* **strict viscosity subsolution** *of* (HJ)*,*

(ii) *there exists $C : \mathcal{P}_2(\mathbb{R}^d) \to [0,+\infty[$ bounded on bounded sets, such that: for all $\varepsilon > 0$, all $(t_0,\mu_0) \in [0,T[\times\mathcal{P}_2(\mathbb{R}^d)$ and all ε-supertest function v of u at (t_0,μ_0)*

$$\partial_t v(t_0,\mu_0) + \mathcal{H}(\mu_0, D_\mu v(t_0,\mu_0)) \ge -C(\mu_0)\varepsilon.$$

Remark 3.15. As done in [24], we can also define strict viscosity solutions using rearrangement invariant ε-test functions $V : [0,T] \times L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d) \to \mathbb{R}$ and the lift H of $\mathcal{H}$ defined by (3).

In order to prove the equivalence between strict and non-strict viscosity solutions, we need more regular test functions:

Definition 3.16 (Regular test functions)**.** Let $u : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be continuous and $(t_0,\mu_0) \in [0,T[\times\mathcal{P}_2(\mathbb{R}^d)$. We denote by $S^+(u,t_0,\mu_0)$ the set of functions $v : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ which write as:

$$v(t,\nu) = u(t_0,\mu_0) + p_t \times (t-t_0) + \int_{\mathbb{R}^d} \varphi \, d(\nu-\mu_0) + \xi\left(\sqrt{|t-t_0|^2 + W_2^2(\nu,\mu_0)}\right)$$

such that:

$$\left\{\begin{array}{l} p_t \in \mathbb{R},\ \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d) \\ \exists R > 0 \text{ and } \sigma \in C^1([0,R[) \cap C([0,+\infty[) \text{ such that } \xi(t) = t\sigma(t), \\ \xi \text{ is non decreasing and } \xi'(0) = \sigma(0) = 0. \end{array}\right. \tag{9}$$

Similarly we define $S^-(u,t_0,\mu_0)$ the set of functions $v : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ which write as:

$$v(t,\nu) = u(t_0,\mu_0) + p_t \times (t-t_0) + \int_{\mathbb{R}^d} \varphi \, d(\nu-\mu_0) - \xi\left(\sqrt{|t-t_0|^2 + W_2^2(\nu,\mu_0)}\right)$$

with again (9).

Remark 3.17. Note that a function v of $S^+(u,t_0,\mu_0)$ with $R > 0$ as above satisfy:

- for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ with $W_2^2(\mu,\mu_0) < R$, $v(\cdot,\mu)$ is in $C^1(]t_0-\alpha, t_0+\alpha[)$ with $\alpha = \sqrt{R^2 - W_2^2(\mu,\mu_0)}$,
- $\partial_t v(t_0,\mu_0) = p_t$, $D_\mu v(t_0,\mu_0) = \nabla\varphi$.

We will see (Proposition 3.20) that v has non-empty superdifferential around (t_0, μ_0) thanks to the regularity of ξ and $W_2^2(\cdot, \mu_0)$.

The following result holds:

Proposition 3.18. *Let $u : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be continuous. Assume (4) holds, then the following assertions are equivalent:*

(i) *u is a* **strict viscosity subsolution** *of* (HJ),

(ii) *there exists $C : \mathcal{P}_2(\mathbb{R}^d) \to [0, +\infty[$, bounded on bounded sets, such that: for all $\varepsilon > 0$, all $(t_0, \mu_0) \in]0, T[\times \mathcal{P}_2(\mathbb{R}^d)$:*

$$\partial_t v(t_0, \mu_0) + \mathcal{H}(\mu_0, D_\mu v(\mu_0)) \geq -C(\mu_0)\varepsilon,$$

for all ε-supertest function v of u at (t_0, μ_0) belonging to $\mathcal{S}^+(u, t_0, \mu_0)$.

The symmetric result holds for supersolution using $\mathcal{S}^-(u, t_0, \mu_0)$.

The proof is very similar to the proof of Proposition 3.14 and uses the following lemma (Lemma 3.1.8. in [5]).

Lemma 3.19. *Let $R > 0$ and $\omega :]0, R] \to \mathbb{R}$ be a non-decreasing measurable bounded map such that $\lim_{t \to 0^+} \omega(t) = 0$. Then it exists $\omega_0 : [0, +\infty[\to \mathbb{R}$ continuous such that:*

a) *$\omega(\tau) \leq \omega_0(\tau)$ for all $\tau \in]0, \frac{R}{2}]$,*

b) *$\omega_0(0) = 0$,*

c) *the function $\rho(r) = r\omega_0(r)$ is in $\mathcal{C}^1([0, R[)$ and $\rho'(0) = 0$.*

Sketch of Proposition 3.18:
We prove only $(ii) \Rightarrow (i)$. Using Propostion 3.11, it is enough to prove that for all $(p_t, \nabla\varphi) \in D_\varepsilon^+(t_0, \mu_0)$ we have:

$$p_t + \mathcal{H}(\mu_0, \nabla\varphi) \geq -C(\mu_0)\varepsilon.$$

The aim is then to build a function v of $S^+(u, t_0, \mu_0)$ which is an ε-supertest function and such that:

$$\partial_t v(t_0, \mu_0) = p_t, \quad D_\mu v(t_0, \mu_0) = \nabla\varphi.$$

Set

$$\alpha := \limsup_{|t-t_0|^2 + W_2^2(\mu_0,\nu) \to 0} \frac{u(t,\nu) - \left(u(t_0,\mu_0) + p_t(t-t_0) + \displaystyle\int \varphi \, d(\nu - \mu_0)\right)}{\sqrt{|t-t_0|^2 + W_2^2(\mu_0,\nu)}} - \varepsilon.$$

If $\alpha \leq 0$ then setting $v(t,\nu) := u(t_0,\mu_0) + p_t(t-t_0) + \displaystyle\int \varphi \, d(\nu - \mu_0)$, the proof is concluded.

Assume now $\alpha > 0$ and set for all $r > 0$

$$\omega(r) = \sup_{|t-t_0|^2 + W_2^2(\mu_0,\nu) \leq r^2} \frac{u(t,\nu) - \left(u(t_0,\mu_0) + p_t(t-t_0) + \displaystyle\int \varphi \, d(\nu - \mu_0)\right)}{\sqrt{|t-t_0|^2 + W_2^2(\mu_0,\nu)}} - \varepsilon$$

This function is non-decreasing, bounded on some $]0, R[$, and it satisfies $\lim_{r \to 0^+} \omega(r) = 0$. Moreover, we can prove it is measurable. Then, we set for all (t, ν) with $\sqrt{|t-t_0|^2 + W_2^2(\mu_0,\nu)} < \frac{R}{2}$:

$$v(t,\nu) := u(t_0,\mu_0) + p_t(t-t_0) + \int \varphi d(\nu - \mu_0) + \rho\left(\sqrt{|t-t_0|^2 + W_2^2(\mu_0,\nu)}\right)$$

where ρ is given by Lemma 3.19 Then v belongs to $S^+(u,t_0,\mu_0)$ and satisfies:

$$\partial_t v(t_0,\mu_0)=p_t, \quad D_\mu v(t_0,\mu_0)=\nabla\varphi, \quad v(t_0,\mu_0)=u(t_0,\mu_0)$$

$\forall\nu$ such that $\sqrt{|t-t_0|^2+W_2^2(\mu_0,\nu)}<\dfrac{R}{2}: \quad u(t,\nu)\leq v(t,\nu)+\varepsilon\sqrt{|t-t_0|^2+W_2^2(\mu_0,\nu)}.$

The result is proved.

QED.

Reducing the set of test functions to $\mathcal{S}^+(u,t_0,\mu_0)$ is interesting because of the nice regularity properties contained in the following result:

Proposition 3.20. *Let $v\in\mathcal{S}^+(u,t_0,\mu_0)$ and R as in the definition 3.16. Then:*

(i) *For all $(t,\mu)\in[0,T[\times\mathcal{P}_2(\mathbb{R}^d)$ such that $|t-t_0|^2+W_2^2(\mu,\mu_0)<R^2$ the superdifferential $D^+v(t,\mu)$ is not empty. More precisely, taking $\gamma\in\Pi_o(\mu,\mu_0)$, we have $(q_t,q_\mu)\in D^+v(t,\mu)$ with:*

$$q_t:=p_t+\frac{\xi'\left(\sqrt{|t-t_0|^2+W_2^2(\mu,\mu_0)}\right)}{\sqrt{|t-t_0|^2+W_2^2(\mu,\mu_0)}}(t-t_0),$$

$$q_\mu(x):=\nabla\varphi(x)+\frac{\xi'\left(\sqrt{|t-t_0|^2+W_2^2(\mu,\mu_0)}\right)}{\sqrt{|t-t_0|^2+W_2^2(\mu,\mu_0)}}\left(x-\int_{\mathbb{R}^d}y\,d\gamma^x(y)\right).$$

(ii) *Considering a sequence $(t_n,\mu_n)_{n\in\mathbb{N}}$ converging to (t_0,μ_0), it exists $N\in\mathbb{N}$ and $(q_{t,n},q_{\mu,n})\in D^+v(t_n,\mu_n)$ defined for $n>N$ such that:*

$$\lim_{\substack{n\to+\infty\\ n>N}} W_2((Id\times q_{\mu,n})\sharp\mu_n,(Id\times D_\mu v(t_0,\mu_0))\sharp\mu_0)=0,$$

$$\lim_{\substack{n\to+\infty\\ n>N}} |q_{t,n}-\partial_t v(t_0,\mu_0)|=0.$$

To prove this proposition, we need the following lemma:

Lemma 3.21. *Let $u:[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, $(t_0,\mu_0)\in[0,T[\times\mathcal{P}_2(\mathbb{R}^d)$ and $h:I\to\mathbb{R}$ where $I\subset\mathbb{R}$ is an open intervall containing $u(t_0,\mu_0)$. We assume $(p_t,p_\mu)\in D^+u(t_0,\mu_0)$ and h is non-increasing, derivable in $u(t_0,\mu_0)$. Then:*

$$h'(u(t_0,\mu_0))\times(p_t,p_\mu)\in D^+(h\circ u)(t_0,\mu_0).$$

Proof: As h is derivable at $u(t_0,\mu_0)$ and $h'(u(t_0,\mu_0))\geq 0$, for any $(t,\mu)\in\,]0,T[\times\mathcal{P}_2(\mathbb{R}^d)$ and any $\gamma\in\Pi_o(\mu_0,\mu)$:

$$\begin{aligned}h(u(t,\mu))&=h(u(t_0,\mu_0))+h'(u(t_0,\mu_0))\,(u(t,\mu)-u(t_0,\mu_0))+o\,(u(t,\mu)-u(t_0,\mu_0))\\&\leq h(u(t_0,\mu_0))+h'(u(t_0,\mu_0))\left[p_t(t-t_0)+\int_{\mathbb{R}^d\times\mathbb{R}^d}p_\mu(x)\cdot(y-x)\,d\gamma(x,y)\right]\\&\quad+h'(u(t_0,\mu_0))o\left(\sqrt{|t-t_0|^2+W_2^2(\mu,\mu_0)}\right)+o\left(\sqrt{|t-t_0|^2+W_2^2(\mu,\mu_0)}\right).\end{aligned}$$

QED

Remark 3.22. In particular if $h(t) = \xi(\sqrt{t})$ with ξ like in the definition 3.16, then setting $w(t,\mu) = h\left(|t-t_0|^2 + W_2^2(\mu,\mu_0)\right)$ and taking $\gamma(x,y) \in \Pi_o(\mu,\mu_0)$, we get:

$$\frac{\xi'\left(\sqrt{|t-t_0|^2 + W_2^2(\mu,\mu_0)}\right)}{\sqrt{|t-t_0|^2 + W_2^2(\mu,\mu_0)}} \times \left[t-t_0\,,\, \left(x - \int y d\gamma^x(y)\right)\right] \in D^+w(t,\mu)$$

for all $(t,\mu) \in]0,T[\times \mathcal{P}_2(\mathbb{R}^d)$ satisfying $\sqrt{|t-t_0|^2 + W_2^2(\mu,\mu_0)} < R$.

Proof of Proposition 3.20: Let v as in the proposition.

(i) Take $(t,\mu) \in [0,T[\times \mathcal{P}_2(\mathbb{R}^d)$ satisfying $\sqrt{|t-t_0|^2 + W_2^2(\mu,\mu_0)} < R$ and $\gamma(x,y) \in \Pi_o(\mu,\mu_0)$. Using Remark 3.22, we get (i).

(ii) Let (t,μ) as above and (q_t, q_μ) defined as in (i). Clearly:

$$\lim_{t\to t_0} |q_t - p_t| = 0.$$

Moreover, by Jensen inequality:

$$\|x - \int_{\mathbb{R}^d} y\, d\gamma^x(y)\|_{L^2_\mu} := \left(\int_{R^d} |x - \int_{\mathbb{R}^d} y d\gamma_n^x(y)|^2 d\mu(x)\right)^{1/2}$$

$$\leq \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\gamma(x,y)\right)^{1/2} = W_2(\mu,\mu_0).$$

Then :

$$W_2((Id\times q_\mu)\sharp\mu, (Id\times\nabla\varphi)\sharp\mu)) \leq \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 + |q_\mu(x) - \nabla\varphi(y)|^2 d\gamma(x,y)\right)^{1/2}$$

$$\leq W_2(\mu,\mu_0) + \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |q_\mu(x) - \nabla\varphi(x) + \nabla\varphi(x) - \nabla\varphi(y)|^2\, d\gamma_n(x,y)\right)^{1/2}$$

$$\leq W_2(\mu,\mu_0) + \frac{\xi'\left(\sqrt{|t-t_0|^2 + W_2^2(\mu,\mu_0)}\right)}{\sqrt{|t-t_0|^2 + W_2^2(\mu,\mu_0)}} \|x - \int_{\mathbb{R}^d} y\, d\gamma^x(y)\|_{L^2_\mu} \int_{\mathbb{R}^d} + Lip(\nabla\varphi) W_2(\mu,\mu_0)$$

$$\leq W_2(\mu,\mu_0)\left[1 + (Lip(\nabla\varphi))\right] + \frac{\xi'\left(\sqrt{|t-t_0|^2 + W_2^2(\mu,\mu_0)}\right)}{\sqrt{|t-t_0|^2 + W_2^2(\mu,\mu_0)}} W_2(\mu,\mu_0)$$

which tends to 0 as $n \to +\infty$. We conclude by Remark 3.17

QED.

3.3. **Equivalence between strict and non-strict viscosity solutions.** Using the last results of the previous section, we are able to show, under certain regularity assumptions of $\mathcal{H}$, that every viscosity solution of (HJ) is a strict solution.

Theorem 3.23. *Assume $\mathcal{H}$ is continuous in the following sense:*

$$\lim_{n\to+\infty} W_2((Id\times p_n)\sharp\mu_n, (Id\times p)\sharp\mu) = 0 \Rightarrow \lim_{n\to+\infty} \mathcal{H}(\mu_n, p_n) = \mathcal{H}(\mu, p)$$

for any $\mu, \mu_n \in \mathcal{P}_2(\mathbb{R}^d)$, $p_n \in L^2_{\mu_n}(\mathbb{R}^d,\mathbb{R}^d)$, $p \in L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$.

(i) *Assume moreover the following assumption:*

(A_1') *It exists* $k : [0,+\infty[\to \mathbb{R}$ *a modulus of continuity such that for all for all* $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $p \in L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$, $q \in L^2_\nu(\mathbb{R}^d,\mathbb{R}^d)$ *with* $m_2(\mu) \le R^2$, *it holds:*

$$|\mathcal{H}(\mu,p) - \mathcal{H}(\mu,q)| \le k(R)\|p-q\|_{L^2_\mu}.$$

Then, $u : [0,T]\times\mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ *(continuous) is a strict viscosity subsolution (resp. supersolution) of* (HJ), *if and only if it is a viscosity subsolution (resp. supersolution) of* (HJ).

(ii) *Assume now, we have only assumption* (A_1): *It exists* $k : [0,+\infty[\to \mathbb{R}$ *a modulus of continuity such that for all for all* $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $p \in L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$, $q \in L^2_\nu(\mathbb{R}^d,\mathbb{R}^d)$ *with*

$$m_2(\mu) \le R^2, \quad \|p\|_{L^2_\mu} \le R, \quad \|q\|_{L^2_\mu} \le R,$$

it holds:

$$|\mathcal{H}(\mu,p) - \mathcal{H}(\mu,q)| \le k(R)\|p-q\|_{L^2_\mu}.$$

Moreover assume $u : [0,T]\times\mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ *is such for all* $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, *it exists* $L(\mu_0) > 0$ *and* $\mathcal{O}(\mu_0)$ *a neighborhood of* μ_0 *such that:*

$$|u(t,\mu_0) - u(t,\mu)| \le L(\mu_0)W_2(\mu_0,\mu) \quad \forall t \in [0,T],\, \forall \mu \in \mathcal{O}(\mu_0).$$

Then u *is a strict viscosity subsolution (resp. supersolution) of* (HJ), *if and only if it is a viscosity subsolution (resp. supersolution) of* (HJ).

The proof of this result is based upon the appendix of [12].

Proof:

1. We first show that if u is a viscosity subsolution and $\mathcal{H}$ satisfies $(A1')$, it is a strict viscosity subsolution.

By Proposition 3.18, it is enough to show that it exists $C : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^+$, bounded on bounded sets, such that, taking an ε-supertest $v \in \mathcal{S}^+(u,t_0,\mu_0)$, it holds

$$\partial_t v(t_0,\mu_0) + \mathcal{H}(\mu_0, D_\mu v(t_0,\mu_0)) \ge -C(\mu_0)\varepsilon,$$

for any $(t_0,\mu_0) \in [0,T[\times\mathcal{P}_2(\mathbb{R}^d)$ and $\varepsilon > 0$.

Let $(t_0,\mu_0) \in]0,T[\times\mathcal{P}_2(\mathbb{R}^d)$, $\varepsilon > 0$, $R > 0$ and $v \in \mathcal{S}^+(u,t_0,\mu_0)$ such that:

$$\left\{\begin{array}{l} u(t,\nu) \le v(t,\nu) + \varepsilon\sqrt{|t-t_0|^2 + W_2^2(\nu,\mu_0)} \\ \qquad \forall (t,\nu) \in]0,T[\times\mathcal{P}_2(\mathbb{R}^d),\ \sqrt{|t-t_0|^2 + W_2^2(\nu,\mu_0)} < R \\ u(t_0,\mu_0) = v(t_0,\mu_0). \end{array}\right. \tag{10}$$

Note that this easily implies for all $(t,\nu) \in [0,T[\times\mathcal{P}_2(\mathbb{R}^d)$ with $\sqrt{|t-t_0|^2 + W_2^2(\nu,\mu_0)} < R$:

$$[u(t_0,\mu_0) - v(t_0,\mu_0)] - [u(t,\nu) - v(t_0,\mu_0)] \ge -\varepsilon\sqrt{|t-t_0|^2 + W_2^2(\nu,\mu_0)}. \tag{11}$$

We aim to build an element $(\mathbf{p}_t,\mathbf{p}_\mu)$ of the **non approximate** superdifferential $D^+u(t_\delta,\mu_\delta)$ where (t_δ,μ_δ) is close to $(\partial v(t_0,\mu_0), D_\mu v(t_0,\mu_0))$ and μ_δ is close to μ_0. This will lead to:

$$\mathbf{p}_t + \mathcal{H}(\mu_\delta,\mathbf{p}_\mu) \ge 0$$

and we will conclude using the regularity properties of $\mathcal{H}$.

Step 1: Let $\alpha \in]0,1[$ such that

$$\alpha < \min\left\{\frac{t_0}{R}, \frac{T-t_0}{R}\right\} \tag{12}$$

so that $|t-t_0| \le R\alpha \Rightarrow t \in]0,T[$. Moreover choose δ such that:

$$(\delta+\varepsilon)\delta \le \frac{R\alpha}{2}. \tag{13}$$

Denote by U and V the lifts of u and v respectively and $X_0 \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ such that $X_0\sharp\mathbb{P} = \mu_0$. We introduce the following continuous functional defined for all $Y \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$, $t \in [0,T]$:

$$\Phi_\delta(t,Y) := U(t,Y) - V(t,Y) - \varepsilon\sqrt{|t-t_0|^2 + c^2(Y,X_0) + \delta} - \frac{|t-t_0|^2}{\delta} - \frac{\|Y-X_0\|^2}{\delta}$$

where we have use the notation: $c(X,Y) = W_2^2(X\sharp\mathbb{P}, Y\sharp\mathbb{P})$ for all $X, Y \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$.
Then set:

$$W_\delta(Y) := \begin{cases} \max_{t\in[0,T]}\left\{\Phi_\delta(t,Y) : \sqrt{|t-t_0|^2 + \|Y-X_0\|^2} \le R\alpha\right\} \text{ if } \|Y-X_0\| \le R\alpha, \\ \\ -\infty \text{ elsewhere.} \end{cases}$$

We notice that W_δ is upper semi-continuous and satisfies $\lim_{\|Y\|\to+\infty} W_\delta(Y) = -\infty$. By Stegall's variational principle (see for instance [6]), it exists X_δ, ξ_δ in $L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ such that:

$$\begin{aligned} &\forall Y \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d), \quad W_\delta(Y) \le W_\delta(X_\delta) + \langle \xi_\delta, Y - X_\delta\rangle, \\ &\qquad\qquad\qquad\qquad\quad \text{with } \|\xi_\delta\| \le \delta\ . \end{aligned} \tag{14}$$

We set $\mu_\delta := X_\delta\sharp\mathbb{P}$ and

$$t_\delta \in argmax_{t\in[0,T]}\left\{\Phi_\delta(t,Y) : \sqrt{|t-t_0|^2 + \|Y-X_0\|^2} \le R\alpha\right\}.$$

By the choice of α, we have $t_\delta \in]0,T[$.
Note that $\sqrt{|t_\delta-t_0|^2 + \|X_\delta - X_0\|^2} \le R\alpha$, in the next section we prove a better estimate.

Step 2: Estimate of $\sqrt{|t_\delta-t_0|^2 + \|X_\delta-X_0\|^2}$.
By (14), we have:

$$[U(t_0,X_0) - V(t_0,X_0)] - [U(t_\delta,X_\delta) - V(t_\delta,X_\delta)]$$
$$\le -\varepsilon\sqrt{|t_\delta-t_0|^2 + c(X_\delta,X_0)^2 + \delta} - \frac{|t_0-t_\delta|^2}{\delta} - \frac{\|X_0-X_\delta\|^2}{\delta} + \langle \xi_\delta, X_0 - X_\delta\rangle$$

which implies:

$$[u(t_0,\mu_0) - v(t_0,\mu_0)] - [u(t_\delta,\mu_\delta) - v(t_\delta,\mu_\delta)] + \frac{|t_0-t_\delta|^2}{\delta} + \frac{\|X_0-X_\delta\|^2}{\delta} \le \delta\|X_0 - X_\delta\|.$$

Then, applying (11):

$$-\varepsilon\sqrt{|t_\delta-t_0|^2 + W_2^2(\mu_\delta,\mu_0)} + \frac{|t_0-t_\delta|^2}{\delta} + \frac{\|X_0-X_\delta\|^2}{\delta} \le \delta\|X_0 - X_\delta\|$$

which implies:

$$\frac{|t_0-t_\delta|^2}{\delta}+\frac{\|X_0-X_\delta\|^2}{\delta}\leq(\delta+\varepsilon)\sqrt{|t_\delta-t_0|^2+\|X_\delta-X_0\|^2}.$$

We conclude by (13):

$$\sqrt{|t_\delta-t_0|^2+\|X_\delta-X_0\|^2}\leq(\delta+\varepsilon)\delta\leq\frac{R\alpha}{2}. \tag{15}$$

Step 3: Building an element of the superdifferential of $u(t_\delta,\mu_\delta)$.

Let $(t,Y)\in[0,T[\times\mathcal{P}_2(\mathbb{R}^d)$ such that: $\sqrt{|t_\delta-t|^2+\|X_\delta-Y\|^2}<\frac{R\alpha}{2}$. Then, by (15), $\sqrt{|t_0-t|^2+\|X_0-Y\|^2}<R\alpha$, and, by (14):

$$\Phi_\delta(t,Y)\leq\Phi_\delta(t_\delta,Y_\delta)+\langle\xi_\delta,Y-X_\delta\rangle.$$

Then, as by derivation:

$$\begin{aligned}&|t_0-t|^2+\|X_0-Y\|^2-|t_0-t_\delta|^2-\|X_0-X_\delta\|^2\\&=2\frac{t_\delta-t_0}{\delta}(t-t_\delta)+2\langle\frac{X_\delta-X_0}{\delta},Y-X_\delta\rangle+o\left(|t-t_\delta\|^2+\|X_\delta-Y\|^2\right)\end{aligned}$$

we get for all $(t,Y)\in]0,T[\times\mathcal{P}_2(\mathbb{R}^d)$:

$$\begin{aligned}&U(t,Y)-V(t,Y)-\varepsilon\sqrt{|t-t_0|^2+c^2(Y,X_0)+\delta}\\&\leq U(t_\delta,X_\delta)-V(t_\delta,X_\delta)-\varepsilon\sqrt{|t_\delta-t_0|^2+c^2(X_\delta,X_0)+\delta}\\&+\langle 2\frac{X_\delta-X_0}{\delta}+\xi_\delta,Y-X_\delta\rangle+2\frac{t_\delta-t_0}{\delta}(t-t_\delta)+o\left(\sqrt{|t-t_\delta|^2+\|Y-X_\delta\|^2}\right).\end{aligned}$$

In other words $\left(2\frac{t_\delta-t_0}{\delta},2\frac{X_\delta-X_0}{\delta}+\xi_\delta\right)$ belongs to the superdifferential at (t_δ,X_δ) of the following map:

$$(t,Y)\mapsto U(t,Y)-V(t,Y)-\varepsilon\sqrt{|t-t_0|^2+c^2(Y,X_0)+\delta}$$

wich is rearrangement invariant as U, V and c are.

Then, by Proposition 3.5, setting $q_\delta:=p^o_{X_\delta}\left(2\frac{X_\delta-X_0}{\delta}+\xi_\delta\right)\in\mathcal{T}_{\mu_\delta}(\mathbb{R}^d)$ (see subsection 2.2), it holds:

$$\|q_\delta\|_{L^2_{\mu_\delta}}\leq\left\|2\frac{X_\delta-X_0}{\delta}+\xi_\delta\right\|_{L^2_{\mathbb{P}}} \tag{16}$$

and for all $(t,\nu)\in[0,T[\times\mathcal{P}_2(\mathbb{R}^d)$, for all $\pi\in\Pi(\mu_\delta,\nu)$:

$$\begin{aligned}&u(t,\nu)-v(t,\nu)-\varepsilon\sqrt{|t-t_0|^2+W_2^2(\nu,\mu_0)+\delta}\\&\leq u(t_\delta,\mu_\delta)-v(t_\delta,\mu_\delta)-\varepsilon\sqrt{|t_\delta-t_0|^2+W_2^2(\mu_\delta,\mu_0)+\delta}\\&+\int_{\mathbb{R}^d\times\mathbb{R}^d}q_\delta(x)\cdot(y-x)d\pi(x,y)+2\frac{t_\delta-t_0}{\delta}(t-t_\delta)+o\left(\sqrt{|t-t_\delta|^2+W_2^2(\nu,\mu_\delta)}\right).\end{aligned}$$

Set $\Delta_\delta:=\sqrt{|t_\delta-t_0|^2+W_2^2(\mu_\delta,\mu_0)+\delta}$ and take $\gamma_\delta(x,y)$ in $\Pi_o(\mu_\delta,\mu_0)$, denote by $\bar{\gamma}_\delta$ the following element of $\mathcal{T}_{\mu_\delta}(\mathbb{R}^d)$:

$$\bar{\gamma}_\delta(x):=x-\int_{\mathbb{R}^d}yd\gamma^x_\delta(y).$$

Then, by Example 3.3, the vector $\frac{1}{\Delta_\delta}(t_\delta - t_0, \bar{\gamma}_\delta)$ belongs to the superdifferential at (t_δ, ν_δ) of

$$(t,\nu) \mapsto \sqrt{|t-t_0|^2 + W_2^2(\nu,\mu_0)+\delta}.$$

We then get, for all $(t,\nu) \in]0,T[\times \mathcal{P}_2(\mathbb{R}^d)$, for all $\pi \in \Pi(\mu_\delta,\nu)$:

$$\begin{aligned} u(t,\nu) \le u(t_\delta,\mu_\delta) + v(t,\nu) - v(t_\delta,\mu_\delta) + \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(q_\delta(x) + \varepsilon\frac{\bar{\gamma}_\delta(x)}{\Delta_\delta}\right)\cdot(y-x)d\pi(x,y) \\ + \left(2\frac{t_\delta - t_0}{\delta} + \varepsilon\frac{t_\delta - t_0}{\Delta_\delta}\right)(t-t_\delta) + o\left(\sqrt{|t-t_\delta|^2 + W_2^2(\nu,\mu_\delta)}\right). \end{aligned}$$

Finally by Proposition 3.20 and Estimate (15), it exists $(p_{t,\delta}, p_{\mu,\delta}) \in D^+v(t_\delta,\mu_\delta)$ such that:

$$\lim_{\delta\to 0} W_2((Id\times p_{\mu,\delta})\sharp\mu_\delta, (Id, D_\mu v(t_0,\mu_0))\sharp\mu_0) = 0, \ \lim_{\delta\to 0} p_{t,\delta} = \partial_t v(t_0,\mu_0). \tag{17}$$

We end this Step by concluding:

$$\left(\left[2\frac{t_\delta - t_0}{\delta} + \varepsilon\frac{t_\delta - t_0}{\Delta_\delta} + p_{t,\delta}\right] \ ; \ \left[q_\delta + \varepsilon\frac{\bar{\gamma}_\delta}{\Delta_\delta} + p_{\mu,\delta}\right]\right) \in D^+u(t_\delta,\mu_\delta). \tag{18}$$

Step 4: Estimates.
Here, we give some estimates in order to understand the behavior of the element of $D^+u(t_\delta,\mu_\delta)$ build in the next section when δ will go to zero.

By (15) and the definition of Δ_δ, we have:

$$\left\|2\frac{(t_\delta - t_0)}{\delta}\right\| \le 2(\delta+\varepsilon), \quad \left\|\varepsilon\frac{(t_\delta - t_0)}{\Delta_\delta}\right\| \le \varepsilon. \tag{19}$$

Again by (15), the definition of q_δ and (14):

$$\|q_\delta\|_{L^2_{\mu_\delta}} \le \left\|2\frac{(X_\delta - X_0)}{\delta} + \xi_\delta\right\| \le 3\delta + 2\varepsilon. \tag{20}$$

Finally, by Jensen's inequality (similarly to the proof of Proposition 3.20, (ii)):

$$\left\|\varepsilon\frac{\bar{\gamma}_\delta}{\Delta_\delta}\right\|_{L^2_{\mu_\delta}} \le \varepsilon\frac{W_2(\mu_0,\mu_\delta)}{\Delta_\delta} \le \varepsilon. \tag{21}$$

Step 5: Conclusion.
By (18), as u is a viscosity subsolution of (HJ), it holds:

$$\left[2\frac{t_\delta - t_0}{\delta} + \varepsilon\frac{t_\delta - t_0}{\Delta_\delta} + p_{t,\delta}\right] + \mathcal{H}\left(\mu_\delta, \left[q_\delta + \varepsilon\frac{\bar{\gamma}_\delta}{\Delta_\delta} + p_{\mu,\delta}\right]\right) \ge 0.$$

Setting $R_\delta = \sqrt{m_2(\mu_\delta)}$, by assumption $(A1')$ and by (20) and (21) the last inequality implies:

$$\begin{aligned} p_{t,\delta} + \mathcal{H}(\mu_\delta, p_{\mu_\delta}) &\ge -k(R_\delta)(3\delta+3\varepsilon) - \left[2\frac{t_\delta - t_0}{\delta} + \varepsilon\frac{t_\delta - t_0}{\Delta_\delta}\right] \\ &\ge (-2 - 3k(R_\delta))(\delta+\varepsilon) \end{aligned}$$

where we used (19) for the last inequality. Then it exists $c > 0$ such that:

$$p_{t,\delta} + \mathcal{H}(\mu_\delta, p_{\mu_\delta}) \geq -ck(R_\delta)(\delta + \varepsilon).$$

Then, sending δ to 0, using the continuity of k and $\mathcal{H}$ and (17) :

$$\partial_t v(t_0, \mu_0) + \mathcal{H}(\mu_0, D_\mu v(t_0, \mu_0)) \geq -ck\left[\sqrt{m_2(\mu_\delta)}\right]\varepsilon$$

the result follow by setting for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$: $C(\mu) := ck\left[\sqrt{m_2(\mu_\delta)}\right]$.

2. In the case (ii), we adapt Step 5 as follows. Arguing as in [24], Proposition 3.23, for δ and ε small enough:

$$p \in D^+_\varepsilon u(t_\delta, \mu_\delta) \Rightarrow \|p\|_{L^2_{\mu_\delta}} \leq L(\mu_0) + \varepsilon + 1 \leq L(\mu_0) + 2.$$

Then, setting, $R_\delta = \sqrt{m_2(\mu_\delta)}$, by assumption $(A1)$:

$$\mathcal{H}\left(\mu_\delta, \left[q_\delta + \varepsilon\frac{\bar{\gamma}_\delta}{\Delta_\delta} + p_{\mu,\delta}\right]\right) - \mathcal{H}(\mu_\delta, p_{\mu_\delta}) \leq 3k(L(\mu_0) + 2 + R_\delta)(\delta + \varepsilon).$$

The sequel follows by the same arguments as above.
QED.

4. Regular $L^2_{\mathbb{P}}$-extension of the Hamilton-Jacobi equation and rearrangement invariant viscosity solutions

4.1. **Building a suitable regular Hamiltonian on $L^2_{\mathbb{P}} \times L^2_{\mathbb{P}}$.** As seen, in the introduction, when $\mathcal{H}$ is regular enough, we aim to build $\overline{\mathrm{H}} : L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \to \mathbb{R}$ such that:

(a) $\overline{\mathrm{H}}(X, p \circ X) = \mathcal{H}(X\sharp\mathbb{P}, p)$ for all $X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$, $p \in L^2_{X\sharp\mathbb{P}}(\mathbb{R}^d, \mathbb{R}^d)$,
(b) $\overline{\mathrm{H}}$ is continuous in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$,
(c) $\overline{\mathrm{H}}$ is suitable to treat some control problem in $\mathcal{P}_2(\mathbb{R}^d)$.

The point (c) will be considered in section 4.3. Some properties of the corresponding equation

$$(\mathrm{HJ}_2) \quad \partial_t U(t, X) + \overline{\mathrm{H}}(X, D_X U(t, X)) = 0, \ \forall t \in [0, T[, \ \forall X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$$

will be studied in section 4.2.

We consider the following distance on $\mathcal{F}_2(\mathbb{R}^d)$ (recall 1):

$$d_{\mathcal{F}_2}((\mu, p), (\nu, q)) = W_2((Id \times p)\sharp\mu, (Id \times q)\sharp\nu) \quad \forall(\mu, p), (\nu, q) \in \mathcal{F}_2(\mathbb{R}^d).$$

The regularity of $\mathcal{H}$ with respect to $d_{\mathcal{F}_2}$ already appeared to be interesting (Theorem 3.23). Moreover we have seen that both $\mathcal{H}_1$ and $\mathcal{H}_2$ are indeed locally uniformly continuous for this distance. We may also introduce $\overline{\mathcal{H}}$ defined on transport plans supported on graphs:

$$\overline{\mathcal{H}}((Id \times p)\sharp\mu) := \mathcal{H}(\mu, p) \quad \forall(\mu, p) \in \mathcal{F}_2(\mathbb{R}^d).$$

The idea is now to extend $\overline{\mathcal{H}}$ to all $\mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ (for the distance W_2). This is possible if $\mathcal{H}$ is regular enough with respect to $d_{\mathcal{F}_2}$. Then, denoting the extension again by $\overline{\mathcal{H}}$, we will set:

$$\overline{\mathrm{H}}(X, Z) := \overline{\mathcal{H}}((X, Z)\sharp\mathbb{P}) \text{ for all } (X, Z) \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d).$$

The following proposition states that $d_{\mathcal{F}_2}$-regularity of $\mathcal{H}$ is mandatory to get (b), moreover the strategy above works to build $\overline{\mathrm{H}}$ with all the desired properties.

Proposition and Definition 4.1. *Let $\mathcal{H} : \mathcal{F}_2(\mathbb{R}^d) \to \mathbb{R}$.*

1) *Assume it exists $\overline{\mathrm{H}} : L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \to \mathbb{R}$ such that (a) and (b) holds. Then $\mathcal{H}$ is $d_{\mathcal{F}_2}$-continuous.*
2) *Assume (A_0) holds (see subsection 2.4):*
   (i) *Then it exists a unique $\overline{\mathrm{H}}$ satisfying (a) and (b), it is defined by:*
   $$\overline{\mathrm{H}}(X, Z) := \lim_{n\to+\infty} \mathcal{H}(\mu_n, p_n)$$
   for any sequence $(\mu_n, p_n)_n$ converging to $(X, Z)\sharp\mathbb{P}$ in $(\mathcal{F}_2(\mathbb{R}^d), d_{\mathcal{F}_2})$. We call $\overline{\mathrm{H}}$ the regular $L^2_{\mathbb{P}}$-extension of $\mathcal{H}$.
   (ii) *$\overline{\mathrm{H}}$ is rearrangement invariant in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d \times \mathbb{R}^d)$, that is for all $X,\ Z,\ X',\ Z' \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$:*
   $$\overline{\mathrm{H}}(X, Z) = \overline{\mathrm{H}}(X', Z') \text{ if } (X, Z)\sharp\mathbb{P} = (X', Z')\sharp\mathbb{P}.$$
   (iii) *The following map that we call the plan extension of $\mathcal{H}$ is also well defined:*
   $$\overline{\mathcal{H}}(\gamma) := \overline{\mathrm{H}}(X, Z) \text{ for any } (X, Z) \text{ such that } (X, Z)\sharp\mathbb{P} = \gamma.$$
   It is the unique continuous map on $\mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\overline{\mathcal{H}}((Id \times p)\sharp\mu) = \mathcal{H}(\mu, p)$ for every $(\mu, p) \in \mathcal{F}_2(\mathbb{R}^d)$. Moreover the lift of $\overline{\mathcal{H}}$ is $\overline{\mathrm{H}}$.
   (iv) *$\overline{\mathcal{H}}$ is W_2-uniformly continuous on balls and $\overline{\mathrm{H}}$ is $L^2_{\mathbb{P}}$-uniformly continuous on balls.*

The key point to prove the proposition is the following density result:

Lemma 4.2. *(Transport maps and plans) It holds:*

(i) *Plans supported by graphs are dense in the set of plans:*
$$\mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d) = \overline{\{(Id \times p)\sharp\mu : \ \mu \in \mathcal{P}_2(\mathbb{R}^d),\ p \in L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)\}}^{W_2},$$
(ii) *or equivalently:*
$$L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) = \overline{\{(X, p \circ X) : \ X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d),\ p \in L^2_{X\sharp\mathbb{P}}(\mathbb{R}^d, \mathbb{R}^d)\}}^{\|\cdot\|_{L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)^2}}.$$

Proof of the lemma: (i) can be proved similarly as Lemma 5.1. The main ideas are taken from Theorem 1.32 p24 of [25]: If $\mu,\ \nu \in \mathcal{P}_2(\mathbb{R}^d)$ are compactly supported and μ is atomless, then:
$$\Pi(\mu, \nu) = \overline{\{(Id \times T)\sharp\mu : T \in L^2_\mu(\mathbb{R}^d),\ T\sharp\mu = \nu\}}^{W_2}.$$

(ii) let us prove the equivalence with (i). The implication (ii)$\Rightarrow$ (i) is quite clear, let us prove the opposite. Let $(X, Y) \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ and set $\gamma := (X, Y)\sharp\mathbb{P}$, by (i), it exists $(\mu_n, p_n) \in \mathcal{F}_2(\mathbb{R}^d)$ such that
$$\lim_{n\to+\infty} W_2((Id \times p_n)\sharp\mu_n, \gamma) = 0.$$
Then, by Lemma 2.3, it exists $(X_n)_n$ in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ and $(\bar{X}, \bar{Y}) \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ such that:
$$X_n\sharp\mathbb{P} = \mu_n, \quad \lim_{n\to+\infty} \|(X_n, p_n \circ X_n) - (\bar{X}, \bar{Y})\| = 0, \quad (\bar{X}, \bar{Y})\sharp\mathbb{P} = \gamma.$$

Moreover, as $(\bar X,\bar Y)$ and (X,Y) have the same law, by Lemma 2.2, it exists $\tau_n:\Omega\to\Omega$, bijective, such that $\tau\sharp\mathbb{P}=\tau^{-1}\sharp\mathbb{P}=\mathbb{P}$ and:

$$\|(X,Y)-(\bar X\circ\tau_n,\bar Y\circ\tau_n)\|\leq 1/n.$$

Finally:

$$\begin{aligned}
&\lim_{n\to+\infty}\|(X_n\circ\tau_n,p_n\circ X_n\circ\tau_n)-(X,Y)\|\\
&\leq\lim_{n\to+\infty}\|(X_n\circ\tau_n,p_n\circ X_n\circ\tau_n)-(\bar X\circ\tau_n,\bar Y\circ\tau_n)\|+\|(\bar X\circ\tau_n,\bar Y\circ\tau_n)-(X,Y)\|\\
&\leq\lim_{n\to+\infty}\left(\int_\Omega|X_n(\tau_n(\omega))-\bar X(\tau_n(\omega))|^2+|p_n\circ X_n(\tau_n(\omega))-\bar Y(\tau_n(\omega))|^2d\mathbb{P}(\omega)\right)^{1/2}+1/n\\
&\leq\lim_{n\to+\infty}\left(\int_\Omega|X_n(\omega)-\bar X(\omega)|^2+|p_n\circ X_n(\omega)-\bar Y(\omega)|^2d(\tau_n\sharp\mathbb{P})(\omega)\right)^{1/2}\\
&\leq\lim_{n\to+\infty}\|(X_n,p_n\circ X_n)-(\bar X,\bar Y)\|=0.
\end{aligned}$$

To conclude, a general $(X,Y)\in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)\times L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$, can be approximated in $L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ by a sequence $(Z_n,p_n\circ Z_n)_n=(X_n\circ\tau_n,p_n\circ X_n\circ\tau_n)_n$ with $Z_n\in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ and $p_n\in L^2_{\mathbb{P}}(\mathbb{R}^d,\mathbb{R}^d)$.

QED.

Proof of the Proposition 4.1: 1) Assume (μ_n,p_n) converges to (μ,p) in $(\mathcal{F}_2,d_{\mathcal{F}_2})$. Then it exists $(X_n)_n$ in $L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ such that:

$$\lim_{n\to+\infty}\|X_n-X\|^2_{L^2_{\mathbb{P}}}+\|p_n\circ X_n-p\circ X\|^2_{L^2_{\mathbb{P}}}=0,\quad X_n\sharp\mathbb{P}=\mu_n,\ X\sharp\mathbb{P}=\mu.$$

By regularity of $\overline{\mathrm{H}}$ and (a), it holds:

$$\lim_{n\to+\infty}\mathcal{H}(\mu_n,p_n)=\lim_{n\to+\infty}\overline{\mathrm{H}}(X_n,p_n\circ X_n)=\overline{\mathrm{H}}(X,p\circ X)=\mathcal{H}(\mu,p).$$

2) First define the following application:

$$\begin{array}{rlcl}\overline{\mathcal{H}}: & \{(Id\times p)\sharp\mu:\ \mu\in\mathcal{P}_2(\mathbb{R}^d),\ p\in L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)\} & \to & \mathbb{R}\\ & (Id\times T)\sharp\nu & \mapsto & \mathcal{H}(\nu,T)\end{array}$$

This map is W_2-uniformly continuous on balls and using Lemma 4.2, it can be extended to a W_2 continuous map, again denoted $\overline{\mathcal{H}}$ on $\mathcal{P}_2(\mathbb{R}^d\times\mathbb{R}^d)$ which is uniformly continuous on balls. Then we denote by $\overline{\mathrm{H}}$ the lift of $\overline{\mathcal{H}}$. By construction, it is rearrangement invariant and $L^2_{\mathbb{P}}$-uniformly continuous on balls. QED.

Remark 4.3. In [17], the author introduced a slightly different notion of rearrangement invariance for Hamiltonians. Namely $\overline{\mathrm{H}}:L^2_{\mathbb{P}}(\Omega\times\mathbb{R}^d)\times L^2_{\mathbb{P}}(\Omega\times\mathbb{R}^d)\to\mathbb{R}$ is said to be rearrangement invariant in the sense of [17] if for all $X,Z\in L^2_{\mathbb{P}}(\Omega\times\mathbb{R}^d)$:

$$\overline{\mathrm{H}}(X\circ\tau,Z\circ\tau)=\overline{\mathrm{H}}(X,Z)\text{ for all }\tau:\Omega\to\Omega,\text{ one to one with }\tau\sharp\mathbb{P}=\tau^{-1}\sharp\mathbb{P}=\mathbb{P}.$$

By Lemma 2.2, this notion is equivalent to the classic one on $L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d\times\mathbb{R}^d)$ when $\overline{\mathrm{H}}$ is continuous.

Remark 4.4. Like in Section 3, it is also possible to define strict solutions for (HJ_2) and to adapt the proof of Theorem 3.23.

Example 4.5. Turning back to Example 2.7:

$$\mathcal{H}_1(\mu,p) := \frac{1}{2}\int_{R^d} |p(x)|^2\, d\mu(x) + \mathcal{V}(\mu).$$

Then, with the assumptions of Example 2.7, the regular $L^2_{\mathbb{P}}$-extension $\overline{\mathrm{H}}_1$ and the plan extension $\overline{\mathcal{H}}_1$ of $\mathcal{H}_1$ are:

$$\overline{\mathrm{H}}_1(X,Z) = \frac{1}{2}\|Z\|^2 + \mathcal{V}(X\sharp\mathbb{P}), \quad \overline{\mathcal{H}}_1(\gamma) = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} |z|^2\, d\gamma(x,z) + \mathcal{V}(\pi_1\sharp\gamma).$$

Moreover, it exists a local modulus of continuity $(R,t)\mapsto \omega_R(t)$ such that:

$$\overline{\mathcal{H}}_1(\gamma_1) - \overline{\mathcal{H}}_1(\gamma_2) \le RW_2(\pi_2\sharp\gamma_1, \pi_2\sharp\gamma_2) + \omega_R(W_2(\pi_1\sharp\gamma_1, \pi_1\sharp\gamma_2))$$

for all $R>0$ and all $\gamma_1,\gamma_2 \in \mathcal{P}_2(\mathbb{R}^d\times\mathbb{R}^d)$ such that $m_2(\pi_i\sharp\gamma_j) \le R^2$ for $i,j=1,2$,

$$\overline{\mathrm{H}}_1(X_1,Z_1) - \overline{\mathrm{H}}_1(X_2,Z_2) \le R\|Z_1 - Z_2\| + \omega_R(\|X_1 - X_2\|)$$

for all $X_1,X_2,Z_1,Z_2 \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ such that $\|X_i\|, \|Z_i\| \le R$ for $i=1,2$.

In the example 2.9, the extension is a bit more difficult to guess:

Lemma 4.6. *Let us consider the following Hamiltonian on* $\mathcal{F}_2(\mathbb{R}^d)$*:*

$$\mathcal{H}_2(\mu,p) := \inf\left\{ \int_{\mathbb{R}^d} f(x,\mathbf{u}(x),\mu)\cdot p(x)\ d\mu(x) :\ \mathbf{u}:\mathbb{R}^d \to \mathbf{U}\ \textit{Borel} \right\}$$

with the assumptions of Example 2.9. Then, the plan extension of $\mathcal{H}_2$ *is given for all* $\gamma\in\mathcal{P}_2(\mathbb{R}^d\times\mathbb{R}^d)$ *by:*

$$\overline{\mathcal{H}}_2(\gamma) = \inf\left\{ \int_{\mathbb{R}^{2d}} f(x,\mathbf{v}(x,z),\pi_1\sharp\gamma)\cdot z\ d\gamma(x,z) :\ \mathbf{v}:\mathbb{R}^{2d} \to \mathbf{U}\ \textit{Borel} \right\}$$

and its lift $\overline{\mathrm{H}}_2$ *is defined for all* $(X,Z)\in(L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d))^2$ *by:*

$$\overline{\mathrm{H}}_2(X,Z) = \inf\left\{ \int_{\Omega} f(X,\mathbf{v}(X,Z),X\sharp\mathbb{P})\cdot Z\ d\mathbb{P} :\ \mathbf{v}:\mathbb{R}^{2d} \to \mathbf{U}\ \textit{Borel} \right\}.$$

Moreover it exists $M>0$ *such that we have the regularity properties:*

$$\overline{\mathcal{H}}_2(\gamma_1) - \overline{\mathcal{H}}_2(\gamma_2) \le (M+4LR)W_2(\gamma_1,\gamma_2)$$

for all $R>0$ *and all* γ_1, $\gamma_2\in\mathcal{P}_2(\mathbb{R}^d\times\mathbb{R}^d)$ *such that* $m_2(\pi_i\sharp\gamma_j)\le R^2$, *for* $i,j=1,2$,

$$\overline{\mathrm{H}}_2(X_1,Z_1) - \overline{\mathrm{H}}_2(X_2,Z_2) \le (M+4LR)\|(X_1,Z_1)-(X_2,Z_2)\|$$

for all $R>0$ *and all* $X_i, Z_i \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ *such that* $\|X_i\|$, $\|Z_i\|\le R$ *for* $i=1,2$.

Proof: Clearly $\overline{\mathcal{H}}_2((Id\times p)\sharp\mu) = \mathcal{H}_2(\mu,p)$ for every $(\mu,p)\in\mathcal{F}_2(\mathbb{R}^d)$. Then, proving the continuity of $\overline{\mathcal{H}}_2$ is enough to show that $\overline{\mathcal{H}}_2$ is the plan extension of $\mathcal{H}_2$. Similarly to the proof of Lemma 2.10, we prove the regularity property above for the following functional on $\mathcal{P}_2(\mathbb{R}^d\times\mathbb{R}^d)$:

$$G(\gamma) = \inf\left\{ \int_{\mathbb{R}^{2d}} f(x,\mathbf{v}(x,z),\pi_1\sharp\gamma)\cdot z\ d\gamma(x,z) :\ \mathbf{v}:\mathbb{R}^{2d} \to \mathbf{U}\ \text{Borel} \right\}$$

so that it is the unique plan extension of $\mathcal{H}_2$. The sequel is straightforward.
QED.

Remark 4.7. Let us consider the extended equation (HJ$_2$) together with the final condition $U(T,X) = \mathcal{G}(X\sharp\mathbb{P})$. A consequence of the rearrangement invariance of $\overline{\mathrm{H}}$ that was proved in [17] is the following: if the extended equation has a unique solution U, then it is rearrangement invariant. Then, setting $u(t,\mu) = U(t,X)$ for any $X \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ of law μ, we get a solution for the equation (HJ) (in the sense of Gangbo and Tudorascu), together with the final condition $u(T,\mu) = \mathcal{G}(\mu)$.

In view of the previous remark, using the work of M. Crandall and P.-L. Lions ([13]) and the above construction of $\overline{\mathrm{H}}$, we get the existence Theorem below for (HJ) together with the final condition $u(T,\mu) = \mathcal{G}(\mu)$ but only in the open time interval $]0,T[$.

Theorem 4.8 (Existence Theorem)**.** *Assume $\mathcal{H} : \mathcal{F}_2(\mathbb{R}^d) \to \mathbb{R}$ satisfy the following assumptions:*

(A_0) *It exists a local modulus $(R,t) \mapsto \omega_R(t)$ such that:*

$$|\mathcal{H}(\mu,p) - \mathcal{H}(\nu,q)| \leq \omega_R(W_2((Id\times p)\sharp\mu,(Id\times q)\sharp\nu))$$

for all (μ,p), (ν,q) in $\mathcal{F}_2(\mathbb{R}^d)$ such that:

$$m_2(\mu) + \|p\|^2_{L^2_\mu} \leq R^2, \quad m_2(\nu) + \|q\|^2_{L^2_\nu} \leq R^2.$$

(A_1'') *It exists a local modulus $(R,t) \mapsto \sigma_R(t)$ such that:*

$$|\mathcal{H}(\mu,p) - \mathcal{H}(\mu,q)| \leq \sigma_R(\|p-q\|_{L^2_\mu})$$

for all (μ,p), (μ,q) in $\mathcal{F}_2(\mathbb{R}^d)$ such that:

$$\|p\|_{L^2_\mu} + \|p-q\|_{L^2_\mu} \leq R.$$

(A_2) *It exists a modulus of continuity ξ such that, for all $R \geq 0$ and $\lambda > 0$, it exists a second modulus of continuity $\xi_{\lambda,R}$ such that:*

$$\mathcal{H}(\nu,\lambda(Id-S)) - \mathcal{H}(\mu,\lambda(T-Id))$$
$$\leq \xi\left(\lambda\|S-Id\|^2_{L^2_\nu} + \|S-Id\|_{L^2_\nu}\right) + \xi_{\lambda,R}\left(W_2((Id\times T)\sharp\mu,(S\times Id)\sharp\nu)\right)$$

for all (μ,T), (ν,S) in $\mathcal{F}_2(\mathbb{R}^d)$ such that:

$$m_2(\mu) + \|T\|^2_{L^2_\mu} \leq R^2, \quad m_2(\nu) + \|S\|^2_{L^2_\nu} \leq R^2.$$

Assume moreover $\mathcal{G}$ is uniformly continuous. Then, it exists u a solution of (HJ) *in the sense of Definition 3.6 on $]0,T[\times\mathcal{P}_2(\mathbb{R}^d)$ satisfying $u(T,\mu) = \mathcal{G}(\mu)$ for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and m a modulus of continuity, $(R,t) \mapsto m_R(t)$ a local modulus such that:*

$$|u(t,\mu) - u(s,\nu)| \leq m(W_2(\mu,\nu)) + m_R(|t-s|)$$

for all $\mu,\nu \in \mathcal{P}_2(\mathbb{R}^d)$, $s,t \in]0,T]$ and $m_2(\nu) \leq R^2$.

Proof: Step 1: By (A_0) and Proposition and Definition 4.1, 2), $\mathcal{H}$ admits a rearrangement invariant $\overline{H}$ such that:

$$|\overline{\mathrm{H}}(X,Z) - \overline{\mathrm{H}}(Y,Z')| \leq \omega_R(\|X-Y\| + \|Z-Z'\|) \tag{22}$$

for all X,Y,Z,Z' in $L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ such that:

$$\|X\|,\ \|Y\|,\ \|Z\|,\ \|Z'\| \leq R.$$

Denote by $\overline{\mathcal{H}}$ its lift. We aim to apply Theorem 1.1. in [13] to (HJ$_2$).

Step 2: We start by checking assumption (H2) of p 373 of [13].
Let X, Z_1, Z_2 in $L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ with $\|Z_1\|+\|Z_1-Z_2\|\leq R$, setting $\varpi := (X,Z_1,Z_2)\sharp\mathbb{P}$, applying Lemma 5.1, it exists $(\mu_n)_n$ in $\mathcal{P}_2(\mathbb{R}^d)$, $p_n, q_n \in L^2_{\mu_n}(\mathbb{R}^d,\mathbb{R}^d)$ for all $n\in\mathbb{N}$ such that:

$$\lim_{n\to+\infty} W_2((Id\times p_n\times q_n)\sharp\mu_n,\varpi)=0,$$
$$\|p_n-q_n\|_{L^2_{\mu_n}}\leq R.$$

Note that: $\lim_{n\to+\infty}\|p_n-q_n\|^2_{L^2_{\mu_n}}=\int_{R^{3d}}|z_1-z_2|^2 d\varpi(x,z_1,z_2)$. By (A1") and the definition of $\overline{\mathcal{H}}$, we get:

$$|\overline{\mathcal{H}}(\pi_{1,2}\sharp\omega)-\overline{\mathcal{H}}(\pi_{1,3}\sharp\omega)|\leq\sigma_R\left(\left[\int_{R^{3d}}|z_1-z_2|^2 d\varpi(x,z_1,z_2)\right]^{1/2}\right)$$

or, equivalently:

$$|\overline{\mathrm{H}}(X,Z_1)-\overline{\mathrm{H}}(X,Z_2)|\leq\sigma_R(\|Z_1-Z_2\|).$$

This implies:

$$|\overline{\mathrm{H}}(X,Y)-\overline{\mathrm{H}}(X,Y+\lambda\frac{X}{\|X\|})|\leq\omega_R(\lambda)$$

for all $\lambda>0$, $X,Y\in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$, $X\neq 0$, $\lambda+\|Y\|\leq R$. Following [12], (1,15), let us take $\zeta\in\mathcal{C}^1(\mathbb{R})$ such that

$$\zeta(r)=0 \text{ for } r\leq 1,\ \zeta(r)=2r-4 \text{ for } r\geq 3,\quad \text{ and } 0\leq\zeta'\leq 2.$$

Then take for any $X\in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$: $v(X)=\zeta(\|X\|)$. We have:

$$\liminf_{\|X\|\to+\infty}\frac{v(X)}{\|X\|}=2>1,\quad \|Dv(X)\|\leq 2\frac{X}{\|X\|}\ \forall X\neq 0,\quad Dv(0)=0.$$

It then holds:

$$|\overline{\mathrm{H}}(X,Y)-\overline{\mathrm{H}}(X,Y+\lambda Dv(X))|\leq\omega_R(2\lambda) \tag{23}$$

for all $\lambda>0$, $X,Y\in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$, $\|Y\|+\lambda\leq R$.

Step 3: Now check assumption (H3) of p 373 of [13].
Take $\gamma=\gamma(x,y)\in\mathcal{P}_2(\mathbb{R}^d\times\mathbb{R}^d)$ and $R=\frac{1}{2}m_2(\gamma)$. By Lemma 4.2, it exists sequences $(\mu_n,T_n)_n$ and $(\nu_n,S_n)_n$ in $\mathcal{F}_2(\mathbb{R}^d)$ such that:

$$\lim_{n\to+\infty}W_2((Id\times T_n)\sharp\mu_n,\gamma)=\lim_{n\to+\infty}W_2((S_n\times Id)\sharp\nu_n,\gamma)=0.$$

Then assumption (A_2) implies for n big enough:

$$\lim_{n\to+\infty}|\mathcal{H}(\nu_n,\lambda(S_n-Id))-\mathcal{H}(\mu_n,\lambda(Id-T_n))|\leq\xi\left(\lambda\|x-y\|^2_{L^2_\gamma}+\|x-y\|_{L^2_\gamma}\right)$$

for all $\lambda>0$. It exists a sequence $(X_n)_n$ and X, Y in $L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ such that:

$$\lim_{n\to+\infty}\|X_n-X\|^2+\|T_n\circ X_n-Y\|^2=0$$
$$(X,Y)\sharp\mathbb{P}=\gamma,\ X_n\sharp\mathbb{P}=\mu_n.$$

Then $\lim_{n\to+\infty}\|\lambda(T\circ X_n-X_n)-\lambda(Y-X)\|=0$ which implies

$$\lim_{n\to+\infty}W_2((Id\times\lambda(Id-T_n))\sharp\mu_n,(\pi_1,\lambda(\pi_1-\pi_2))\sharp\gamma)=0.$$

In the same way $\lim_{n\to+\infty} W_2((Id\times\lambda(S_n-Id)\sharp\nu_n,(\pi_2,\lambda(\pi_1-\pi_2)\sharp\gamma)=0$. So finally we get:

$$|\overline{\mathcal{H}}((\pi_2,\lambda(\pi_1-\pi_2))\sharp\gamma)-\overline{\mathcal{H}}((\pi_1,\lambda(\pi_1-\pi_2))\sharp\gamma))|\le\xi\bigg(\lambda\|x-y\|^2_{L^2_\gamma}+\|x-y\|_{L^2_\gamma}\bigg).$$

As this is true for any $\gamma\in\mathcal{P}_2(\mathbb{R}^d\times\mathbb{R}^d)$, we have also:

$$|\overline{H}(Y,\lambda(X-Y))-\overline{H}(X,\lambda(X-Y))|\le\xi\bigg(\lambda\|X-Y\|^2+\|X-Y\|\bigg)$$

for all $\lambda>0$, $X,Y\in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$.

Step 4: By step 1, 2 and 3, all assumptions of [13], Theorem 1.1 are satisfied. Then, (HJ_2) together with $U(T,X)=\mathcal{G}(X\sharp\mathbb{P})$ has a unique solution U_0. Moreover it exists m a modulus of continuity, $(R,t)\mapsto m_R(t)$ a local modulus such that:

$$|U_0(t,X)-U_0(s,Y)|\le m(\|X-Y\|)+m_R(|t-s|)$$

for all $s,t\in[0,T]$, X,Y in $L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$. As recalled in Remark 4.7, it is rearrangement invariant, the sequel follows. QED.

Example 4.9. The Hamiltonian $\mathcal{H}_1$ of Example 2.7 satisfies (A_1''), let us prove it satisfies (A_2) provided

$$\mathcal{V}(\mu)-\mathcal{V}(\nu)\le\omega(W_2(\mu,\nu))\quad\forall\mu,\nu\in\mathcal{P}_2(\mathbb{R}^d)$$

for some modulus of continuity ω. Indeed, take μ,ν,S,T,λ,R as in the theorem and $\pi\in\Pi_o((Id\times T)\sharp\mu,(S\times Id)\sharp\nu)$. Note that π has the following shape:

$$\pi(x,x',y,y')=\delta_{T(x)}(x')\otimes\delta_{S(y)}(y')\otimes\gamma(x,y)\text{ with }\gamma\in\Pi(\mu,\nu)$$

and $W_2((Id\times T)\sharp\mu,(S\times Id)\sharp\nu)^2=\displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-Sy|^2+|Tx-y|^2\,d\gamma(x,y)\ge W_2^2(\mu,S\sharp\nu)$.

Then, denoting by C all positive constants independent of λ and R:

$$\mathcal{H}_1(\nu,\lambda(S-Id))-\mathcal{H}_1(\mu,\lambda(Id-T))=\lambda^2(\|S-Id\|^2_{L^2_\nu}-\|T-Id\|^2_{L^2_\mu})+\mathcal{V}(\nu)-\mathcal{V}(\mu)$$

$$\begin{aligned}
&\le\lambda^2\int_{\mathbb{R}^d\times\mathbb{R}^d}|Sy-x+x-Tx+Tx-y|^2d\gamma(x,y)\\
&\quad-\lambda^2\|T-Id\|^2_{L^2_\mu}+\mathcal{V}(\nu)-\mathcal{V}(S\sharp\nu)+\mathcal{V}(S\sharp\nu)-\mathcal{V}(\mu)\\
&\le C\lambda^2W_2((Id\times T)\sharp\mu,(S\times Id)\sharp\nu)^2+\omega(W_2(\nu,S\sharp\nu))+\omega(W_2(\mu,S\sharp\nu))\\
&\le C\lambda^2W_2((Id\times T)\sharp\mu,(S\times Id)\sharp\nu)^2+\omega\left(\left[2\int|Sy|^2+|y|^2\,d\nu(y)\right]^{1/2}\right)\\
&\quad+\omega(W_2((Id\times T)\sharp\mu,(S\times Id)\sharp\nu))\\
&\le C\lambda^2W_2((Id\times T)\sharp\mu,(S\times Id)\sharp\nu)^2+\omega(R)+\omega(W_2((Id\times T)\sharp\mu,(S\times Id)\sharp\nu)).
\end{aligned}$$

Example 4.10. The Hamiltonian $\mathcal{H}_2$ of Example 2.9 satisfies (A_1'') provided f is bounded, moreover it also satisfies (A_2). Again, take $\mu, \nu, S, T, \lambda, R$ as in the theorem and $\pi \in \Pi_o((Id \times T)\sharp\mu, (S \times Id) \times \nu)$ with:

$$\pi(x, x', y, y') = \delta_{T(x)}(x') \otimes \delta_{S(y)}(y') \otimes \gamma(x, y).$$

Set $M := \sup_{\mathbf{w}\in\mathbf{U}} |f(0, \mathbf{w}, \delta_0)|$. Take $\varepsilon > 0$ and $\mathbf{u}$ an ε-optimal control for $\mathcal{H}_2(\mu, \lambda(Id-T))$. Set $\mathbf{v}(y) = \int \mathbf{u}(x)d\gamma^y(x)$. Then, denoting by C or C' all positive constants independent of λ and R:

$$\mathcal{H}_2(\nu, \lambda(S - Id)) \leq \int_{\mathbb{R}^d} f(y, \mathbf{v}(y), \nu) \cdot \lambda(y - S(y))\, d\nu(y)$$

$$\leq \int_{\mathbb{R}^d} f(S(y), \mathbf{v}(y), \nu) \cdot \lambda(y - S(y))\, d\nu(y) + L\lambda\|Id - S\|^2_{L^2_\nu}$$

(by use of Cauchy-Schwarz and the Lipschitz property of f)

$$\leq \int f(S(y), \mathbf{u}(x), \nu) \cdot \lambda(y - S(y)) d\gamma(x, y) + L\lambda\|Id - S\|^2_{L^2_\nu}$$

(by definition of $\mathbf{v}$ and affinity of f)

$$\leq \int f(x, \mathbf{u}(x), S\sharp\nu) \cdot \lambda(y - S(y)) d\gamma(x, y) + CLR\lambda W_2^2((Id \times T)\sharp\mu, (S \times Id)\sharp\nu) + 2L\lambda\|Id - S\|^2_{L^2_\nu}$$

$$\leq \int f(x, \mathbf{u}(x), \mu) \cdot \lambda(y - S(y)) d\gamma(x, y) + C'LR\lambda W_2^2((Id \times T)\sharp\mu, (S \times Id)\sharp\nu) + C'L\lambda\|Id - S\|^2_{L^2_\nu}$$

(again by use of Cauchy-Schwarz and the Lipschitz property of f two times to change ν in $S\sharp\nu$, $S(y)$ in x first and then $S\sharp\nu$ into μ in the argument of f)

$$\leq \int f(x, \mathbf{u}(x), \mu) \cdot \lambda(T(x) - x) d\mu(x) + \int f(x, \mathbf{u}(x), \mu) \cdot \lambda(x - S(y) + y - T(x)) d\mu(x)$$

$$+ C'LR\lambda W_2^2((Id \times T)\sharp\mu, (S \times Id)\sharp\nu) + 2CL\lambda\|Id - S\|^2_{L^2_\nu}$$

$$\leq \mathcal{H}_2(\mu, \lambda(Id - T)) + \varepsilon + C(M + 2RL)\lambda W_2^2((Id \times T)\sharp\mu, (S \times Id)\sharp\nu) + 2CL\lambda\|Id - S\|^2_{L^2_\nu}$$

using (8). The result follows by making ε tend to 0.

4.2. **Rearrangement invariant viscosity solutions in $L^2_{\mathbb{P}}$.** Here we consider two types of Hamiltonians: $\overline{\mathcal{H}} : \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d) \to \mathbb{R}$ and its lift $\overline{\mathrm{H}} : L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)^2 \to \mathbb{R}$. We are interested in rearrangement invariant viscosity solutions of (HJ_2).

We will use the following result appearing in [24]:

Lemma 4.11. *Let $U : [0, T] \times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \to \mathbb{R}$ be rearrangement invariant. Set $X_0, Z \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$, $(t_0, p_t) \in [0, T[\times\mathbb{R}$ such that $(p_t, Z) \in D^+U(t_0, X_0)$.*

Then for any couple $(X_0', Z') \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)^2$ such that $(X_0', Z')\sharp\mathbb{P} = (X_0, Z)\sharp\mathbb{P}$, it holds:

$$(p_t, Z') \in D^+U(t_0, X_0').$$

This lemma emphasizes that it is meaningful to look at elements $Z \in DU^+(X)$ as couples (X, Z) and more precisely to consider their joint laws $\gamma = (X, Z)\sharp\mathbb{P}$. It justifies the introduction of $\overline{\mathcal{H}}$ and the assumption of rearrangement invariance for extended Hamiltonians starting from $\mathcal{H} : \mathcal{F}_2(\mathbb{R}^d) \to \mathbb{R}$ and their lifts H. In the same spirit, we will use the following notion of sub/superdifferential which was introduced in the book [1] of Ambrosio, Gigli and Savaré (see Definition 10.3.1. p241):

Definition 4.12. Let $u : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$. Let $(t_0, \mu_0) \in [0,T[\times\mathcal{P}_2(\mathbb{R}^d)$ and $\gamma \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$.

(i) We say that (p_t, γ) belongs to the AGS-superdifferential of u at (t_0, μ_0) iff: *for all $(t,\nu) \in [0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, and $\varpi(x,y,z) \in \mathcal{P}_2(\mathbb{R}^{3d})$ such that $\pi_{1,3}\sharp\varpi = \gamma$, $\pi_2\sharp\varpi = \nu$, it holds*

$$u(t,\nu)-u(t_0,\mu_0) \le p_t(t-t_0)+\int_{\mathbb{R}^{3d}} z\cdot(y-x)\, d\varpi(x,y,z)+o\left(\sqrt{|t-t_0|^2+\|x-y\|^2_{L^2_\varpi}}\right).$$

We denote the set of (p_t, γ) satisfying this property by $D^+_{AGS}u(t_0, \mu_0)$.

(ii) We say that (p_t, γ) belongs to the AGS-subdifferential of u at (t_0, μ_0) iff *for all $(t,\nu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$, and $\varpi(x,y,z) \in \mathcal{P}_2(\mathbb{R}^{3d})$ such that $\pi_{1,3}\sharp\varpi = \gamma$, $\pi_2\sharp\varpi = \nu$, it holds*

$$u(t,\nu)-u(t_0,\mu_0) \ge p_t(t-t_0)+\int_{\mathbb{R}^{3d}} z\cdot(y-x)\, d\varpi(x,y,z)+o\left(\sqrt{|t-t_0|^2+\|x-y\|^2_{L^2_\varpi}}\right).$$

We denote the set of (p_t, γ) satisfying this property by $D^-_{AGS}u(t_0, \mu_0)$.

Example 4.13. Let $p_t \in \mathbb{R}$, $t_0 \in [0,T[$, $X, Z \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$, $\gamma(x,z) = [(X\times Z)\sharp\mathbb{P}](x,z) \in \Pi(\mu_0,\nu)$.

- As noticed in [24], if U is the lift of u:

$$(p_t, Z) \in D^\pm U(t_0, X) \Leftrightarrow (p_t, \gamma) \in D^\pm_{AGS}u(t_0, \mu_0).$$

- Moreover, by Proposition 3.5, if $\gamma \in D^\pm_{AGS}u(t_0,\mu)$, then:

$$(p_t, (Id \times p_X(Z))\sharp\mu_0)\,,\ \text{and}\ \ (p_t, (Id \times p^o_X(Z))\sharp\mu_0)\ \text{belong to}\ D^\pm_{AGS}u(t_0,\mu).$$

- Fix $\lambda > 0$ and assume $\gamma(x,z) = [(X \times Z)\sharp\mathbb{P}](x,z) \in \Pi_o(\mu_0,\nu)$ is an optimal transport plan. Abusing slightly notations we will denote by π_x and π_z the first and second projections from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}^d$. Let $u(\mu) := \lambda W_2^2(\mu,\nu)$, then, (see Theorem 10.2.2. p236 of [1]):

$$\bar\gamma = (\pi_x \times (2\lambda(\pi_x - \pi_z)))\sharp\gamma = (X \times 2\lambda(X-Z))\sharp\mathbb{P} \in D^+_{AGS}u(\mu_0).$$

Also by the remark above:

$$\left(Id \times 2\lambda(Id - \int z\, d\gamma^{\cdot}(z))\right)\sharp\mu_0 = (X \times 2\lambda(X - p_X(Z)))\sharp\mathbb{P} \in D^+_{AGS}u(\mu_0)$$

where $\int z\, d\gamma^{\cdot}(z)$ denotes the map $x \mapsto \int z\, d\gamma^x(z)$.

The AGS-superdifferential satisfies two nice convexity properties:

Lemma 4.14. *(Convexity properties of $D^+_{AGS}u$) Let $u : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$. Let $(t_0,\mu_0) \in [0,T[\times\mathcal{P}_2(\mathbb{R}^d)$ and (p_0,γ_0), (p_1,γ_1) in $D^+_{AGS}u(t_0,\mu_0)$.*

(i) *(Convexity) Let $\lambda \in [0,1]$, set $(p_\lambda,\gamma_\lambda) = (1-\lambda)(p_0,\gamma_0) + \lambda(p_1,\gamma_1)$, then $(p_\lambda,\gamma_\lambda) \in D^+_{AGS}u(t_0,\mu_0)$.*

(ii) *(Displacement convexity) Set* $\omega(x, z_0, z_1) := \gamma_1^x(z_1) \otimes \gamma_0^x(z_0) \otimes \mu_0(x)$ *and take* $\lambda \in [0,1]$. *Then, defining*

$$\bar{\gamma}_\lambda := (\pi_x \times ((1-\lambda)\pi_{z_0} + \lambda \pi_{z_1}))\sharp\omega$$

we have $((1-\lambda)p_0 + \lambda p_1, \bar{\gamma}_\lambda) \in D^+_{AGS}u(t_0, \mu_0)$. *Here we have denoted by* π_{z_0} *and* π_{z_1} *the second and third projections of* $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$.

Note that, for any regular φ, by definition of $\bar{\gamma}_\lambda$:

$$\int \varphi(x,z)\, d\bar{\gamma}_\lambda(x,z) = \int \varphi(x, (1-\lambda)z_0 + \lambda z_1)\, d\omega(x, z_0, z_1).$$

Proof: Assume for simplicity that u does not depend on the time variable.
(i) Take $\omega(x,y,z) \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ and such that $\pi_{1,3}\sharp\omega = \gamma_\lambda$ set $\nu = \pi_2\sharp\omega$. Then ω can be written as a convex combination:

$$\omega(x,y,z) = \omega^{x,z}(y) \otimes \gamma_\lambda(x,z) = (1-\lambda)\omega^{x,z}(y) \otimes \gamma_0(x,z) + \lambda\omega^{x,z}(y) \otimes \gamma_1(x,z).$$

Then as $\gamma_i \in D^+_{AGS,\varepsilon}u(t_0,\mu_0)$ $(i = 0,1)$, we have:

$$u(\nu) \leq u(\mu) + \int_{\mathbb{R}^{3d}} z \cdot (y-x) d\omega^{x,z}(y) d\gamma_i(x,z) + o\left(\|y-x\|_{L^2_{\omega^{x,z}\otimes\gamma_i}}\right).$$

By convex combination:

$$u(\nu) \leq u(\mu) + \int_{\mathbb{R}^{3d}} z\cdot(y-x) d\omega(x,y,z) + o\left((1-\lambda)\|y-x\|_{L^2_{\omega^{x,z}\otimes\gamma_0}} + \lambda\|y-x\|_{L^2_{\omega^{x,z}\otimes\gamma_1}}\right).$$

We conclude as by concavity of $\tau \mapsto \sqrt{\tau}$:

$$(1-\lambda)\|y-x\|_{L^2_{\omega^{x,z}\otimes\gamma_0}} + \lambda\|y-x\|_{L^2_{\omega^{x,z}\otimes\gamma_1}}$$

$$\leq \sqrt{(1-\lambda)\int_{\mathbb{R}^{3d}} |y-x|^2 d\omega^{x,z}(y) d\gamma_0(x,z) + \lambda \int_{\mathbb{R}^{3d}} |y-x|^2 d\omega^{x,z}(y) d\gamma_1(x,z)}$$

$$= \left(\int_{\mathbb{R}^{3d}} |y-x|^2 d\omega(x,y,z)\right)^{1/2}.$$

(ii) It exists $(X, Z_0, Z_1) \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)^3$ such that $\omega = (X, Z_0, Z_1)\sharp\mathbb{P}$. Then $\bar{\gamma}_\lambda = (X \times ((1-\lambda)Z_1 + \lambda Z_2))$ and, setting U the lift of U: $Z_i \in D^+U(X)$ for $i = 1,2$. By convexity of $D^+U(X)$, we also have that $((1-\lambda)Z_1 + \lambda Z_2)$ belongs to $D^+U(X)$ and we conclude by the first point of Example 4.13.

QED.

An easy consequence of the first point of Example 4.13 is the following result:

Proposition 4.15. *Let* $U : [0,T] \times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \to \mathbb{R}$ *continuous. Assume* U *is rearrangement invariant and let* $u : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ *associated* ($u(t,\mu) = U(t,X)$ *for any* X *such that* $X\sharp\mathbb{P} = \mu$).
U *is a viscosity subsolution of* (HJ$_2$) *iff for all* $(t_0,\mu_0) \in]0,T[\times\mathcal{P}_2(\mathbb{R}^d)$ *an all* $(p_t, \gamma) \in D^+_{AGS}u(t_0,\mu_0)$:

$$p_t + \overline{\mathcal{H}}(\gamma) \geq 0.$$

The symmetric property holds for supersolutions.

The regularity of the extension built in the previous section allows to use classic results of uniqueness for the equation (HJ_2) together with some final condition. As shown by the result below, we can also prove some results specific to rearrangement invariant solutions.

Proposition 4.16 (Comparison principle for rearrangement invariant solutions). *Let $\overline{\mathrm{H}} : L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \to \mathbb{R}$ be a rearrangement invariant Hamiltonian. We assume moreover that:*

- *($\bar{A}_1$) It exists $k > 0$ such that for all $R > 0$, and all $X, Z_1, Z_2 \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ with $\|Z_1\|, \|Z_2\| \leq R$, $\|X\| \leq R$:*

$$\overline{\mathrm{H}}(X, Z_1) - \overline{\mathrm{H}}(X, Z_2) \leq k(1 + R)\|Z_1 - Z_2\|, \tag{24}$$

- *($\bar{A}_2$) it exists $\omega_{\overline{\mathrm{H}}} : \mathbb{R}^+ \to \mathbb{R}^+$ a modulus of continuity such that for all $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, we can find a pair $X, Y \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ with $(X, Y)\sharp\mathbb{P} \in \Pi_o(\mu, \nu)$ and:*

$$\overline{\mathrm{H}}(Y, \lambda(Y - X)) - \overline{\mathrm{H}}(X, \lambda(Y - X)) \leq \omega_{\overline{\mathrm{H}}}(\|X - Y\| + \lambda\|X - Y\|^2) \quad \forall \lambda > 0. \tag{25}$$

Let U_1, U_2 : $[0, T] \times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \to \mathbb{R}$ two rearrangement invariant, uniformly continuous mapping. Namely it exists ω_1, ω_2 two modulus of continuity such that:

$$U_i(t, X) - U_i(s, Y) \leq \omega_i\left((\|X - Y\|^2 + |t - s|^2)^{1/2}\right), \; i = 1, 2.$$

Assume that U_1 (resp. U_2) is a viscosity subsolutions (resp. supersolution) of (HJ_2) *such that:*

$$U_i(T, X) = \mathcal{G}(X\sharp\mathbb{P}) \quad \forall X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d), \; i = 1, 2.$$

for some $\mathcal{G} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$.

Then, it holds: $U_1 \leq U_2$.

The proof will reduce to a classic one by using the following lemma:

Lemma 4.17 (Stegall's Variational Principle in $\mathcal{P}_2(\mathbb{R}^d)$). *Let $v : \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and $V : L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \to \mathbb{R}$ its lift. Assume that V is l.s.c. and coercive. Let $\delta > 0$,*

(i) *it exists μ_δ, ν_δ in $\mathcal{P}_2(\mathbb{R}^d)$, $p_\delta \in L^2_{\mu_\delta}(\Omega, \mathbb{R}^d)$, $q_\delta \in L^2_{\nu_\delta}(\Omega, \mathbb{R}^d)$ such that $\|p_\delta\|_{L^2_{\mu_\delta}} \leq \delta$, $\|q_\delta\|_{L^2_{\nu_\delta}} \leq \delta$ and*

$$v(\mu_\delta, \nu_\delta) \leq v(\mu, \nu) + \int_{\mathbb{R}^{2d}} p_\delta(x) \cdot (y - x)\, d\gamma_\mu(x, y) + \int_{\mathbb{R}^{2d}} q_\delta(x) \cdot (y - x)\, d\gamma_\nu(x, y) \tag{26}$$

for all $\gamma_\mu \in \Pi(\mu_\delta, \mu)$, $\gamma_\nu \in \Pi(\nu_\delta, \nu)$;

(ii) *for all X_δ ,Y_δ in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ of law μ_δ, ν_δ,*

$$V(X_\delta, Y_\delta) \leq V(X, Y) + \langle p_\delta \circ X_\delta, X - X_\delta\rangle + \langle q_\delta \circ Y_\delta, Y - Y_\delta\rangle \tag{27}$$

for all X, Y in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$;

(iii) *it exists $\bar{p}_\delta \in \mathcal{T}_{\mu_\delta}(\mathbb{R}^d)$, $\bar{q}_\delta \in \mathcal{T}_{\nu_\delta}(\mathbb{R}^d)$ such that, replacing p_δ by $\bar{p}_\delta$ and q_δ by $\bar{q}_\delta$, (26) holds for all optimal $\gamma_\mu \in \Pi_o(\mu_\delta, \mu)$, $\gamma_\nu \in \Pi_o(\nu_\delta, \nu)$ and (27) holds for all X, Y with $(X_\delta, X)\sharp\mathbb{P} \in \Pi_o(\mu_\delta, X\sharp\mathbb{P})$, $(Y_\delta, Y)\sharp\mathbb{P} \in \Pi_o(\nu_\delta, Y\sharp\mathbb{P})$;*

(iv) *for all* $\gamma_\mu \in \Pi(\mu_\delta, \mu)$, $\gamma_\nu \in \Pi(\nu_\delta, \nu)$*:*

$$v(\mu_\delta, \nu_\delta) \le v(\mu, \nu) + \int_{\mathbb{R}^{2d}} \bar{p}_\delta(x) \cdot (y-x)\, d\gamma_\mu(x,y) + \int_{\mathbb{R}^{2d}} \bar{q}_\delta(x) \cdot (y-x)\, d\gamma_\nu(x,y)$$
$$+ o\left(\|y-x\|_{L^2_{\gamma_\mu}}\right) + o\left(\|y-x\|_{L^2_{\gamma_\nu}}\right);$$

(v) *for all* X_δ *,*Y_δ *in* $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ *of law* μ_δ*,* ν_δ*,*

$$V(X_\delta, Y_\delta) \le V(X,Y) + \langle \bar{p}_\delta \circ X_\delta, X - X_\delta\rangle + \langle \bar{q}_\delta \circ Y_\delta, Y - Y_\delta\rangle$$
$$+ o(\|X - X_\delta\|) + o(\|Y - Y_\delta\|);$$

for all X*,* Y *in* $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$*.*

Proof of Lemma 4.17: We prove only (i), (iii) and (iv). Statement (ii) follows from (i) and (v) from (iv). **Proof of (i).** Let $\delta_0 > 0$. By Stegall's variational principle (see for instance [6], Theorem 8.8.), it exists X_0, Y_0, ξ_0, ζ_0 in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ such that

$$\|\xi_0\|, \|\zeta_0\| \le \delta_0,$$
$$V(X_0, Y_0) + \langle \xi_0, X_0\rangle + \langle \zeta_0, Y_0\rangle \le V(X,Y) + \langle \xi_0, X\rangle + \langle \zeta_0, Y\rangle \,\forall X, Y \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d).$$

Set $\mu_\delta = X_0 \sharp \mathbb{P}$, $\nu_\delta = Y_0 \sharp \mathbb{P}$, then, for all $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, $\varpi_\mu(x,y,z)$, $\varpi_\nu(x,y,z) \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ such that

$$\pi_{1,3}\sharp \varpi_\mu = (X_0, \xi_0)\sharp \mathbb{P},\ \pi_{1,3}\sharp\varpi_\nu = (Y_0, \zeta_0)\sharp\mathbb{P}, \quad \pi_2\sharp\varpi_\mu = \mu,\ \pi_2\sharp\varpi_\nu = \nu,$$

it holds:

$$v(\mu_\delta, \nu_\delta) \le v(\mu,\nu) + \int_{\mathbb{R}^3} z\cdot(y-x)\, d\varpi_\mu(x,y,z) + \int_{\mathbb{R}^{3d}} z \cdot (y-x)\, d\varpi_\nu(x,y,z).$$

Denote by γ_{μ_δ} the plan $(X_0, \xi_0)\sharp\mathbb{P}$ and by γ_{ν_δ} the plan $(Y_0, \zeta_0)\sharp\mathbb{P}$ which disintegrate in $\gamma_{\mu_\delta}(x,z) = \gamma^x_{\mu_\delta}(z) \otimes \mu_\delta(x)$ and $\gamma_{\nu_\delta}(x,z) = \gamma^x_{\nu_\delta}(z) \otimes \nu_\delta(x)$. Let $\gamma_\mu \in \Pi(\mu_\delta, \mu)$, $\gamma_\nu \in \Pi(\nu_\delta, \nu)$ and set:

$$\varpi_\mu(x,y,z) = \gamma^x_{\mu_\delta}(z) \otimes \gamma_\mu(x,y),\ \varpi_\nu(x,y,z) = \gamma^x_{\nu_\delta}(z) \otimes \gamma_\nu(x,y).$$

Then, it holds:

$$v(\mu_\delta,\nu_\delta) \le v(\mu,\nu) + \int_{\mathbb{R}^3} z\cdot(y-x)\, d\gamma^x_{\mu_\delta}(z) d\gamma_\mu(x,y) + \int_{\mathbb{R}^{3d}} z\cdot(y-x)\, d\gamma^x_{\nu_\delta}(z) d\gamma_\nu(x,y)$$
$$= \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^d} z\, d\gamma^x_{\mu_\delta}(z)\right]\cdot(y-x)\, d\gamma_\mu(x,y) + \int_{\mathbb{R}^{2d}} \left[\int_{\mathbb{R}^d} z\, d\gamma^x_{\nu_\delta}(z)\right]\cdot(y-x)\, d\gamma_\nu(x,y).$$

(i) follows by taking $p_\delta(x) := \int_{\mathbb{R}^d} z\, d\gamma^x_{\mu_\delta}(z)$ and $q_\delta := \int_{\mathbb{R}^d} z\, d\gamma^x_{\nu_\delta}(z)$.

Proof of (iii). Let $\bar{p}_\delta$ the projection of p_δ on $\mathcal{T}_{\mu_\delta}(\mathbb{R}^d)$, then for all $\gamma_\mu \in \Pi_o(\mu_\delta, \mu)$, $\gamma_\nu \in \Pi_o(\nu_\delta, \nu)$, disintegrating γ_μ, it holds:

$$v(\mu_\delta,\nu_\delta) \le v(\mu,\nu) + \int_{\mathbb{R}^{2d}} p_\delta(x)\cdot(y-x)\, d\gamma^x_\mu(y) d\mu_\delta(x) + \int_{\mathbb{R}^{2d}} q_\delta(x)\cdot(y-x)\, d\gamma_\nu(x,y)$$
$$\le v(\mu,\nu) + \int_{\mathbb{R}^d} p_\delta(x) \cdot \left(\left[\int_{\mathbb{R}^d} y d\gamma^x_\mu(y)\right] - x\right)\, d\mu_\delta(x) + \int_{\mathbb{R}^{2d}} q_\delta(x)\cdot(y-x)\, d\gamma_\nu(x,y)$$

and as $x \mapsto \left[\int_{\mathbb{R}^d} y d\gamma_\mu^x(y)\right] - x$ belongs to $\mathcal{T}_{\mu_\delta}(\mathbb{R}^d)$:

$$v(\mu_\delta,\nu_\delta) \leq v(\mu,\nu) + \int_{\mathbb{R}^d} \bar{p}_\delta(x) \cdot \left(\left[\int_{\mathbb{R}^d} y d\gamma_\mu^x(y) \right] - x \right) d\mu_\delta(x) + \int_{\mathbb{R}^{2d}} q_\delta(x) \cdot (y-x)\, d\gamma_\nu(x,y)$$

$$\leq v(\mu,\nu) + \int_{\mathbb{R}^{2d}} \bar{p}_\delta(x) \cdot (y-x)\, d\gamma_\mu^x(y) d\mu_\delta(x) + \int_{\mathbb{R}^{2d}} q_\delta(x) \cdot (y-x)\, d\gamma_\nu(x,y).$$

Repeating this idea for q_δ gives the result.

Proof of (iv). Take now $\pi_\mu \in \Pi(\mu_\delta,\mu)$, $\pi_\nu \in \Pi(\nu_\delta,\nu)$, γ_μ, γ_ν as above and $\varepsilon > 0$, $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d,\mathbb{R}^d)$ such that $\|\nabla\varphi - \bar{p}_\delta\|_{L^2_{\mu_\delta}} \leq \varepsilon$. Then, using Lemma 3.3 of [17]:

$$v(\mu_\delta,\nu_\delta) \leq v(\mu,\nu) + \int_{\mathbb{R}^{2d}} \bar{p}_\delta(x) \cdot (y-x)\, d\gamma_\mu(x,y) + \int_{\mathbb{R}^{2d}} \bar{q}_\delta(x) \cdot (y-x)\, d\gamma_\nu(x,y)$$

$$\leq v(\mu,\nu) + \int_{\mathbb{R}^{2d}} \nabla\varphi(x) \cdot (y-x)\, d\gamma_\mu(x,y) + \varepsilon\|y-x\|_{L^2_{\gamma_\mu}} + \int_{\mathbb{R}^{2d}} \bar{q}_\delta(x) \cdot (y-x)\, d\gamma_\nu(x,y)$$

$$\leq v(\mu,\nu) + \int_{\mathbb{R}^{2d}} \nabla\varphi(x) \cdot (y-x) d\pi_\mu(x,y) + o(\|y-x\|^2_{L^2_{\pi_\mu}}) + \varepsilon\|y-x\|_{L^2_{\pi_\mu}}$$
$$+ \int_{\mathbb{R}^{2d}} \bar{q}_\delta(x) \cdot (y-x)\, d\gamma_\nu(x,y)$$

$$\leq v(\mu,\nu) + \int_{\mathbb{R}^{2d}} \bar{p}_\delta(x) \cdot (y-x) d\pi_\mu(x,y) + o(\|y-x\|^2_{L^2_{\pi_\mu}}) + 2\varepsilon\|y-x\|_{L^2_{\pi_\mu}}$$
$$+ \int_{\mathbb{R}^{2d}} \bar{q}_\delta(x) \cdot (y-x)\, d\gamma_\nu(x,y).$$

Repeating the same idea for $\bar{q}_\delta$ and making ε go to zero gives the result.
QED.

Proof of Proposition 4.16: We set:

$$M := \max_{t\in[0,T]} (|U_1(t,0)| + |U_2(t,0)|).$$

All along the proof we will denote by C any constant depending only on k and M. Assume by contradiction that

$$-\xi := \inf_{[0,T]\times L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)} (U_2 - U_1) < 0. \tag{28}$$

We set, for any $t_1,t_2 \in [0,T]$, $X_1,X_2 \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$:

$$\Phi(t_1,t_2,X_1,X_2) :=$$

$$U_2(t_2,X_2) - U_1(t_1,X_1) + \frac{1}{2\varepsilon}\left(W_2^2(X_1\sharp\mathbb{P}, X_2\sharp\mathbb{P}) + |t_1-t_2|^2\right) + \frac{\alpha}{2}\left(\|X_1\|^2 + \|X_2\|^2\right) - \eta t_2$$

depending on $\varepsilon \in]0,1[$, $\alpha > 0$ and $\eta > 0$ such that:

$$\eta > \frac{\xi}{2T}. \tag{29}$$

We introduce also for all $X_1,X_2 \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$:

$$\Psi(X_1,X_2) := \min_{t_1,t_2\in[0,T]} \Phi(t_1,t_2,X_1,X_2).$$

The mapping Ψ is rearrangement invariable separately in both variable, l.s.c. and coercive. Let $\delta > 0$. Then, by applying Lemma 4.17 and thanks to (25), we can find $\bar{X}_1$, $\bar{X}_2$ in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$, $p_1 \in L^2_{\bar{X}_1\sharp\mathbb{P}}(\mathbb{R}^d, \mathbb{R}^d)$, $p_2 \in L^2_{\bar{X}_2\sharp\mathbb{P}}(\mathbb{R}^d, \mathbb{R}^d)$ such that:

$$\|\bar{X}_1 - \bar{X}_2\| = W_2(\bar{X}_1\sharp\mathbb{P}, \bar{X}_2\sharp\mathbb{P}), \quad \|p_1\|_{L^2_{\bar{X}_1\sharp\mathbb{P}}}, \|p_2\|_{L^2_{\bar{X}_2\sharp\mathbb{P}}} \leq \delta, \tag{30}$$

(31)
$$\Psi(\bar{X}_1, \bar{X}_2) + \langle p_1 \circ \bar{X}_1, \bar{X}_1\rangle + \langle p_2 \circ \bar{X}_2, \bar{X}_2\rangle \leq \Psi(X_1, X_2) + \langle p_1 \circ \bar{X}_1, X_1\rangle + \langle p_2 \circ \bar{X}_2, X_2\rangle, \ \forall X_1, X_2,$$

(32)
$$\overline{\mathrm{H}}(\bar{X}_1, \lambda(\bar{X}_1 - \bar{X}_2)) - \overline{\mathrm{H}}(\bar{X}_2, \lambda(\bar{X}_1 - \bar{X}_2)) \leq \omega_H(\|\bar{X}_1 - \bar{X}_2\| + \lambda\|\bar{X}_1 - \bar{X}_2\|^2) \quad \forall \lambda > 0.$$

Then, by definition if Ψ, (30) and (31), it exists $\bar{t}_1, \bar{t}_2 \in [0, T]$ such that for all t_1, t_2, X_1, X_2:
(33)
$$\begin{aligned} U_2(\bar{t}_2, \bar{X}_2) - U_1(\bar{t}_1, \bar{X}_1) \ &+ \ \tfrac{1}{2\varepsilon}\left(\|\bar{X}_1 - \bar{X}_2\|^2 + |\bar{t}_1 - \bar{t}_2|^2\right) + \tfrac{\alpha}{2}\left(\|\bar{X}_1\|^2 + \|\bar{X}_2\|^2\right) \\ &- \ \eta\bar{t}_2 + \langle p_1 \circ \bar{X}_1, \bar{X}_1\rangle + \langle p_2 \circ \bar{X}_2, \bar{X}_2\rangle \\ \leq U_2(t_2, X_2) - U_1(t_1, X_1) \ &+ \ \tfrac{1}{2\varepsilon}\left(\|X_1 - X_2\|^2 + |t_1 - t_2|^2\right) + \tfrac{\alpha}{2}\left(\|X_1\|^2 + \|X_2\|^2\right) \\ &- \ \eta t_2 + \langle p_1 \circ \bar{X}_1, X_1\rangle + \langle p_2 \circ \bar{X}_2, X_2\rangle. \end{aligned}$$

Step 1: We prove some estimates on r and ρ defined by:

$$r := \left(\|\bar{X}_1\|^2 + \|\bar{X}_2\|^2\right)^{1/2}, \quad \rho := \left(\|\bar{X}_1 - \bar{X}_2\|^2 + |\bar{t}_1 - \bar{t}_2|\right)^{1/2}. \tag{34}$$

First, applying (33) with $X_1 = X_2 = 0$, $t_1 = t_2 = \bar{t}_2$ and recalling the definition of $M > 0$, we get:

$$\begin{aligned} U_2(\bar{t}_2, \bar{X}_2) - U_1(\bar{t}_1, \bar{X}_1) \ &+ \ \frac{1}{2\varepsilon}\left(\|\bar{X}_1 - \bar{X}_2\|^2 + |\bar{t}_1 - \bar{t}_2|^2\right) + \frac{\alpha}{2}\left(\|\bar{X}_1\|^2 + \|\bar{X}_2\|^2\right) \\ &- \ \eta\bar{t}_2 + \langle p_1 \circ \bar{X}_1, \bar{X}_1\rangle + \langle p_2 \circ \bar{X}_2, \bar{X}_2\rangle \\ \leq M - \eta\bar{t}_2. \end{aligned}$$

Then, by (30), we get:

$$\frac{1}{2\varepsilon}\left(\|\bar{X}_1 - \bar{X}_2\|^2 + |\bar{t}_1 - \bar{t}_2|^2\right) + \frac{\alpha}{2}\left(\|\bar{X}_1\|^2 + \|\bar{X}_2\|^2\right) \leq 2M + \delta\left(\|\bar{X}_1\| + \|\bar{X}_2\|\right). \tag{35}$$

Firstly this implies that r satisfies $\frac{\alpha}{2}r^2 - \sqrt{2}\delta r - 2M \leq 0$ so that $r \leq \frac{1}{\alpha}(\sqrt{2}\delta + \sqrt{2\delta^2 + 4M\alpha})$ and:

$$r := \left(\|\bar{X}_1\|^2 + \|\bar{X}_2\|^2\right)^{1/2} \leq C\left(\frac{\delta}{\alpha} + \frac{1}{\sqrt{\alpha}}\right), \quad \lim_{\delta \to 0} \alpha r \leq C\sqrt{\alpha}. \tag{36}$$

Secondly, (35) combined with (36) gives $\frac{1}{2\varepsilon}\rho^2 \leq 2M + C\left(\frac{\delta^2}{\alpha} + \frac{\delta}{\sqrt{\alpha}}\right)$. So that:

$$\rho^2 \leq \varepsilon C\left(1 + \frac{\delta^2}{\alpha} + \frac{\delta}{\sqrt{\alpha}}\right), \quad \lim_{\delta \to 0} \frac{\rho}{\varepsilon} \leq \frac{C}{\sqrt{\varepsilon}}, \quad \lim_{\varepsilon \to 0} \lim_{\alpha \to 0} \lim_{\delta \to 0} \rho = 0,. \tag{37}$$

Applying (33) with $(t_2, X_2) = (\bar{t}_2, \bar{X}_2)$, $(t_1, X_1) = (\bar{t}_2, \bar{X}_2)$, we get:

$$\begin{aligned} -U_1(\bar{t}_1, \bar{X}_1) + \frac{1}{2\varepsilon}\left(\|\bar{X}_1 - \bar{X}_2\|^2 + |\bar{t}_1 - \bar{t}_2|^2\right) + \frac{\alpha}{2}\|\bar{X}_1\|^2 + \langle p_1 \circ \bar{X}_1, \bar{X}_1\rangle \\ \leq -U_1(\bar{t}_2, \bar{X}_2) + \frac{\alpha}{2}\|\bar{X}_2\|^2 + \langle p_1 \circ \bar{X}_1, \bar{X}_2\rangle. \end{aligned}$$

So that, using again (36) :

$$\begin{aligned}\frac{\rho^2}{2\varepsilon} &\le \omega_1(\rho)+\frac{\alpha}{2}\left(\|\bar X_1-\bar X_2\|^2+2\|\bar X_1\|\times\|\bar X_1-\bar X_2\|\right)+\delta\|\bar X_1-\bar X_2\|\\ &\le \omega_1(\rho)+\frac{\alpha}{2}\left(\rho^2+2C\rho\left(\frac{\delta}{\alpha}+\frac{1}{\sqrt{\alpha}}\right)\right)+\delta\rho\\ &\le \omega_1(\rho)+C(\alpha\rho^2+\rho(\delta+\sqrt{\alpha})).\end{aligned}$$

Finally we get:

$$\lim_{\varepsilon\to0}\lim_{\alpha\to0}\lim_{\delta\to0}\frac{\rho^2}{\varepsilon}:=\lim_{\varepsilon\to0}\lim_{\alpha\to0}\lim_{\delta\to0}\frac{\|\bar X_1-\bar X_2\|^2+|\bar t_1-\bar t_2|^2}{\varepsilon}=0. \tag{38}$$

Step 2: We now assume that $\bar t_1,\ \bar t_2\neq T$ and get a contradiction. By (33) with $(t_2,X_2)=(\bar t_2,\bar X_2)$ and (t_1,X_1) in $[0,T[\times L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$, we have:

$$\begin{aligned}&U_1(t_1,X_1)-U_1(\bar t_1,\bar X_1)\\ \le\ & \frac{1}{2\varepsilon}\left(\|X_1-\bar X_2\|^2-\|\bar X_1-\bar X_2\|^2+|t_1-\bar t_2|^2-|\bar t_1-\bar t_2|^2\right)\\ &\quad+\frac{\alpha}{2}(\|X_1\|^2-\|\bar X_1\|^2)+\langle p_1\circ\bar X_1,X_1-\bar X_1\rangle\\ \le\ & \frac{1}{2\varepsilon}\left(\|X_1-\bar X_1\|^2+2\langle X_1-\bar X_1,\bar X_1-\bar X_2\rangle+|t_1-\bar t_1|^2+2(t_1-\bar t_1)\times(\bar t_1-\bar t_2)\right)\\ &\quad+\frac{\alpha}{2}(\|X_1-\bar X_1\|^2+2\langle X_1-\bar X_1,\bar X_1\rangle)+\langle p_1\circ\bar X_1,X_1-\bar X_1\rangle\\ \le\ & \langle\frac{\bar X_1-\bar X_2}{\varepsilon}+\alpha\bar X_1+p_1\circ\bar X_1,X_1-\bar X_1\rangle+\frac{\bar t_1-\bar t_2}{\varepsilon}(t_1-\bar t_1)\\ &\quad+o\left((\|X_1-\bar X_1\|^2+|t_1-\bar t_1|^2)^{1/2}\right).\end{aligned}$$

This proves that:

$$\left(\frac{\bar t_1-\bar t_2}{\varepsilon},\frac{\bar X_1-\bar X_2}{\varepsilon}+\alpha\bar X_1+p_1\circ\bar X_1\right)\in D^+U_1(\bar t_1,\bar X_1) \tag{39}$$

Similarly, applying (33) with $(t_1,X_1)=(\bar t_1,\bar X_1)$ and (t_2,X_2) in $[0,T[\times L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$, we get:

$$\left(\frac{\bar t_1-\bar t_2}{\varepsilon}+\eta,\frac{\bar X_1-\bar X_2}{\varepsilon}-\alpha\bar X_2-p_2\circ\bar X_2\right)\in D^-U_2(\bar t_2,\bar X_2) \tag{40}$$

As by assumption, U_1 is a subsolution and U_2 is a supersolution, using (39) and (40):

$$\frac{\bar t_1-\bar t_2}{\varepsilon}+\overline{\mathrm{H}}\left(\bar X_1\,;\,\frac{\bar X_1-\bar X_2}{\varepsilon}+\alpha\bar X_1+p_1\circ\bar X_1\right)\ge0,$$

$$\frac{\bar t_1-\bar t_2}{\varepsilon}+\eta+\overline{\mathrm{H}}\left(\bar X_2\,;\,\frac{\bar X_1-\bar X_2}{\varepsilon}-\alpha\bar X_2-p_2\circ\bar X_2\right)\le0.$$

By substraction, we then have:

$$-\eta+\overline{\mathrm{H}}\left(\bar X_1\,;\,\frac{\bar X_1-\bar X_2}{\varepsilon}+\alpha\bar X_1+p_1\circ\bar X_1\right)-\overline{\mathrm{H}}\left(\bar X_2\,;\,\frac{\bar X_1-\bar X_2}{\varepsilon}-\alpha\bar X_2+p_2\circ\bar X_2\right)\ge0.$$

By use of (24), setting $R(\varepsilon,\alpha,\delta) := \frac{\rho}{\varepsilon} + \alpha r + \delta$ (with the notations (34)), this yields:

$$-\eta + \overline{\mathrm{H}}\left(\bar{X}_1\,;\,\frac{\bar{X}_1-\bar{X}_2}{\varepsilon}\right) - \overline{\mathrm{H}}\left(\bar{X}_2\,;\,\frac{\bar{X}_1-\bar{X}_2}{\varepsilon}\right)$$
$$+2k(1+R(\varepsilon,\alpha,\delta))(\alpha r+\delta) \geq 0,$$

and by (32):

$$(41) \qquad -\eta + \omega_H\left(\|\bar{X}_1-\bar{X}_2\| + \frac{\|\bar{X}_1-\bar{X}_2\|^2}{\varepsilon}\right) + 2k(1+R(\varepsilon,\alpha,\delta))(\alpha r+\delta) \geq 0.$$

Note that, using (36), (37) gives:

$$\lim_{\alpha\to 0}\lim_{\delta\to 0}(1+R(\varepsilon,\alpha,\delta))(\alpha r+\delta) \leq \lim_{\alpha\to 0}(1+\frac{C}{\sqrt{\varepsilon}}+C\sqrt{\alpha})C\sqrt{\alpha} = 0.$$

So that, making $\delta\to 0$, $\alpha\to 0$ and $\varepsilon\to 0$ in (41) gives (remember (37) and (38)):

$$-\eta \geq 0.$$

This yields a contradiction as $\eta > 0$.

Step 3: Now we show that $\bar{t}_1, \bar{t}_2 \neq T$. We do the proof only in case $\bar{t}_2 = T$, the remaining case being similar.
First notice that, by (28), it exists $(t,X) \in [0,T]\times L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ such that:

$$(U_2-U_1)(t,X) < -\frac{\xi}{2}.$$

Then, applying again (33) with $t_1 = t_2 = t$ and $X_1 = X_2 = X$ leads (recall (34)):

$$U_2(\bar{t}_2,\bar{X}_2) - U_1(\bar{t}_1,\bar{X}_1) + \frac{\rho^2}{2\varepsilon} + \frac{\alpha}{2}r^2 - \eta\bar{t}_2 + \langle p_1\circ\bar{X}_1,\bar{X}_1\rangle + \langle p_2\circ\bar{X}_2,\bar{X}_2\rangle$$
$$\leq \frac{-\xi}{2} + \alpha\|X\|^2 - \eta t + \langle p_1\circ\bar{X}_1 + p_2\circ\bar{X}_2, X\rangle$$

which implies by (36):

$$U_2(\bar{t}_2,\bar{X}_2) - U_1(\bar{t}_1,\bar{X}_1) - \delta C\left(\frac{\delta}{\alpha}+\frac{1}{\sqrt{\alpha}}\right) - \eta T \leq \frac{-\xi}{2} + \alpha\|X\|^2 + 2\delta\|X\|.$$

Then, as by assumption $U_2(\bar{T},\cdot) = U_1(\bar{T},\cdot)$ and $\bar{t}_2 = T$:

$$-\omega_1(\rho) - \delta C\left(\frac{\delta}{\alpha}+\frac{1}{\sqrt{\alpha}}\right) - \eta T \leq \frac{-\xi}{2} + \alpha\|X\|^2 + 2\delta\|X\|.$$

We make $\delta\to 0$, $\alpha\to 0$ and $\varepsilon\to 0$ and get by (37):

$$-\eta T \leq \frac{-\xi}{2}$$

which gives a contradiction with (29).
Q.E.D.

The following proposition gives the assumptions on $\mathcal{H}$ that will imply $(\bar{A}_2)$ for the Hamiltonian built in subsection 4.1

Proposition 4.18. *Let $\mathcal{H} : \mathcal{F}_2(\mathbb{R}^d) \to \mathbb{R}$ an Hamiltonian satisfying (A_0) and $\overline{\mathrm{H}}$ the Hamiltonian given by Proposition and Definition 4.1, 2). Assume $\mathcal{H}$ satisfies the following assumption:*

$[(A_2^*)]$ *It exists ω a modulus of continuity such that for all μ, ν in $\mathcal{P}_2(\mathbb{R}^d)$ absolutely continuous with respect to the Lebesgue measure and all $\lambda > 0$:*

$$\mathcal{H}(\mu, \lambda(T - Id)) - \mathcal{H}(\nu, \lambda(Id - S)) \leq \omega(W_2(\mu, \nu) + \lambda W_2^2(\mu, \nu))$$

where $\Pi_o(\mu, \nu) = \{(Id \times T)\sharp\mu\} = \{(S \times Id)\sharp\nu\}$.
Then $\overline{\mathrm{H}}$ satisfies (A_2).

Proof: Let μ, ν in $\mathcal{P}_2(\mathbb{R}^d)$, arguing as in Lemma 5.1, we build $(\mu_n)_n$ and $(\nu_n)_n$ two sequences in $\mathcal{P}_2(\mathbb{R}^d)$ of absolutely continuous probability measures such that:

$$\lim_{n\to+\infty} W_2(\mu_n, \mu) = \lim_{n\to+\infty} W_2(\nu_n, \nu) = 0. \tag{42}$$

Let $\gamma_n = (Id \times T_n)\sharp\mu_n = (S_n \times Id)\sharp\nu_n$ the only element of $\Pi_o(\mu_n, \nu_n)$. Up to a subsequence it converges weakly to some γ and thanks to (42), it holds:

$$\lim_{n\to+\infty} m_2(\gamma_n) = m_2(\mu) + m_2(\nu)$$

so that $(\gamma_n)_n$ is tight, γ is in $\Pi(\mu, \nu)$ and $\lim_{n\to+\infty} W_2(\gamma_n, \gamma) = 0$. Moreover $\gamma \in \Pi_o(\mu, \nu)$ as:

$$W_2^2(\mu, \nu) = \lim_{n\to+\infty} W_2^2(\mu_n, \nu_n) = \lim_{n\to+\infty} \int_{\mathbb{R}^{2d}} |x - y|^2 d\gamma_n(x, y) = \int_{\mathbb{R}^{2d}} |x - y|^2 d\gamma(x, y).$$

The convergence of $(\gamma_n)_n$ also implies (see Lemma 2.3) the existence of a sequence $(X_n)_n$ in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ and X, $Y \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ such that:

$$X_n\sharp\mathbb{P} = \mu_n,\ X\sharp\mathbb{P} = \mu,\ (X_n, T_n \circ X_n)\sharp\mathbb{P} = \gamma_n,\ (X, Y)\sharp\mathbb{P} = \gamma,$$

$$\lim_{n\to+\infty} \|X_n - X\| = \lim_{n\to+\infty} \|T_n \circ X_n - Y\| = 0.$$

From that we deduce that for all $\lambda > 0$:

$$\lim_{n\to+\infty} W_2((Id \times \lambda(T_n - Id))\sharp\mu_n, (\pi_1 \times \lambda(\pi_2 - \pi_1))\sharp\gamma)$$

$$= \lim_{n\to+\infty} \|(X_n, \lambda(T_n \circ X_n - X_n)) - (X, \lambda(Y - X))\| = 0.$$

In the same way:

$$\lim_{n\to+\infty} W_2((Id \times \lambda(Id - S_n))\sharp\nu_n, (\pi_2 \times \lambda(\pi_2 - \pi_1))\sharp\gamma) = 0.$$

By $[(A_2^*)]$, for every $\lambda > 0$, we have for all $n \in \mathbb{N}$:

$$\mathcal{H}(\mu_n, \lambda(T_n - Id)) - \mathcal{H}(\nu_n, \lambda(Id - S_n)) \leq \omega(W_2(\mu_n, \nu_n) + \lambda W_2^2(\mu_n, \nu_n))$$

and by definition of $\overline{\mathcal{H}}$ built in Proposition and Definition 4.1, (iii):

$$\overline{\mathcal{H}}((Id \times \lambda(T_n - Id))\sharp\mu_n) - \overline{\mathcal{H}}((Id \times \lambda(Id - S_n))\sharp\nu_n) \leq \omega(W_2(\mu_n, \nu_n) + \lambda W_2^2(\mu_n, \nu_n)).$$

As $\overline{\mathcal{H}}$ is locally W_2-uniformly continuous, by making $n \to +\infty$, we get:

$$\overline{\mathcal{H}}((\pi_1 \times \lambda(\pi_2 - \pi_1))\sharp\gamma) - \overline{\mathcal{H}}((\pi_2 \times \lambda(\pi_2 - \pi_1))\sharp\gamma) \leq \omega(W_2(\mu, \nu) + \lambda W_2^2(\mu, \nu)),$$

Recalling $\gamma \in \Pi_o(\mu, \nu)$ and the definition of $\overline{\mathrm{H}}$, the result follows. QED.

Discussion about assumptions of Proposition 4.16 and the domain of $\overline{\mathrm{H}}$*:*
The formulations of $(\bar{A}_1)$ and $(\bar{A}_2)$ are not homogeneous since $(\bar{A}_2)$ involves only a subdomain of $\overline{\mathrm{H}}$, namely:

$$\{(X, \lambda(Y-X)) :\ \lambda \in \mathbb{R},\ (X,Y)\sharp\mathbb{P} \in \Pi_o(X\sharp\mathbb{P}, Y\sharp\mathbb{P})\}$$

whereas $(\bar{A}_1)$ involves all $L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)\times L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$. As we are going to see, it is possible to restrict $(\bar{A}_1)$.
We will need the following objects (see [1] p 314):

$$\forall \mu \in \mathcal{P}_2(\mathbb{R}^d), \quad G(\mu) := \{(\pi_1, \lambda(\pi_2-\pi_1))\sharp\gamma :\ \gamma \text{ optimal },\lambda > 0\}$$

and we can define a distance on $\Pi(\mu,\cdot) := \{\gamma \in \mathcal{P}_2(\mathbb{R}^d) :\ \pi_1\sharp\gamma = \mu\}$ by setting for all γ^{12}, γ^{13} in $\Pi(\mu,\cdot)$:

$$W^2_\mu(\gamma^{12},\gamma^{13}) := \min\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d} |x_2-x_3|^2\, d\varpi(x_1,x_2,x_3) :\ \pi_{1,2}\sharp\varpi = \gamma^{12},\ \pi_{1,3}\sharp\varpi = \gamma^{13}\right\}.$$

Definition 4.19. The geometric tangent space in $\mathcal{P}_2(\mathbb{R}^d)$ at μ is defined by:

$$\mathbf{Tan}_\mu(\mathbb{R}^d) := \overline{G(\mu)}^{W_\mu}.$$

Let us introduce moreover for all $X \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ of law μ the following convex cone:

$$\mathbf{T}_X := \{Z :\ (X,Z)\sharp\mathbb{P} \in \mathbf{Tan}_\mu(\mathbb{R}^d)\}.$$

Using Proposition 4.29 p58 of [18] (see also Definition 4.16 p52), $\mathbf{T}_X$ is a vector subspace of $L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$.
Then, we may replace assumption $(\bar{A}_1)$ in Proposition 4.16 by:
$(\bar{A}'_1)$ *It exists* $k > 0$ *such that for all* $R > 0$*, and all* $X \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ $Z_1, Z_2 \in \mathbf{T}_X$ *with* $\|Z_1\|, \|Z_2\| \le R$*,* $\|X\| \le R$*:*

$$\overline{\mathrm{H}}(X,Z_1) - \overline{\mathrm{H}}(X,Z_2) \le k(1+R)\|Z_1-Z_2\|$$

or equivalently:
It exists $k > 0$ *such that for all* $R > 0$*, and all* $\mu \in \mathcal{P}_2(\mathbb{R}^d)$*,* γ_1*,* γ_2 *in* $\mathbf{Tan}_\mu(\mathbb{R}^d)$ *with* $m_2(\mu)$*,* $m_2(\pi_2\sharp\gamma_i) \le R^2$ *for* $i = 1,2$*:*

$$\bar{\mathcal{H}}(\gamma_1) - \bar{\mathcal{H}}(\gamma_2) \le k(1+R)W_\mu(\gamma_1,\gamma_2).$$

Then we modify the proof of Proposition 4.16 as follows.

- Build, thanks to Lemma 4.17, p_1, p_2 as in the proof above and $\bar{p}_1 \in \mathcal{T}_{\bar{X}_1\sharp\mathbb{P}}(\mathbb{R}^d)$, $\bar{p}_2 \in \mathcal{T}_{\bar{X}_2\sharp\mathbb{P}}(\mathbb{R}^d)$
- Double (31) and (33) by:

$$\Psi(\bar{X}_1,\bar{X}_2) + \langle \bar{p}_1\circ\bar{X}_1, \bar{X}_1\rangle + \langle \bar{p}_2\circ\bar{X}_2, \bar{X}_2\rangle$$

$$\le \Psi(X_1,X_2) + \langle \bar{p}_1\circ\bar{X}_1, X_1\rangle + \langle \bar{p}_2\circ X_2, X_2\rangle + o(\|X_1-\bar{X}_1\|) + o(\|X_2-\bar{X}_2\|),\ \forall X_1, X_2,$$

and

(43)
$$\begin{aligned} & U_2(\bar{t}_2,\bar{X}_2) - U_1(\bar{t}_1,\bar{X}_1) + \tfrac{1}{2\varepsilon}\left(\|\bar{X}_1-\bar{X}_2\|^2 + |\bar{t}_1-\bar{t}_2|^2\right) + \tfrac{\alpha}{2}\left(\|\bar{X}_1\|^2 + \|\bar{X}_2\|^2\right) \\ & \quad -\eta\bar{t}_2 + \langle \bar{p}_1\circ\bar{X}_1, \bar{X}_1\rangle + \langle \bar{p}_2\circ\bar{X}_2, \bar{X}_2\rangle \\ \le\ & U_2(t_2,X_2) - U_1(t_1,X_1) + \tfrac{1}{2\varepsilon}\left(\|X_1-X_2\|^2 + |t_1-t_2|^2\right) + \tfrac{\alpha}{2}\left(\|X_1\|^2 + \|X_2\|^2\right) \\ & -\eta t_2 + \langle \bar{p}_1\circ\bar{X}_1, X_1\rangle + \langle \bar{p}_2\circ\bar{X}_2, X_2\rangle + o(\|X_1-\bar{X}_1\|) + o(\|X_2-\bar{X}_2\|). \end{aligned}$$

- Step 1 and 3 are unchanged. Use (43) only in step 2 to show:

$$\left(\frac{\bar t_1-\bar t_2}{\varepsilon},\frac{\bar X_1-\bar X_2}{\varepsilon}+\alpha\bar X_1+\bar p_1\circ\bar X_1\right)\in D^+U_1(\bar t_1,\bar X_1),$$
$$\left(\frac{\bar t_1-\bar t_2}{\varepsilon}+\eta,\frac{\bar X_1-\bar X_2}{\varepsilon}-\alpha\bar X_2-\bar p_2\circ\bar X_2\right)\in D^-U_2(\bar t_2,\bar X_2).$$

Then notice:

$$\left[\frac{\bar X_1-\bar X_2}{\varepsilon}+\alpha\bar X_1+\bar p_1\circ\bar X_1\right]\in\mathbf{T}_{\bar X_1},\quad\left[\frac{\bar X_1-\bar X_2}{\varepsilon}-\alpha\bar X_2-\bar p_2\circ\bar X_2\right]\in\mathbf{T}_{\bar X_2}.$$

So that we can apply $(\bar A_1')$ to conclude the step.

With these modifications the comparison principle will hold for an Hamiltonian defined only on $\{(X,Z)\,:\,(X,Z)\sharp\mathbb{P}\in\mathbf{Tan}_{X\sharp\mathbb{P}}(\mathbb{R}^d)\}$. We could also modify the definition of sudifferential and superdifferential by restricting them to this set. This seems relevant for rearrangement invariant solutions of (HJ$_2$). Moreover it is coherent with the definitions of sub/superdifferential section of 3 which are restricted to $\mathcal{T}_\mu(\mathbb{R}^d)$ and not all $L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$. Indeed we have the following result:

Proposition 4.20. (i) *The following equalities hold:*

$$\overline{\cup_{\mu\in\mathcal{P}_2(\mathbb{R}^d)}\mathbf{Tan}_\mu(\mathbb{R}^d)}^{W_2}=\overline{\{(Id\times p)\sharp\mu\,:\,\mu\in\mathcal{P}_2(\mathbb{R}^d),\,p\in\mathcal{T}_\mu(\mathbb{R}^d)\}}^{W_2}$$
$$=\overline{\{(\pi_1\times\lambda(\pi_2-\pi_1))\sharp\gamma\,:\,\lambda\in\mathbb{R},\,\gamma\in\Pi_o(\pi_1\sharp\gamma,\pi_2\sharp\gamma)\}}^{W_2}.$$

(ii) *Assume the Hamiltonian $\mathcal{H}$ is defined only inside*

$$\{(\mu,p)\,:\,\mu\in\mathcal{P}_2(\mathbb{R}^d),\,p\in\mathcal{T}_\mu(\mathbb{R}^d)\}$$

and satisfies (A_0) on this set. Then it exists $\overline{\mathcal{H}}$ defined on $\overline{\cup_{\mu\in\mathcal{P}_2(\mathbb{R}^d)}\mathbf{Tan}_\mu(\mathbb{R}^d)}^{W_2}$ uniformly continuous on balls and such that:

$$\overline{\mathcal{H}}((Id\times p)\sharp\mu)=\mathcal{H}(\mu,p)\,\forall\mu\in\mathcal{P}_2(\mathbb{R}^d),\,p\in\mathcal{T}_\mu(\mathbb{R}^d).$$

Proof: We show only (i). Note that, for any $\mu_0\in\mathcal{P}_2(\mathbb{R}^d)$, γ_1, γ_2 in $\Pi(\mu_0,\cdot)$, we have $W_2(\gamma_1,\gamma_2)\le W_{\mu_0}(\gamma_1,\gamma_2)$. So that:

$$\overline{\cup_{\mu\in\mathcal{P}_2(\mathbb{R}^d)}\mathbf{Tan}_\mu(\mathbb{R}^d)}^{W_2}=\overline{\cup_{\mu\in\mathcal{P}_2(\mathbb{R}^d)}\overline{G(\mu)}^{W_\mu}}^{W_2}=\overline{\cup_{\mu\in\mathcal{P}_2(\mathbb{R}^d)}G(\mu)}^{W_2}$$
$$=\overline{\{(\pi_1\times\lambda(\pi_2-\pi_1))\sharp\gamma\,:\,\lambda\in\mathbb{R},\,\gamma\in\Pi_o(\pi_1\sharp\gamma,\pi_2\sharp\gamma)\}}^{W_2}.$$

Now let $\pi\in G(\mu_0)$, $\pi=(\pi_1,\lambda(\pi_2-\pi_1))\sharp\gamma$ with $\gamma\in\Pi_o(\mu_0,\pi_2\sharp\gamma)$ and $\lambda>0$. By Lemma 7.2.1 and Theorem 7.2.2 in [1], we can build a constant speed geodesic between μ_0 and $\pi_2\sharp\gamma$ by setting $\mu_t=((1-t)\pi_1+t\pi_2)\sharp\gamma$ for all $t\in[0,1]$. Moreover for all $n\in\mathbb{N}^*$, $\Pi_o(\mu_{1/n},\pi_2\sharp\gamma)=\{(Id\times T_n)\sharp\mu_{1/n}\}$ for some $T_n\in L^2_{\mu_{1/n}}(\mathbb{R}^d,\mathbb{R}^d)$. Then:

$$\lim_{n\to+\infty}W_2((Id\times T_n)\sharp\mu_{1/n},\gamma)\le\lim_{n\to+\infty}W_2(\mu_{1/n},\mu_0)=\lim_{n\to+\infty}1/nW_2(\mu_0,\pi_2\sharp\gamma)=0.$$

Arguing as in Step 3 of Theorem 4.8, we also have:

$$\lim_{n\to+\infty}W_2((Id\times\lambda(T_n-Id))\sharp\mu_{1/n},\pi)=0.$$

From that we have $\pi\in\overline{\{(Id\times p)\sharp\mu\,:\,\mu\in\mathcal{P}_2(\mathbb{R}^d),\,p\in\mathcal{T}_\mu(\mathbb{R}^d)\}}$. As this is true for all $\pi\in G(\mu_0)$ and all $\mu_0\in\mathcal{P}_2(\mathbb{R}^d)$, we get:

$$\overline{\cup_{\mu\in\mathcal{P}_2(\mathbb{R}^d)}G(\mu)}^{W_2}\subset\overline{\{(Id\times p)\sharp\mu\,:\,\mu\in\mathcal{P}_2(\mathbb{R}^d),\,p\in\mathcal{T}_\mu(\mathbb{R}^d)\}}.$$

The opposite inclusion holds as

$$\{(Id \times p)\sharp\mu : \mu \in \mathcal{P}_2(\mathbb{R}^d),\, p \in \mathcal{T}_\mu(\mathbb{R}^d)\} \subset \cup_{\mu\in\mathcal{P}_2(\mathbb{R}^d)} \mathbf{Tan}_\mu(\mathbb{R}^d).$$

QED.

An open question is weather we can build an Hamiltonian for which there will be both a rearrangement invariant solution for (HJ_2) and a non rearrangement invariant solution.

4.3. **An example of application to an Optimal Control problem in $\mathcal{P}_2(\mathbb{R}^d)$.**

4.3.1. *Setting of the problem.* In this section we focus on the following value function related to a multiagent control problem studied in [20], [23] and defined for all $t_0 \in [0, T]$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$:

$$\mathcal{V}(t_0, \mu) := \inf_{(v_t,\mu_t)} \{\mathcal{G}(\mu_T) : t \in [t_0, T] \mapsto (v_t, \mu_t) \text{ admissible and } \mu_{t_0} = \mu\}$$

where (v_t, μ_t) is admissible in $[t_0, T]$ if:

- $(t, x) \in [t_0, T] \times \mathbb{R}^d \mapsto v_t$ is a Borel map;
- $t \in [t_0, T] \mapsto \mu_t \in \mathcal{P}_2(\mathbb{R}^d)$ associated with v_t is in $AC^2([t_0, T], \mathcal{P}_2(\mathbb{R}^d))$, that is:
  $$\partial_t\mu_t + \mathrm{div}(v_t\mu_t) = 0 \text{ in } \mathbb{R}^d \times]t_0, T[$$
  in the distributional sense and
  $$\int_0^T \int_{\mathbb{R}^d} |v_t(x)|^2 \, d\mu_t(x) \, dt < +\infty;$$
- it exists $\mathbf{u} : [0, T] \times \mathbb{R}^d \to \mathbf{U}$ Borel such that:
  $$v_t(x) = f(x, \mathbf{u}(t, x), \mu_t) \text{ a.e. } t,\ \mu_t\text{-a.e.}$$
  where f and $\mathbf{U}$ satisfy the same properties as in Example 2.9.

The map $\mathcal{G} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is assumed to be bounded and uniformly continuous.

Considering an admissible curve $t \mapsto \mu_t$, at each time t, μ_t represents the location of a group with a high number of individuals. This is justified by the Superposition Principle (Theorem 8.2.1. of [1]) and by the following result. As usual, for any $\sigma \in \mathcal{C}([0, T], \mathbb{R}^d)$ and any $t \in [0, T]$, we set: $e_t(\sigma) := \sigma(t)$.

Proposition 4.21. *(see* [1], [23], [9] *and* [24]*)*

(i) *Let $t \in [t_0, T] \mapsto (v_t, \mu_t)$ admissible in $[t_0, T]$ such that $\mu_{t_0} = \mu$ and $(t, x) \mapsto \mathbf{u}(t, x) \in \mathbf{U}$ the associated control.*
It exists $\eta \in \mathcal{P}(\mathcal{C}([t_0, T], \mathbb{R}^d))$ such that $\mu_t = e_t\sharp\eta$ and η is concentrated on $\sigma \in W^{1,2}([t_0, T], \mathbb{R}^d)$ such that:
$$\dot\sigma(t) = v_t(\sigma(t)) = f(\sigma(t), \mathbf{w}(t, \sigma), e_t\sharp\eta) \text{ a.e. } t,$$
with $\mathbf{w}(t, \sigma) := \mathbf{u}(t, \sigma(t))$. Moreover:
$$\int_{t_0}^T \int_{\mathcal{C}([t_0,T],\mathbb{R}^d)} |\dot\sigma(t)|^2 \, d\eta(\sigma) dt < +\infty.$$

(ii) *Let $\eta \in \mathcal{P}(\mathcal{C}([t_0,T],\mathbb{R}^d))$ concentrated on $\sigma \in W^{1,2}([t_0,T],\mathbb{R}^d))$ such that:*

$$\dot\sigma(t) = f(\sigma(t), \mathbf{u}(t,\sigma), e_t\sharp\eta) \text{ a.e. } t,$$

with $\mathbf{u} : [t_0,T] \times \mathcal{C}([0,T],\mathbb{R}^d) \to \mathbf{U}$ Borel. Assume moreover that setting $\mu_t := e_t\sharp\eta$, it holds $\int_{t_0}^T m_2(\mu_t)\,dt < +\infty$. Then set:

$$\mu_t = e_t\sharp\eta, \quad v_t(x) = \int_{\mathcal{C}([t_0,T],\mathbb{R}^d)} \dot\sigma(t) d\eta^{t,x}(\sigma) \text{ a.e. } t,\ \mu_t\text{–a.e.}x$$

where $\eta^{t,x}$ is obtained by disintegration of η: $\eta(\sigma) = \eta^{t,x}(\sigma) \otimes \mu_t(x)$. Then (v_t,μ_t) is admissible for $\mathcal{V}(t_0,\mu_0)$ and:

$$v_t(x) = f(x, \mathbf{w}(t,x), \mu_t) \text{ a.et, } \mu_t\text{–a.e.}x, \quad \mathbf{w}(t,x) := \int \mathbf{u}(t,\sigma)\, d\eta^{t,x}(\sigma).$$

The trajectories σ on which a measure η as above is concentrate can be seen as the trajectories of the individuals of the considered group. Note that for a general $\eta \in \mathcal{P}(\mathcal{C}([t_0,T],\mathbb{R}^d))$, trajectories may cross at a certain point x at time t with different velocities. Nevertheless the corresponding (v_t,μ_t) only sees an average of these velocites. As in [9], considering measures $\eta \in \mathcal{P}(\mathcal{C}([t_0,T],\mathbb{R}^d))$ as in (ii) will be called the Kantorovich point of view while considering $t \to \mu_t$ is the Eulerian point of view.

It was proved in [23] that the infimum in $\mathcal{V}$ is a minimum, moreover $\mathcal{V}$ is Bounded uniformly continuous (Proposition 4.3. of [23]) and it is the unique strict viscosity solution of the following Hamilton-Jacobi-Bellman equation (Theorem 5.8 of [23]):

$$\begin{cases} \partial_t u(t,\mu) + \mathcal{H}_2(\mu, D_\mu u(t,\mu)) = 0 & \forall (t,\mu) \in [0,T[\times\mathcal{P}_2(\mathbb{R}^d) \\ u(T,\mu) = \mathcal{G}(\mu) \quad \forall \mu \in \mathcal{P}_2(\mathbb{R}^d) \end{cases}$$

where $\mathcal{H}_2$ is the Hamiltonian defined in Example 2.9. Note that, as $\mathcal{H}_2$ satisfies all the assumptions of Theorem 3.23 above, it is also the unique (non-strict) viscosity solution of this equation.
At the end of this section, we will get that its lift is also the unique viscosity solution of:

$$\begin{cases} \partial_t U(t,X) + \overline{\mathrm{H}}_2(X, D_X U(t,X)) = 0 & \forall (t,X) \in [0,T[\times L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d) \\ U(T,X) = \mathcal{G}(X\sharp\mathbb{P}) \quad \forall X \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d). \end{cases}$$

4.3.2. *Several types of trajectories and several value functions.*
An admissible trajectory can be represented by a curve in $L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$, this is the Lagrangian points of view. It holds:

Proposition 4.22. *(see [9] and [24]) Let $t \in [t_0,T] \mapsto (v_t,\mu_t)$ admissible in $[t_0,T]$ such that $\mu_{t_0} = \mu$ and $(t,x) \mapsto \mathbf{u}(t,x) \in \mathbf{U}$ the associated control.*

(i) *Let $Y_{t_0} \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ of law μ. Then, for all $\varepsilon > 0$, it exists $t \mapsto X_t \in W^{1,2}([t_0,T], L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d))$ such that $\|Y_{t_0} - X_{t_0}\| \le \varepsilon$ and:*

$$X_t\sharp\mathbb{P} = \mu_t,$$

$$\dot X_t(\omega) = v_t(X_t(\omega)) = f(X_t(\omega), \mathbf{w}(t,\omega), X_t\sharp\mathbb{P}) \text{ a.e. } t,\ \mathbb{P}\text{-a.e.}\omega \in \Omega,$$

with $\mathbf{w}(t,\omega) := \mathbf{u}(t, X_t(\omega))$.

(ii) *Let* $t \mapsto X_t \in W^{1,2}([t_0,T], L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d))$ *such that:*

$$\dot{X}_t(\omega) = f(X_t(\omega), \mathbf{u}(t,\omega), X_t\sharp\mathbb{P}) \text{ a.e. } t,\ \mathbb{P}\text{-a.e.}\omega \in \Omega,$$

with $\mathbf{u} : [t_0,T]\times\Omega \to \mathbf{U}$ *Borel.*
Then, set:

$$\mu_t = X_t\sharp\mathbb{P}, \quad \gamma_t := (X_t, \dot{X}_t)\sharp\mathbb{P}, \quad v_t(x) = \int_{\mathbb{R}^d} z\, d\gamma^{t,x}(z) \text{ a.e.}t,\ \mu_t\text{-a.e.}x$$

where by disintegration $\gamma_t = \gamma^{t,x}\otimes\mu_t(x)$*. Then* (v_t,μ_t) *is admissible for* $\mathcal{V}(t_0,\mu_0)$ *and setting* $m_t = (Id\times X_t)\sharp\mathbb{P}$*:*

$$v_t(x) = f(x, \mathbf{w}(t,x), \mu_t) \text{ a.e.}t,\ \mu_t\text{-a.e.}x$$

$$\text{with } \mathbf{w}(t,x) := \int_\Omega \mathbf{u}(t,\omega)\, dm^{t,x}(\omega) d\mu_t(x)$$

where by disintegration $m_t(\omega,x) = m^{t,x}(\omega)\otimes\mu_t(x)$

Element of Proof of (ii): to prove $v_t(x) = f(x,\mathbf{w}(t,x),\mu_t)$ a.e.t, μ_t-a.e.x, observe that for any $\Phi \in \mathcal{C}_c([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$, by affinity of f:

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d} \Phi(t,x)\cdot v_t(x)\, d\mu_t(x)dt = \int_0^T\int_\Omega \Phi(t,X_t)\cdot\dot{X}_t\, d\mathbb{P}dt\\
&= \int_0^T\int_\Omega \Phi(t,X_t)\cdot f(X_t(\omega),\mathbf{u}(t,\omega),X_t\sharp\mathbb{P})\ d\mathbb{P}(\Omega)dt\\
&= \int_0^T\int_{\Omega\times\mathbb{R}^d} \Phi(t,x)\cdot f(x,\mathbf{u}(t,\omega),\mu_t)\ dm_t(\omega,x)dt\\
&= \int_0^T\int_{\mathbb{R}^d} \Phi(x,t)\cdot f(x,\mathbf{w}(t,x),\mu_t)\ d\mu_t(x)dt.
\end{aligned}$$

QED.

Note that, while we can represent any admissible (or optimal) curve $t\in[t_0,T]\mapsto(v_t,\mu_t)$ as a trajectory in $L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$, we cannot choose any starting point Y_{t_0} of law μ_0.

To be complete, we also state:

Proposition 4.23. (i) *Let* $\eta \in \mathcal{P}(\mathcal{C}([t_0,T],\mathbb{R}^d))$ *concentrated on curves* $\sigma \in W^{1,2}([t_0,T],\mathbb{R}^d)$ *such that:*

$$\dot{\sigma}(t) = f(\sigma(t), \mathbf{u}(t,\sigma), e_t\sharp\eta) \text{ a.e. } t,$$

with $\mathbf{u} : [t_0,T]\times\mathcal{C}([0,T],\mathbb{R}^d) \to \mathbf{U}$ *Borel. Assume moreover that setting* $\mu_t := e_t\sharp\eta$*, it holds* $\int_{t_0}^T m_2(\mu_t)\,dt < +\infty$*.*
Then, it exists $T_\eta : \Omega \to \mathcal{C}([t_0,T],\mathbb{R}^d)$ *such that* $T_\eta\sharp\mathbb{P} = \eta$ *and setting* $X_t = e_t\circ T_\eta$*,* $t\mapsto X_t$ *is in* $W^{1,2}([t_0,T],\mathbb{R}^d)$ *and:*

$$\begin{cases} X_t\sharp\mathbb{P} = e_t\sharp\eta = \mu_t,\\ \dot{X}_t(\omega) = f(X_t(\omega),\mathbf{w}(t,\omega),X_t\sharp\mathbb{P}) \text{ a.e. } t,\ \mathbb{P}\text{-a.e.}\omega\in\Omega,\end{cases}$$

with $\mathbf{w}(t,\omega) := \mathbf{u}(t,T_\eta(\omega))$.

(ii) *Let* $t \mapsto X_t \in W^{1,2}([t_0,T], L_{\mathbb{P}}(\Omega,\mathbb{R}^d))$ *such that:*

$$\dot{X}_t(\omega) = f(X_t(\omega), \mathbf{u}(t,\omega), X_t\sharp\mathbb{P}) \text{ a.e. } t, \ \mathbb{P}\text{-a.e.}\omega \in \Omega,$$

with $\mathbf{u} : [t_0,T]\times\Omega \to \mathbf{U}$ *Borel.*
Set $T_X : \omega \in \Omega \mapsto X_.(\omega) \in \mathbb{R}^d \times \mathcal{C}([0,T],\mathbb{R}^d)$ *and* $\eta := T_X\sharp\mathbb{P}$, *and* $\mu_t := X_t\sharp\mathbb{P} = e_t\sharp\eta$, $m = (Id \times T_X)\sharp\mathbb{P}$.
Then, η *is concentrated on curves* $\sigma \in W^{1,2}([t_0,T],\mathbb{R}^d)$ *such that:*

$$\dot{\sigma}(t) = f(\sigma(t), \mathbf{w}(t,\sigma), e_t\sharp\eta) \text{ a.e. } t,$$

with $\mathbf{w}(t,\sigma) = \int \mathbf{u}(t,\omega)\ dm^\sigma(\omega)$ *where* m^σ *is obtained by disintegration:* $m(\omega,\sigma) = m^\sigma(\omega)\otimes\eta(\sigma)$. *Moreover:* $\int_{t_0}^T m_2(\mu_t)\,dt < +\infty$.

Proof: (i) For all $\varphi \in \mathcal{C}_c(\mathbb{R}^d)$, we have for all $t \in [t_0,T]$:

$$\int_{\mathbb{R}^d} \varphi(x)\,d(X_t\sharp\mathbb{P})(x) = \int_\Omega \varphi(X_t(\omega))\,d\mathbb{P}(\omega) = \int_\Omega \varphi(e_t\circ T_\eta(\omega))\,d\mathbb{P}(\omega)$$
$$= \int_{\mathcal{C}([t_0,T],\mathbb{R}^d)} \varphi(e_t(\sigma))\,d\eta(\sigma) = \int_{\mathbb{R}^d} \varphi(x)\,d\mu_t(x)$$

so that $X_t\sharp\mathbb{P} = \mu_t$. Then taking $Y \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ and setting $m = (T_\eta \times Y)\sharp\mathbb{P}$ we have for all $t \in [t_0,T]$:

$$\begin{aligned}
\langle X_t, Y\rangle &= \int_\Omega e_t(T_\eta)\cdot Y\,d\mathbb{P} = \int_{\mathcal{C}([t_0,T],\mathbb{R}^d)\times\Omega} e_t(\sigma)\cdot y\,dm(\sigma,y)\\
&= \int_{\mathcal{C}([t_0,T],\mathbb{R}^d)\times\Omega} \left[\sigma(t_0) + \int_{t_0}^t \dot{\sigma}(\tau)\,d\tau\right]\cdot y\,dm(\sigma,y)\\
&= \langle X_{t_0}, Y\rangle + \int_{\mathcal{C}([t_0,T],\mathbb{R}^d)\times\Omega} \left[\int_{t_0}^t f(\sigma(\tau), \mathbf{u}(\tau,\sigma), \mu_\tau)\cdot y\,d\tau\right] dm(\sigma,y)\\
&= \langle X_{t_0}, Y\rangle + \int_{t_0}^t \left[\int_{\mathcal{C}([t_0,T],\mathbb{R}^d)\times\Omega} f(e_\tau(\sigma), \mathbf{u}(t,\sigma), \mu_t)\cdot y\,dm(\sigma,y)\right] d\tau\\
&= \langle X_{t_0}, Y\rangle + \int_{t_0}^t \left[\int_\Omega f(e_\tau\circ T_\eta, \mathbf{u}(t,T_\eta), \mu_t)\cdot Y\,d\mathbb{P}\right] d\tau.
\end{aligned}$$

From that, we deduce $t \mapsto X_t$ is in $W^{1,1}([t_0,T], L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d))$ and that, moreover:

$$\dot{X}_t(\omega) = f(X_t(\omega), \mathbf{u}(t,T_\eta(\omega), X_t\sharp\mathbb{P}) \text{ a.e. } t, \ \mathbb{P}\text{-a.e.}\omega \in \Omega.$$

To conclude, we recall that, by Remark 7.2 of [9]:

$$\int_{t_0}^T \int_{\mathcal{C}([t_0,T],\mathbb{R}^d)} |\dot{\sigma}(t)|^2 dt d\eta(\sigma) < +\infty.$$

This inequality implies: $\int_{t_0}^T \int_\Omega |\dot{X}_t(\omega)|^2 dt d\mathbb{P}(\omega) < +\infty$.

(ii) We only prove that η is concentrated on $\sigma \in W^{1,2}([t_0,T],\mathbb{R}^d)$ such that:

$$\dot{\sigma}(t) = f(\sigma(t), \mathbf{w}(t,\sigma), e_t\sharp\eta) \text{ a.e. } t,$$

with $\mathbf{w}$ defined as in the proposition. Indeed let $\varphi \in \mathcal{C}_c([t_0,T]\times\mathbb{R}^d)$ then thanks to the affinity of f in $\mathbf{u}$:

$$\int_{t_0}^T \int_\Omega \varphi(t,\sigma(t))\,d\eta(\sigma)dt = \int_{t_0}^T \int_\Omega \varphi(t, e_t\circ T_X)\,d\mathbb{P}dt = \int_{t_0}^T \int_\Omega \varphi(t,X_t)\,d\mathbb{P}dt$$

$$= \int_{t_0}^T \int_\Omega \varphi\left(t, X_{t_0}(\omega) + \int_{t_0}^t f(X_\tau(\omega),\mathbf{u}(\tau,\omega),\mu_\tau)d\tau\right)\,d\mathbb{P}(\omega)dt$$

$$= \int_{t_0}^T \int_\Omega \varphi\left(t, e_{t_0}\circ T_X(\omega) + \int_{t_0}^t f(e_\tau\circ T_X(\omega),\mathbf{u}(\tau,\omega),\mu_\tau)d\tau\right)\,d\mathbb{P}(\omega)dt$$

$$= \int_{t_0}^T \int_{\Omega\times\mathcal{C}([t_0,T],\mathbb{R}^d)} \varphi\left(t, e_{t_0}(\sigma) + \int_{t_0}^t f(e_\tau(\sigma),\mathbf{u}(\tau,\omega),\mu_\tau)d\tau\right)\,dm(\omega,\sigma)dt$$

$$= \int_{t_0}^T \int_{\Omega\times\mathcal{C}([t_0,T],\mathbb{R}^d)} \varphi\left(t, \sigma(t_0) + \int_{t_0}^t f(\sigma(\tau),\int_\Omega \mathbf{u}(\tau,\omega)\,dm^\sigma(\omega),\mu_\tau)d\tau\right)\,d\eta(\sigma)dt.$$

QED.

Again, in (i), we cannot choose any Y_{t_0} of law $e_{t_0}\sharp\eta$.

Remark 4.24. The Lagrangian point of view is similar to the Kantorovitch one in the sense that different trajectories $t\mapsto X_t(\omega_1)$ and $t\mapsto X_t(\omega_2)$ crossing at (x,t) may have different velocites $\dot{X}_t(\omega_1)$ and $\dot{X}_t(\omega_2)$. In other words $\dot{X}_t$ may not be in H_{X_t}. As we have already noticed, this particularity is not captured by the Eulerian point of view. Nevertheless, due to the convexity of the constaint on v_t in $\mathcal{V}$, all points of view happen to be equivalent, this was proved in [9]. To state this result, we introduce several values.

Set for all $t_0\in[t_0,T]$ and $X\in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$:

$$V_L(t_0,X) = \inf\{\mathcal{G}(X_T\sharp\mathbb{P}) : t\mapsto X_t \text{ is admissible and } X_{t_0}=X\},$$

a map $t\mapsto X_t$ si said to be admissible for $V_L(t_0,X)$ if for some $\mathbf{u}:[t_0,T]\times\Omega\to\mathbf{U}$ Borel:

- $t\mapsto X_t$ is in $W^{1,2}([0,T],L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d))$,
- $\dot{X}_t(\omega) = f(X_t(\omega),\mathbf{u}(t,\omega),X_t\sharp\mathbb{P})$ a.e. t, $\mathbb{P}$-a.e. $\omega\in\Omega$.

Set for all $t_0\in[t_0,T]$ and $\mu\in\mathcal{P}_2(\mathbb{R}^d)$:

$$\mathcal{V}_K(t_0,\mu) = \inf\{\mathcal{G}(e_T\sharp\eta) : \eta \text{ is admissible and } e_{t_0}\sharp\eta=\mu\},$$

a probability measure $\eta\in\mathcal{P}(\mathcal{C}([t_0,T],\mathbb{R}^d))$ is admissible for $\mathcal{V}_K(t_0,\mu)$ if for some $\mathbf{u}:[t_0,T]\times\mathcal{C}([0,T],\mathbb{R}^d)\to\mathbf{U}$ Borel:

- η is concentrated on $\sigma\in W^{1,2}([t_0,T],\mathbb{R}^d)$ such that:
$$\dot\sigma(t) = f(\sigma(t),\mathbf{u}(t,\sigma),e_t\sharp\eta) \text{ a.e. } t,$$
- moreover, setting $\mu_t := e_t\sharp\eta$, it holds $\int_{t_0}^T m_2(\mu_t)\,dt < +\infty$.

We state:

Proposition 4.25. *For any $\mu\in\mathcal{P}_2(\mathbb{R}^d)$ and any $X\in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ of law μ:*

$$\mathcal{V}(t_0,\mu) = \mathcal{V}_K(t_0,\mu) = V_L(t_0,X) = \mathcal{V}(t_0,X\sharp\mathbb{P}).$$

We refer to [9] for the proof in a quite general case. We give a quick alternative proof in our case. As for $\mathcal{G}$, we will only assume its continuity (it is even more simple with $\mathcal{G}$ uniformly continuous as above).

Alternative proof: First notice that, by propositions 4.21 et 4.22, it holds:

$$\mathcal{V}_K(t_0,\mu)=\mathcal{V}(t_0,\mu)\le V_L(t_0,X) \tag{44}$$

for all $t_0\in[0,T]$, $\mu\in\mathcal{P}_2(\mathbb{R}^d)$ and $X\in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ of law μ.
Let us prove the opposite inequality of (44). Consider $(v_t,\mu_t)_t$ optimal for $\mathcal{V}(t_0,\mu)$, $\mathbf{u}$ the associated control and $t\mapsto Y_t$ given by Proposition 4.21, (i) with:

$$\begin{cases} \dot{Y}_s(\omega)=f(Y_s(\omega),\mathbf{u}(s,Y_s(\omega)),Y_s\sharp\mathbb{P}) \text{ a.e. } s\in[t_0,T],\ \mathbb{P}\text{-a.e.}\omega\in\Omega \\ Y_s\sharp\mathbb{P}=\mu_s \text{ a.e. } s\in[t_0,T], \quad Y_{t_0}\sharp\mathbb{P}=\mu=X\sharp\mathbb{P}. \end{cases}$$

By Lemma 2.2, for all $n\in\mathbb{N}$ it exists $\tau_n:\Omega\to\Omega$ a bijection with $\tau_n^{-1}\sharp\mathbb{P}=\tau_n\sharp\mathbb{P}=\mathbb{P}$ and

$$\|Y_{t_0}\circ\tau_n-X\|_{L^\infty_{\mathbb{P}}}\le\frac{1}{n}.$$

Note that $(Y_t-Y_{t_0})\circ\tau_n=Y_t\circ\tau_n-Y_{t_0}\circ\tau_n$ for all $t\in[0,T]$, indeed, for all $Z\in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$:

$$\begin{aligned}\langle(Y_t-Y_{t_0})\circ\tau_n,Z\rangle&=\langle(Y_t-Y_{t_0}),Z\circ\tau_n^{-1}\rangle=\langle Y_t,Z\circ\tau_n^{-1}\rangle-\langle Y_{t_0},Z\circ\tau_n^{-1}\rangle\\&=\langle Y_t\circ\tau_n,Z\rangle-\langle Y_{t_0}\circ\tau_n,Z\rangle=\langle Y_t\circ\tau_n-Y_{t_0}\circ\tau_n,Z\rangle.\end{aligned}$$

From this we deduce that $t\mapsto Y_t\circ\tau_n$ is admissible for $V_L(t_0,Y_{t_0}\circ\tau_n)$, indeed, a.e. t, $\mathbb{P}$-a.e.:

$$\begin{aligned}Y_t\circ\tau_n-Y_{t_0}\circ\tau_n&=(Y_t-Y_{t_0})\circ\tau_n=\int_{t_0}^t f(Y_s\circ\tau_n,\mathbf{u}(s,Y_s\circ\tau_n,Y_s\sharp\mathbb{P})ds\\&=\int_{t_0}^t f\left(Y_s\circ\tau_n,\mathbf{u}(s,Y_s\circ\tau_n),(Y_s\circ\tau_n)\sharp\mathbb{P}\right)ds.\end{aligned}$$

Moreover $\mathcal{G}(Y_T\sharp\mathbb{P})=\mathcal{G}(\mu)=\mathcal{G}((Y_T\circ\tau_n)\sharp\mathbb{P})$ so that:

$$\mathcal{V}(t_0,\mu)=V_L(t_0,Y_{t_0})=V_L(t_0,Y_{t_0}\circ\tau_n).$$

Now, as $\mathcal{G}$ is continuous, for all $\varepsilon>0$, it exists $\eta>0$ (depending on μ_T) such that for all $\nu\in\mathcal{P}_2(\mathbb{R}^d)$:

$$W_2(\nu,\mu_T)\le\eta\Rightarrow|\mathcal{G}(\mu_T)-\mathcal{G}(\nu)|\le\varepsilon. \tag{45}$$

Let $t\mapsto X_t$ in $W^{1,2}([0,T],L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d))$ satisfying:

$$\begin{cases} \dot{X}_t(\omega)=f(X_t(\omega),\mathbf{u}(t,Y_t(\omega)),X_t\sharp\mathbb{P}) \text{ a.e. } t\in[t_0,T],\ \mathbb{P}\text{-a.e.}\omega\in\Omega \\ \dot{X}_{t_0}=\tilde{X}. \end{cases}$$

Thanks to Gronwall's Lemma, it exists $C>0$ such that:

$$W_2(\mu_T,X_T\sharp\mathbb{P})\le\|X_T-Y_T\circ\tau_n\|\le C\|X-Y_0\circ\tau_n\|.$$

Then, for n big enough, $W_2(\mu_T,X_T\sharp\mathbb{P})\le\eta$, and as $t\mapsto X_t$ is admissible, using (45):

$$V_L(t_0,X)\le\mathcal{G}(X_T\sharp\mathbb{P})\le\mathcal{G}(\mu_T)+\varepsilon=\mathcal{V}(t_0,\mu)+\varepsilon.$$

As this is true for all $\varepsilon>0$, making ε go to zero gives the desired result.
QED.

4.3.3. *Dynamic programming principles and viscosity solution.*
The following dynamic programming principle was proved in [23]: for all $s < t$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$

$$\mathcal{V}(s,\mu) = \inf_{(v_\tau,\mu_\tau)} \{\mathcal{V}(t,\mu_t) : (v_\tau,\mu_\tau) \text{ is admissible for } \mathcal{V}(s,\mu) \text{ and } \mu_s = \mu\}.$$

Remark 4.26. Thanks to the results of the section above, the dynamic programming principle above implies a dynamic programming principle in the Lagrangian setting:

$$V_L(s,X) = \inf\{\mathcal{V}(t,X_t) : \tau \mapsto X_\tau \text{ is admissible for } V_L(s,X) \text{ and } X_s = X\}$$

for all $s < t$ and $X \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$. Note that the admissible trajectories of the right side of the inequalities do not necessarily satisfy $\dot{X}_t \in H_{X_t}$.

We are now able to show:

Proposition 4.27. *The lift V_L of $\mathcal{V}$ is the unique viscosity solution of the following equation:*

$$\begin{cases} \partial_t U(t,X) + \overline{\mathrm{H}}_2(X, D_X U(t,X)) = 0 \ \forall (X,t) \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d) \times [0,T[\\ U(T,X) = \mathcal{G}(X\sharp\mathbb{P}) \ \forall X \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d). \end{cases}$$

Proof: We only prove that V_L is a subsolution. The rest being similar. Let $(t_0,X) \in [t_0,T[\times L^2_{\mathbb{P}}(\Omega)$, $(p_t,Z) \in D^+V_L(t_0,X)$ and $\mathbf{u} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbf{U}$ a Borel map. Let $t \mapsto X_t$ such that:

$$\begin{cases} \dot{X}_t(\omega) = f(X_t(\omega), \mathbf{u}(X(\omega), Z(\omega)), X_t\sharp\mathbb{P}) \text{ a.e. } t \in [t_0,T], \ \mathbb{P}\text{-a.e.}\omega \in \Omega \\ \dot{X}_{t_0} = X. \end{cases}$$

By the dynamic programming principle above

$$0 = V_K(t_0+h, X_{t_0+h}) - V_K(t_0, X_{t_0}).$$

Then, as $(p_t,Z) \in D^+V_L(t_0,X)$, it holds:

$$0 \le \langle Z, X_{t_0+h} - X_{t_0}\rangle + p_t h + o\left(\sqrt{h^2 + \|X_{t_0+h} - X_{t_0}\|^2}\right)$$

$$= \int_\Omega \int_{t_0}^{t_0+h} f(X_t(\omega), \mathbf{u}(X(\omega), Z(\omega)), X_t\sharp\mathbb{P}) \cdot Z(\omega)\, d\mathbb{P}(\omega)\, dt + p_t h$$

$$+ o\left(\sqrt{h^2 + \|X_{t_0+h} - X_{t_0}\|^2}\right).$$

Dividing by h and making h tends to 0 gives:

$$0 \le \int_\Omega f(X(\omega), \mathbf{u}(X(\omega), Z(\omega)), X\sharp\mathbb{P}) \cdot Z(\omega)\, d\mathbb{P}(\omega) + p_t.$$

As this is true for any Borel map $\mathbf{u} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbf{U}$, recalling the definition of $\overline{\mathrm{H}}_2$:

$$0 \le \overline{\mathrm{H}}_2(X,Z) + p_t.$$

QED.

Conclusion:

- the extension $\overline{\mathrm{H}}$ allows to take into account the different velocities of crossing trajectories,

- moreover, thanks to the convexity of the constraint of $\mathcal{V}$, the dynamic programming principle in the Eulerian point of view implies a richer one in the Lagrangian point of view,
- finally, thanks to this second dynamic programming principle, the lift $V = V_L$ of $\mathcal{V}$ satisfies the extended equation (HJ$_2$).

5. Appendix

Lemma 5.1. *Let $\varpi \in \mathcal{P}_2(\mathbb{R}^{3d})$, it exists $\mu_n \in \mathcal{P}_2(\mathbb{R}^d)$ and $p_n, q_n \in L^2_{\mu_n}(\mathbb{R}^d, \mathbb{R}^d)$ such that:*

$$\lim_{n\to+\infty} W_2((Id \times p_n \times q_n)\sharp\mu_n\,,\,\varpi) = 0.$$

Proof: This proof is based upon the proof of Theorem 1.32 of [25].
Step 1: Let $\varpi \in \mathcal{P}_2(\mathbb{R}^{3d})$, we first build $(\varpi_n)_n$ absolutely continuous with respect to the Lebesgue measure such that:

$$\lim_{n\to+\infty} W_2(\varpi_n\,,\,\varpi) = 0.$$

Indeed, for all $n \in \mathbb{N}$ consider a partition $(Q_{i,n})_{i\in I_n}$ of $\mathbb{R}^d$ where each $Q_{i,n}$ is a cube of edge $1/n$ and set:

$$\varpi_n := \sum_{i,j,k\in I_n} n^{3d}\varpi(Q_{i,n} \times Q_{j,n} \times Q_{k,n})\,\mathcal{L}^{3d}\lfloor Q_{i,n} \times Q_{j,n} \times Q_{k,n}$$

the measure $\mathcal{L}^{3d}$ being the Lebesgue measure on $\mathbb{R}^{3d}$. Then, by construction, for all $i, j, k \in I_n$, $\varpi_n(Q_{i,n} \times Q_{j,n} \times Q_{k,n}) = \varpi(Q_{i,n} \times Q_{j,n} \times Q_{k,n})$ and:

$$W_2^2(\varpi_n, \varpi) \leq \sum_{i,j,k\in I_n} W_2^2(\varpi_n\lfloor Q_{i,n} \times Q_{j,n} \times Q_{k,n}\,,\,\varpi\lfloor Q_{i,n} \times Q_{j,n} \times Q_{k,n})$$

$$\leq \sum_{i,j,k\in I_n} [diam(Q_{i,n})^2 + diam(Q_{j,n})^2 + diam(Q_{k,n})^2]\varpi(Q_{i,n} \times Q_{j,n} \times Q_{k,n})$$

$$\leq \sum_{i,j,k\in I_n} \frac{3d}{n^2}\varpi(Q_{i,n} \times Q_{j,n} \times Q_{k,n}) = \frac{3d}{n^2}.$$

So that $\lim_{n\to+\infty} W_2(\varpi_n\,,\,\varpi) = 0$.
Step 2: Assume now $\varpi \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ is absolutely continuous with respect to the Lebesgue measure and denote by μ its first marginal. We now prove that it exists $(p_n)_n$, $(q_n)_n$ in $L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$ such that:

$$\lim_{n\to+\infty} W_2((Id \times p_n \times q_n)\sharp\mu, \varpi) = 0.$$

For all $n \in \mathbb{N}$, consider a partition $(A_{i,n})_{i\in I_n}$ of $\mathbb{R}^d$ made of sets of diameter less than $1/n$. We set:

$$\varpi_{i,n} = \varpi\lfloor A_{i,n} \times R^{2d}. \tag{46}$$

The first marginal $\mu\lfloor A_{i,n}$ of $\varpi_{i,n}$ is absolutely continuous with respect to the Lebesgue measure so that it exists $T_{i,n} : A_{i,n} \to \mathbb{R}^d \times \mathbb{R}^d$ such that $T_{i,n}\sharp\mu\lfloor A_{i,n} = \pi_{2,3}\sharp\varpi_{i,n}$. We also set $T_n := \sum_{i\in I_n} \mathbf{1}_{A_{i,n}} T_{i,n}$ and $\varpi_n = (Id \times T_n)\sharp\mu$. It holds for all $i, j, k \in I_n$:

$$\varpi_n(A_{i,n} \times A_{j,n} \times A_{k,n}) = [\mu\lfloor_{A_{i,n}}\otimes\delta_{T_{i,n}}](A_{i,n} \times A_{j,n} \times A_{k,n})$$

$$= [\mu\lfloor_{A_{i,n}}\otimes\delta_{T_{i,n}}](\mathbb{R}^d \times A_{j,n} \times A_{k,n}) = \pi_{2,3}\sharp\varpi_{i,n}(A_{j,n} \times A_{k,n})$$

$$= \varpi(A_{i,n} \times A_{j,n} \times A_{k,n}) \text{ (recalling (46)).}$$

Then, again:

$$W_2^2(\varpi_n, \varpi) \leq \sum_{i,j,k \in I_n} W_2^2(\varpi \lfloor A_{i,n} \times A_{j,n} \times A_{k,n} \,,\, \varpi_n \lfloor A_{i,n} \times A_{j,n} \times A_{k,n})$$

$$\leq \sum_{i,j,k \in I_n} \varpi(A_{i,n} \times A_{j,n} \times A_{k,n})[diam(A_{i,n})^2 + diam(A_{j,n})^2 + diam(A_{k,n})^2])$$

$$\leq \sum_{i,j,k \in I_n} \varpi(A_{i,n} \times A_{j,n} \times A_{k,n}) \frac{3d}{n^2} \leq \frac{3d}{n^2}.$$

To conclude the step, it is enough to remark that it exists $p_n, q_n \in L^2_{\mu_n}(\mathbb{R}^d, \mathbb{R}^d)$ such that:

$$T_n(x) = (p_n(x), q_n(x)) \, \mu\text{-a.e.}x, \quad \varpi_n = (Id \times p_n \times q_n)\sharp\mu.$$

Conclusion: Putting together Step 1 and 2 gives the lemma.
QED

Acknowledgements: The author wants to thank Pierre Cardaliaguet and Guillaume Carlier for interesting discussions, questions and suggestions.
This research was funded, in part, by l'Agence Nationale de la Recherche (ANR), project ANR-22-CE40-0010.

References

[1] L. Ambrosio, N. Gigli, G. Savaré, Gradient flows in metric spaces and in the space of probability measures, 2nd ed., Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008.
[2] L. Ambrosio, J. Feng, *On a class of first order Hamilton-Jacobi equations in metric spaces*, J. Differ. Equations 256, No. 7, 2194-2245 (2014)
[3] Z. Badreddine, H. Frankowska, *Hamilton-Jacobi inequalities on a metric space*, J. Differ. Equations 271, 1058-1091 (2021)
[4] Z. Badreddine, H. Frankowska, *Solutions to Hamilton-Jacobi equation on a Wasserstein space*, Calc. Var. Partial Differ. Equ. 61, No. 1, Paper No. 9, 41 p. (2022)
[5] P. Cannarsa, C. Sinestrari, Semiconcave functions, Hamilton-Jacobi equations, and optimal control. Progress in Nonlinear Differential Equations and their Applications 58. Boston, MA: Birkhäuser. xii, 304 p. (2004).
[6] P. Cardaliaguet, *Notes on Mean Field Games*, (2013).
[7] P. Cardaliaguet, S. Daudin, J. Jackson, P.E. Souganidis *An algebraic convergence rate for the optimal control of McKean-Vlasov dynamics*, prepint (2023), arXiv:2203.14554
[8] P. Cardaliaguet and M. Quincampoix, Deterministic differential games under probability knowledge of initial condition, Int. Game Theory Rev. (2008) 10(1):1-16.
[9] G. Cavagnari, S. Lisini, C. Orrieri and G. Savaré, *Lagragian, Eulerian and Kantorovitch formulations of multi-agent optimal control problems: equivalence and Gamma-convergence*BLABLA
[10] G. Cavagnari, A. Marigonda, M. Quincampoix *Compatibility of state constraints and dynamics for multiagent control systems*, J. Evol. Equ. 21, No. 4, 4491-4537 (2021).
[11] G. Conforti, R. C. Kraaij, D. Tonon, *Hamilton-Jacobi equations for controlled gradient flows: the comparison principle.* J. Funct. Anal. 284, No. 9, Article ID 109853, 53 p. (2023)
[12] M.G. Crandall and P.-L. Lions *Hamilton-Jacobi Equations in Infinite Dimensions I. Uniqueness of Viscosity Solutions*, J. Funct. Anal. 62, (1985) 379-396.
[13] M.G. Crandall and P.-L. Lions *Hamilton-Jacobi Equations in Infinite Dimensions II. Existence of Viscosity Solutions*, J. Funct. Anal.n 65, (1986) 368-405.
[14] M.G. Crandall and P.-L. Lions *Hamilton-Jacobi Equations in Infinite Dimensions III.*, J. Funct. Anal. 68, No. 2, 214-247 (1986).

[15] W. Gangbo, S. Mayorga, A. Święch *Finite dimensional approximations of Hamilton-Jacobi-Bellman equations for stochastic particle systems with common noise*, SIAM J. Control Optim. 61, No. 2, 820-851 (2023).

[16] W. Gangbo, T. Nguyen and A. Tudorascu, *Hamilton-Jacobi equations in the Wasserstein space*, Methods Appl. Anal. 15, No. 2, 155-184 (2008)

[17] W. Gangbo, A. Tudorascu, *On differentiability in the Wasserstein space and well-posedness for Hamilton-Jacobi equations*, Journal de Mathématiques Pures et Appliquées, 125, p. 119–174, (2019).

[18] N. Gigli, *On the geometry of the space of probability measures in $\mathbb{R}^n$ endowed with the quadratic optimal transport distance*, Ph. D. Thesis, (2004), available at `https://cvgmt.sns.it/paper/491/`

[19] Lions, P.-L. *Cours au Collège de France.* `www.college-de-france.fr`.

[20] A. Marigonda, M. Quincampoix, *Mayer control problem with probabilistic uncertainty on initial positions*, J. Differential Equations, 264, 2018, 5, p. 3212–3252.

[21] O. Jerhaoui, *Théorie de viscosité des équations de Hamilton Jacobi du premier ordre sur certains espaces métriques.*, PHD thesis 2022.

[22] O. Jerhaoui, H. Zidani, *Viscosity solutions of Hamilton-Jacobi equations in proper CAT(0) spaces*, preprint 2022, hal-03899199

[23] C. Jimenez, A. Marigonda, M. Quincampoix *Optimal Control of Multiagent Systems in the Wasserstein Space*, (2020) Calculus of Variations and Partial Differential Equations.

[24] C. Jimenez, A. Marigonda, M. Quincampoix *Dynamical systems and Hamilton-Jacobi-Bellman equations on the Wasserstein space and their L^2 representations*, to be published in SIAM Journal of Mathematical Analysis (2023).

[25] F. Santambrogio, "Optimal transport for applied mathematicians," Calculus of variations, PDEs, and modeling. Progress in Nonlinear Differential Equations and their Applications, 87. Birkhäuser/Springer, Cham, 2015.

Chloé Jimenez: Univ Brest, CNRS UMR 6205,
Laboratoire de Mathématiques de Bretagne Atlantique,
6, avenue Victor Le Gorgeu, 29200 Brest, France.

E-mail address: `chloe.jimenez@univ-brest.fr`