# Differential operators on a reductive Lie algebra

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**DIFFERENTIAL OPERATORS ON A REDUCTIVE LIE ALGEBRA**

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## 1. Differential operators

Let $\mathcal{X}$ be an affine complex algebraic variety. Denote by $\mathcal{O}(\mathcal{X})$ the algebra of regular functions, and by $\mathcal{D}(\mathcal{X})$ the algebra of differential operators (on $\mathcal{X}$). Recall that $\mathcal{D}(\mathcal{X})$ is a filtered $\mathbb{C}$-algebra (by the order of differential operators): one defines, inductively,

$$\mathcal{D}_0(\mathcal{X}) = \mathcal{O}(\mathcal{X}), \quad \mathcal{D}_m(\mathcal{X}) = \{P \in \operatorname{End}_{\mathbb{C}}(\mathcal{O}(\mathcal{X})) \,:\, [P, \mathcal{O}(\mathcal{X})] \subset \mathcal{D}_{m-1}(\mathcal{X})\}.$$

Then $\mathcal{D}(\mathcal{X}) = \bigcup_m \mathcal{D}_m(\mathcal{X})$ and we denote by

$$\operatorname{gr}\mathcal{D}(\mathcal{X}) = \bigoplus_m \mathcal{D}_m(\mathcal{X})/\mathcal{D}_{m-1}(\mathcal{X})$$

the associated graded algebra. The principal symbol of an element $P \in \mathcal{D}(\mathcal{X})$ is denoted by $\operatorname{gr}(P)$.

Assume that $\mathcal{X}$ is smooth. Then, $\mathcal{D}(\mathcal{X})$ is generated by $\mathcal{O}(\mathcal{X})$ and $\operatorname{Der}\mathcal{O}(\mathcal{X})$ (the module of $\mathbb{C}$-linear derivations on $\mathcal{O}(\mathcal{X})$). Furthermore, $\operatorname{gr}\mathcal{D}(\mathcal{X}) = \mathrm{S}_{\mathcal{O}(\mathcal{X})}(\operatorname{Der}\mathcal{O}(\mathcal{X}))$. Here $\mathrm{S}_{\mathcal{O}(\mathcal{X})}(\operatorname{Der}\mathcal{O}(\mathcal{X}))$ is the symmetric algebra of the module $\operatorname{Der}\mathcal{O}(\mathcal{X})$, that we identify with $\mathcal{O}(T^*\mathcal{X})$, the ring of regular functions on the cotangent bundle of $\mathcal{X}$.

For any affine algebraic subvariety $\mathcal{X} \subset \mathbb{C}^n$, let $\mathcal{A}(\mathcal{X})$ the radical ideal defining $\mathcal{X}$. Conversely if $E \subset \mathcal{O}(\mathbb{C}^n)$ is a subset, let $\mathcal{V}(E) \subseteq \mathbb{C}^n$ be the variety of zeroes of $E$. In particular, for any subset $E$ of $\mathcal{D}(\mathfrak{g})$, $\mathcal{V}(\operatorname{gr} E)$ is an affine subvariety of $T^*\mathcal{X}$.

Let $\mathcal{Y}$ be a smooth affine algebraic variety, and $\varphi : \mathcal{X} \to \mathcal{Y}$ be a morphism. Recall that $\varphi$ is étale at $x \in \mathcal{X}$, if $\varphi$ yields an isomorphism $d_x\varphi : T_x\mathcal{X} \xrightarrow{\sim} T_{\varphi(x)}\mathcal{Y}$. The following result is classical.

**Proposition 1.1.** *Assume that $\varphi : \mathcal{X} \to \mathcal{Y}$ is étale. Then, for all $m \in \mathbb{N}$, one has natural identifications*

$$\mathcal{O}(\mathcal{X}) \otimes_{\mathcal{O}(\mathcal{Y})} \mathrm{S}^m(\operatorname{Der}\mathcal{O}(\mathcal{Y})) \xrightarrow{\sim} \mathrm{S}^m(\operatorname{Der}\mathcal{O}(\mathcal{X})), \quad \mathcal{O}(\mathcal{X}) \otimes_{\mathcal{O}(\mathcal{Y})} \mathcal{D}_m(\mathcal{Y}) \xrightarrow{\sim} \mathcal{D}_m(\mathcal{X}).$$

**Remark .** Assume that $\mathcal{X} = V$ is an $n$-dimensional complex vector space. Then $\mathcal{D}(V)$ is a Weyl algebra on $2n$ generators. We have $\mathcal{O}(V) = \mathrm{S}(V^*)$ and we will identify $\mathrm{S}(V)$ with the algebra of constant coefficient differential operators. If we fix a coordinate basis $\{x_i, \partial_i; 1 \le i \le n\}$, we then have

$$\mathrm{S}(V) = \mathbb{C}[\partial_1, \dots, \partial_n] = \mathbb{C}[\partial(v)\,;\, v \in V],$$

where $\partial(v)$ is the derivation given by $\partial(v)(f)(x) = \frac{d}{dt}_{|t=0} f(x + tv)$. Note that $\mathcal{D}(V) = \mathrm{S}(V^*) \otimes_{\mathbb{C}} \mathrm{S}(V)$ as an $\mathcal{O}(V)$-module.

Let $G$ be a complex reductive algebraic group with Lie algebra $\mathfrak{g}$. Assume that $\mathcal{X}$ is a $G$-variety[1]. We denote by $\mathcal{X}/G$ the affine variety whose ring of regular functions is the ring of invariants $\mathcal{O}(\mathcal{X})^G$. Recall that $\mathcal{X}/G$ can be identified with the variety of closed orbits in $\mathcal{X}$ and that we have a natural surjective morphism $p : \mathcal{X} \twoheadrightarrow \mathcal{X}/G$. For $x \in \mathcal{X}$ we denote by $G^x$ its stabilizer in $G$ and we set $\mathfrak{g}^x = \operatorname{Lie}(G^x)$. Recall (Matsushima's theorem) that if $G.x$ is closed, then $G^x$ is reductive.

[1] $G$ acts rationally on $\mathcal{X}$.

The action of $G$ induces a morphism of Lie algebras $\tau_{\mathcal{X}} : \mathfrak{g} \to \operatorname{Der} \mathcal{O}(\mathcal{X})$, given by $\tau_{\mathcal{X}}(\xi)(f) = \frac{d}{dt}_{|t=0}(\exp(t\xi).f)$.

**Example .** Consider the adjoint action of $G$ on its Lie algebra $\mathfrak{g}$. Set (for simplicity) $\tau_{\mathfrak{g}} = \tau$ in this case. Since $\mathfrak{g}$ is reductive, we can fix a nondegenerate invariant bilinear symmetric form $\kappa$ on $\mathfrak{g}$. Then $\mathfrak{g}$ and $\mathfrak{g}^*$ can be identified through $\kappa$ by $x \mapsto \kappa_x = \kappa(\ ,x)$. It follows easily that $\tau(\xi)(\kappa_x) = \kappa_{[\xi,x]}$, for all $\xi \in \mathfrak{g}$. The elements of $\mathcal{O}(\mathfrak{g})\tau(\mathfrak{g})$ will be called "adjoint vector fields" on $\mathfrak{g}$. An easy computation also shows that the principal symbol of $\tau(\xi)$, denoted by $\sigma(\xi)$, is the function on $T^*\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^* \equiv \mathfrak{g} \times \mathfrak{g}$, given by $\sigma(\xi)(a,b) = \kappa([b,a],\xi)$ for all $a, b \in \mathfrak{g}$.

In this situation an orbit $G.x$ is closed if and only if $G^x$ is reductive, if and only if $x$ is semisimple.

Return now to the general situation. The group $G$ acts on $\mathcal{D}(\mathcal{X})$ by $(g.P)(f) = g.(P(g^{-1}.f))$ for all $g \in G$, $P \in \mathcal{D}(\mathcal{X})$ and $f \in \mathcal{O}(\mathcal{X})$. It is not difficult to see that this $G$-action is rational and that $G.\mathcal{D}_m(\mathcal{X}) \subseteq \mathcal{D}_m(\mathcal{X})$ for all $m$. Denote by $\mathcal{D}(\mathcal{X})^G$ the ring of invariant differential operators, that we filter by the $\mathcal{D}_m(\mathcal{X})^G$. Since $G$ is reductive, it follows that

$$\operatorname{gr}[\mathcal{D}(\mathcal{X})^G] = [\operatorname{gr} \mathcal{D}(\mathcal{X})]^G = \mathcal{O}(T^*\mathcal{X})^G = \mathcal{O}(T^*\mathcal{X}/G).$$

By restriction we obtain a morphism

$$\psi : \mathcal{D}(\mathcal{X})^G \to \mathcal{D}(\mathcal{X}/G), \quad \psi(P)(f) = P(f) \text{ for all } f \in \mathcal{O}(\mathcal{X}/G).$$

It is clear that $\psi(\mathcal{D}_m(\mathcal{X})^G) \subseteq \mathcal{D}_m(\mathcal{X}/G)$. Note that $\mathcal{O}(\mathcal{X})^G \subseteq \{f \in \mathcal{O}(\mathcal{X}) : \tau_{\mathcal{X}}(\mathfrak{g})(f) = 0\}$, with equality when $G$ is connected. Moreover the differential of the action of $G$ on $\mathcal{D}(\mathcal{X})$ is given by: $\xi.P = [\tau_{\mathcal{X}}(\xi), P]$ for all $\xi \in \mathfrak{g}$, $P \in \mathcal{D}(\mathcal{X})$. Set

$$\mathcal{J}(\mathcal{X}) = \{D \in \mathcal{D}(\mathcal{X}) : D(\mathcal{O}(\mathcal{X})^G) = 0\}, \quad \mathcal{I}(\mathcal{X}) = \mathcal{J}(\mathcal{X}) \cap \mathcal{D}(\mathcal{X})^G.$$

Clearly $\operatorname{Ker} \psi = \mathcal{I}(\mathcal{X})$ and $\mathcal{J}(\mathcal{X}) \supseteq \mathcal{D}(\mathcal{X})\tau_{\mathcal{X}}(\mathfrak{g})$.

Assume now that $G = W$ is a finite sugroup of $\mathrm{GL}(V)$, where $V$ is a complex vector space of dimension $\ell$. Then, the morphism $p : V \twoheadrightarrow V/W$ is finite and every orbit is closed. Define a $W$-stable open subset of $V$ by

$$V' := \{v \in V \mid p \text{ étale at } v\}.$$

Hence, $V' = \{v \in V \mid \operatorname{rk}_v p = \ell \text{ and } p(v) \text{ is a smooth point}\}$.

Note that if the action of $W$ is not faithful, we may decompose $V = V_W \oplus V^W$ so that $V/W = (V_W/W) \oplus V^W$ and $(V_W)^W = 0$. Therefore the analysis of the situation always reduces to the case of a faithful action of $W$ on $V$. In this case, it is a classical result that $V' = \{v \in V \mid W^v = \{1\}\}$.

Recall that $\mathcal{D}(V)$ is a simple ring, and, since $W$ is finite, $\mathcal{D}(V)^W$ is also simple [10]. Hence $\psi : \mathcal{D}(V)^W \hookrightarrow \mathcal{D}(V/W)$ is an embedding. The following result is well known[2].

**Theorem 1.2.** *The following are equivalent*

(1) *$\psi$ is a (filtered) isomorphism;*
(2) $\operatorname{codim}(V \setminus V') \geq 2$*;*
(3) *$W$ does not contain any pseudoreflection ($\neq 1$).*

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[2]We shall not use this result.

Recall that $V/W$ is smooth if and only if $W$ is generated by pseudoreflections. Therefore, if $W \neq \{1\}$ and $V/W$ is smooth, $\psi : \mathcal{D}(V)^W \hookrightarrow \mathcal{D}(V/W)$ is not surjective. Actually, if $W$ acts faithfully on $V$ and $\mathrm{S}(V^*)^W = \mathbb{C}[p_1, \dots, p_\ell]$ is a polynomial ring, it is not difficult to see that there does not exist any $d \in \mathcal{D}(V)^W$ such that $\psi(d) = \frac{\partial}{\partial p_i}$.

**Example .** The following case is obvious, but will prove useful in the sequel. Assume that $\dim V = 1$ and set

$$\mathrm{S}(V^*) = \mathbb{C}[z], \qquad \mathrm{S}(V) = \mathbb{C}[\partial_z].$$

Let $W = \{\pm 1\}$ act on $V$ by multiplication. Then

$$\mathrm{S}(V^*)^W = \mathbb{C}[z^2], \quad \mathrm{S}(V) = \mathbb{C}[\partial_z^2], \quad \mathcal{D}(V)^W = \mathbb{C}[z^2, z\partial_z, \partial_z^2]^3.$$

Set $t = z^2$. Then $\mathcal{D}(V/W) = \mathbb{C}[t, \partial_t]$ and the morphism $\psi : \mathcal{D}(V)^W \hookrightarrow \mathcal{D}(V/W)$ is given by

$$\psi(z^2) = t, \quad \psi(z\partial_z) = 2t\partial_t, \quad \psi(\partial_z^2) = 4t\partial_t^2 + 2\partial_t.$$

Note that $\partial_t \notin \operatorname{Im}\psi$. We have $V' = V \setminus \{0\}$, and if we localize at the invariant function $t = z^2$, we obtain

$$\psi : \mathcal{D}(V)^W_{z^2} = \mathbb{C}[z^{\pm 2}, z^{-1}\partial_z] \xrightarrow{\sim} \mathcal{D}(V/W)_t = \mathbb{C}[t^{\pm 1}, \partial_t],$$

since $\psi(\frac{1}{2}z^{-1}\partial_z) = \partial_t$. Thus $\mathcal{D}(V')^W \xrightarrow{\sim} \mathcal{D}(V'/W)$.

## 2. The map $\delta$: definition

Let $G$ be a *connected* reductive algebraic group with maximal torus $H$. Set $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{h} = \mathrm{Lie}(H)$ and denote by $W = W(\mathfrak{g}, \mathfrak{h})$ the associated Weyl group. Let $R$ be the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$. Fix a basis $B$ of $R$ and let $R^+$ be the set of positive roots. We set $\mathfrak{n}^\pm = \oplus_{\{\pm\alpha \in R^+\}} \mathfrak{g}_\alpha$, $\mathfrak{g}_{\pm\alpha} = \mathbb{C}X_{\pm\alpha}$. If $\mathfrak{z}$ is the centre of $\mathfrak{g}$ and $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$, we have

$$\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}, \quad \mathfrak{h} = \mathfrak{t} \oplus \mathfrak{z}, \quad \mathfrak{s} = \mathfrak{t} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

where $\mathfrak{t}$ is a Cartan subalgebra of the semisimple Lie algebra $\mathfrak{s}$. We set $n = \dim \mathfrak{g}$, $\ell = \dim \mathfrak{h}$ and $k = \dim \mathfrak{t}$. As in §1, we denote by $\kappa$ an invariant symmetric form on $\mathfrak{g}$. Recall that the discriminant of $\mathfrak{g}$ is the invariant function $d_\ell$ defined by

$$\det(t\mathrm{Id} - \operatorname{ad} x) = t^n + \cdots + (-1)^\ell d_\ell(x) t^\ell.$$

The set of generic[4] elements is $\mathfrak{g}' = \{x \in \mathfrak{g} \mid d_\ell(x) \neq 0\}$. Then $\mathfrak{g}'$ is the set of points where the morphism $p : \mathfrak{g} \twoheadrightarrow \mathfrak{g}/G$ is smooth.

Recall the fundamental result of Chevalley:

**Theorem 2.1.** *There is a natural isomorphism $\mathfrak{h}/W \xrightarrow{\sim} \mathfrak{g}/G$: the restriction of functions from $\mathfrak{g}$ to $\mathfrak{h}$ yields an isomorphism of algebras,*

$$\phi : \mathrm{S}(\mathfrak{g}^*)^G \xrightarrow{\sim} \mathrm{S}(\mathfrak{h}^*)^W, \quad \phi(f) = f_{|\mathfrak{h}}.$$

*Similarly, there exists an isomorphism $\phi : \mathrm{S}(\mathfrak{g})^G \to \mathrm{S}(\mathfrak{h})^W$, induced by the projection of $\mathfrak{g}$ onto $\mathfrak{h}$ given by the decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\mathfrak{n}^+ \oplus \mathfrak{n}^-)$.*

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[3]Observe that $[z^2, \partial_z^2] = 4z\partial_z + 2$, and thus $\mathcal{D}(V)^W = \mathbb{C}[z^2, \partial_z^2]$.

[4]An element $x$ is called *generic* if it is semisimple and $\dim \mathfrak{g}^x = \ell$.

For sake of simplicity, all the isomorphisms related to the previous Chevalley isomorphisms will be denoted by the same symbol, $\phi$.

Note that we may write $\mathrm{S}(\mathfrak{g}^*)^G = \mathbb{C}[u_1, \dots, u_k, u_{k+1}, \dots, u_\ell]$, where $u_i \in \mathrm{S}(\mathfrak{s}^*)^{\mathfrak{s}}$ for $i = 1, \dots, k$ and $u_j \in \mathfrak{z}^*$ for $i = k+1, \dots, \ell$ (hence $\mathrm{S}(\mathfrak{z}^*) = \mathbb{C}[u_{k+1}, \dots, u_\ell]$). We set $p_j = u_{j|\mathfrak{h}}$ and we denote by $p : \mathfrak{h} \twoheadrightarrow \mathfrak{h}/W$ the associated morphism. Then $\mathrm{S}(\mathfrak{h}^*)^W = \mathrm{S}(\mathfrak{t}^*)^W \otimes \mathrm{S}(\mathfrak{z}^*) = \mathbb{C}[p_1, \dots, p_\ell]$. Define an element of $\mathrm{S}(\mathfrak{h}^*)$ by

$$\pi = \prod_{\alpha \in R^+} \alpha.$$

The following are well known, see [3, Proposition 3.13]:

- Let $\epsilon(w)$ be the signature of $w \in W$, then,
$$\mathrm{S}(\mathfrak{h}^*)^W \pi = \{f \in \mathrm{S}(\mathfrak{h}^*) \mid \forall w \in W,\, w.f = \epsilon(w) f\};$$
- $\phi(d_\ell) = (\pm)\pi^2 \in \mathrm{S}(\mathfrak{h}^*)^W$;
- up to a nonzero constant, $\pi(x) = \det \mathrm{Jac}(p)(x)$ and $p$ is étale at $h \in \mathfrak{h}$ if, and only if, $h \in \mathfrak{h}' = \{x \in \mathfrak{h} : \pi(x) \neq 0\}$.

Recall [16, Corollary 3.11] that if $x \in \mathfrak{g}$ is semisimple, then $G^x$ is a connected reductive subgroup of $G$. One can conjugate $x$ and assume that $x \in \mathfrak{h}$. If we set $\Gamma = \{\alpha \in B : \alpha(x) = 0\}$, then: $\mathfrak{g}^x = \mathfrak{h} \oplus \left(\sum_{\{\beta \in \mathbb{Z}\Gamma \cap R\}} \mathfrak{g}_\beta\right)$, $[x, \mathfrak{g}] = \oplus_{\{\beta \notin \mathbb{Z}\Gamma\}} \mathfrak{g}_\beta$.

The Chevalley isomorphism $\phi$ induces an isomorphism

$$\phi : \mathcal{D}(\mathfrak{g}/G) \xrightarrow{\sim} \mathcal{D}(\mathfrak{h}/W), \quad \phi(P)(f) = \phi(P(\phi^{-1}(f)))$$

for all $P \in \mathcal{D}(\mathfrak{g}/G)$, $f \in \mathcal{O}(\mathfrak{h}/W) = \mathrm{S}(\mathfrak{h}^*)^W$. By composing with the natural morphism $\psi : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{g}/G)$, we obtain the morphism

$$r = \psi \circ \phi : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h}/W), \quad r(P)(f) = \phi(P(\phi^{-1}(f))).$$

The element $r(P)$ is called the *radial component* of $P$. It is clear that

$$\mathrm{Ker}\, r = \mathcal{I} = \{P \in \mathcal{D}(\mathfrak{g})^G \,:\, P\big(\mathrm{S}(\mathfrak{g}^*)^G\big) = 0\}.$$

Since the morphism $p : \mathfrak{h}' \twoheadrightarrow \mathfrak{h}'/W$ is étale, it follows from Proposition 1.1 that we can identify $\mathcal{D}(\mathfrak{h}')^W$ with $\mathcal{D}(\mathfrak{h}'/W)$ (observe that $\mathcal{D}(\mathfrak{h}') = \mathcal{O}(\mathfrak{h}') \otimes_{\mathcal{O}(\mathfrak{h}'/W)} \mathcal{D}(\mathfrak{h}'/W)$ and take the $W$-invariants). Therefore

$$\mathrm{Im}\, r \subset \mathcal{D}(\mathfrak{h}/W) \subset \mathcal{D}(\mathfrak{h}'/W) \equiv \mathcal{D}(\mathfrak{h}')^W \subset \mathcal{D}(\mathfrak{h}').$$

Inside $\mathcal{D}(\mathfrak{h}')$ we can consider the inner automorphism

$$\iota : D \mapsto \pi \circ D \circ \pi^{-1}, \text{ i.e. } \iota(D)(f) = \pi D(\pi^{-1} f) \text{ for all } f \in \mathcal{O}(\mathfrak{h}').$$

From $w.\iota(D) = \pi \circ w.D \circ \pi^{-1}$, we get that $\iota(\mathcal{D}(\mathfrak{h}')^W) = \mathcal{D}(\mathfrak{h}')^W$.

**Definition 2.2.** *The Harish-Chandra map $\delta : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h}')^W$ is defined to be $\delta = \iota \circ r$, i.e.*

$$\forall\, D \in \mathcal{D}(\mathfrak{g})^G, \forall\, f \in \mathcal{O}(\mathfrak{h})^W, \quad \delta(D)(f) = \pi r(D)(\pi^{-1} f).$$

In the next two sections we will sketch a proof of the following result of Harish-Chandra.

**Theorem 2.3.** (1) $\mathrm{Im}\, \delta \subseteq \mathcal{D}(\mathfrak{h})^W$.

(2) $\delta$ coincides with the Chevalley isomorphisms on $\mathrm{S}(\mathfrak{g}^*)^G$ and $\mathrm{S}(\mathfrak{g})^G$.

We end this section by the following slight generalization of the definition of $\delta$. Let $U \subseteq \mathfrak{g}$ be a $G$-stable open subset. Set $\tilde{\mathfrak{h}} = U \cap \mathfrak{h}$ and $\tilde{\mathfrak{h}}' = U \cap \mathfrak{h}'$. Then the Chevalley isomorphism yields $U/G \xrightarrow{\sim} \tilde{\mathfrak{h}}/W$, and we can define in a similar way the "radial component" of elements of $\mathcal{D}(U)^G$. We then have a morphism

$$r : \mathcal{D}(U)^G \to \mathcal{D}(\tilde{\mathfrak{h}}/W) \hookrightarrow \mathcal{D}(\tilde{\mathfrak{h}}'/W) \equiv \mathcal{D}(\tilde{\mathfrak{h}}')^W.$$

After composition with $\iota$ (i.e. conjugation by the restriction of $\pi$ on $\tilde{\mathfrak{h}}'$), we obtain a morphism

$$\delta = \iota \circ r : \mathcal{D}(U)^G \to \mathcal{D}(\tilde{\mathfrak{h}}')^W$$

which extends the previously defined $\delta$.

## 3. The map $\delta$ in the $\mathfrak{sl}(2)$-case

In this section we assume that $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) = \mathbb{C}e + \mathbb{C}f + \mathbb{C}h$, where as usual $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then $\mathfrak{h} = \mathbb{C}h$, $R = \{\pm\alpha\}$ where $\alpha(h) = 2$. We choose $\kappa(a,b) = \operatorname{tr}(ab)$, hence $\kappa(e,f) = 1$, $\kappa(h,h) = 2$. Let $\{x,y,z\}$ be the dual basis of $\{e,f,h\}$, thus $x = \kappa_f$, $y = \kappa_e$ and $z = \frac{1}{2}\kappa_h$. Furthermore $\partial(e) = \partial_y$, $\partial(f) = \partial_x$ and $\partial(h) = \partial_z$. Then

$$\mathrm{S}(\mathfrak{g}^*)^G = \mathbb{C}[z^2 + xy], \qquad \mathrm{S}(\mathfrak{g})^G = \mathbb{C}[\partial_z^2 + 4\partial_x\partial_y].$$

We set

$$\zeta = z^2 + xy, \;\; \omega = \partial_z^2 + 4\partial_x\partial_y, \;\; \varepsilon_{\mathfrak{g}} = x\partial_x + y\partial_y + z\partial_z, \;\; \varepsilon_{\mathfrak{h}} = z\partial_z.$$

Observe that $E_{\mathfrak{g}} := \varepsilon_{\mathfrak{g}} + 3/2 = [-\frac{1}{4}\zeta, \omega]$.

Recall that $W = \{1, s\}$, where $s : h \mapsto -h$. Therefore we are in the situation of the example $W = \{\pm 1\}$ given in §1. Hence, if $t = z^2$,

$$\psi : \mathcal{D}(\mathfrak{h})^W = \mathbb{C}[z^2, \partial_z^2] \hookrightarrow \mathcal{D}(\mathfrak{h}/W) = \mathbb{C}[t, \partial_t]$$

is given by $\psi(z^2) = t$, $\psi(\partial_z^2) = 4t\partial_t^2 + 2\partial_t$. The Chevalley isomorphisms are determined by $\phi(\zeta) = z^2 = t$, $\phi(\omega) = \partial_z^2$. Recall that $r : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h}/W)$.

**Lemma 3.1.** *We have:*

(1) $\mathcal{D}(\mathfrak{g})^G = \mathbb{C}[\zeta, \omega] \cong U(\mathfrak{sl}(2))$;

(2) $r(\zeta) = t$, $r(\omega) = 4t\partial_t^2 + 6\partial_t$.

*Proof.* (1) By an usual argument of associated graded ring, we will obtain generators of $\mathcal{D}(\mathfrak{g})^G$ by computing

$$[\operatorname{gr} \mathcal{D}(\mathfrak{g})]^G = \mathrm{S}(\mathfrak{g}^* \times \mathfrak{g})^G \equiv \mathrm{S}(\mathfrak{g}^* \times \mathfrak{g}^*)^G.$$

Here, $G$ acts diagonally on $\mathfrak{g}^* \times \mathfrak{g}^*$ by $g.(a,b) = (g.a, g.b)$ and we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through $\kappa$. Under this identification, $\partial_z \leftrightarrow 2z$, $\partial_x \leftrightarrow y$ and $\partial_y \leftrightarrow x$. Therefore $\operatorname{gr} \mathcal{D}(\mathfrak{g}) \equiv \mathrm{S}(\mathfrak{g}^* \times \mathfrak{g}^*) = \mathbb{C}[U, V]$, where $U$ and $V$ are the generic matrices $U = \left[\begin{smallmatrix} z & x \\ y & -z \end{smallmatrix}\right]$, $V = \left[\begin{smallmatrix} \frac{1}{2}\partial_z & \partial_y \\ \partial_x & -\frac{1}{2}\partial_z \end{smallmatrix}\right]$. Then, classical invariant theory gives that $\mathrm{S}(\mathfrak{g}^* \times \mathfrak{g}^*)^G$ is generated by

$$\operatorname{tr}(U^2) = \zeta, \quad \operatorname{tr}(UV) = \varepsilon_{\mathfrak{g}}, \quad \operatorname{tr}(V^2) = \omega/4.$$

Thus $\mathcal{D}(\mathfrak{g})^G = \mathbb{C}[\zeta, \omega, E_{\mathfrak{g}}] = \mathbb{C}[\zeta, \omega]$. Now observe that

$$[E_{\mathfrak{g}}, -\zeta/4] = 2\zeta, \quad [E_{\mathfrak{g}}, \omega] = -2\omega, \quad [-\zeta/4, \omega] = E_{\mathfrak{g}}.$$

Therefore, there exists a surjective morphism $\nu : U(\mathfrak{sl}(2)) \twoheadrightarrow \mathcal{D}(\mathfrak{g})^G$, such that $\nu(e) = -\frac{1}{4}\zeta$, $\nu(f) = \omega$ and $\nu(h) = E_{\mathfrak{g}}$. To prove that $\nu$ is injective[5], one can either show that

[5]We leave the details to the reader.

GKdim $\mathcal{D}(\mathfrak{g})^G = \text{GKdim}\,\text{gr}\,\mathcal{D}(\mathfrak{g})^G = \text{GKdim}\,U(\mathfrak{sl}(2)) = 3$, see Corollary 5.8 (note that the maximal dimension of a $G$-orbit in $\mathfrak{g}\times\mathfrak{g}$ is 3), or prove that, if $\Omega$ is the Casimir element of $U(\mathfrak{sl}(2))$, then $\nu(\Omega - c) \neq 0$ for all $c \in \mathbb{C}$.

(2) The equality $r(\zeta) = t$ is clear. It is easily seen that

$$r(\omega)(1) = 0,\;\; r(\omega)(t) = 6,\;\; r(\omega)(t^2) = 20t.$$

Hence, $r(\omega) = 4t\partial_t^2 + 6\partial_t$ as desired. □

**Remark .** Observe that $r(\omega) = \partial_z^2 + 4\partial_t \notin \mathcal{D}(\mathfrak{h})^W$, since $\partial_t \notin \mathcal{D}(\mathfrak{h})^W$ (see §1). Thus $\text{Im}\, r \not\subset \mathcal{D}(\mathfrak{h})^W$.

**Lemma 3.2.** *$\delta(\omega) = \partial_z^2$ and $\delta(\zeta) = z^2$.*

*Proof.* In the notation of §2, we have $\pi = \alpha = 2z$ and $\mathfrak{h}' = \mathfrak{h} \setminus \{0\}$. Recall that we can identify $\mathcal{D}(\mathfrak{h}'/W) = \mathbb{C}[t^{\pm1}, \partial_t]$ with $\mathcal{D}(\mathfrak{h}')^W = \mathbb{C}[z^{\pm 2}, \frac{1}{2}z^{-1}\partial_z]$. Now, since $z\partial_z z^{-1} = \partial_z - z^{-1}$ and $r(\omega) = 4t\partial_t^2 + 6\partial_t = \partial_z^2 + 2z^{-1}\partial_z$, we obtain

$$\delta(\omega) = \iota(r(\omega)) = (\partial_z - z^{-1})^2 + 2z^{-1}(\partial_z - z^{-1}) = \partial_z^2.$$

The second equality is obvious. □

**Proposition 3.3.** (1) $\delta(\mathcal{D}(\mathfrak{g})^G) = \mathcal{D}(\mathfrak{h})^W$.

(2) $\delta$ coincides with the Chevalley isomorphisms on $\mathrm{S}(\mathfrak{g}^*)^G$ and $\mathrm{S}(\mathfrak{g})^G$.

*Proof.* The claims follow from Lemma 3.1 and Lemma 3.2. □

**Remark .** From $\mathcal{D}(\mathfrak{g})^G \cong U(\mathfrak{sl}(2))$ we get that $\delta$ induces isomorphisms

$$\mathcal{D}(\mathfrak{h})^W \cong \mathcal{D}(\mathfrak{g})^G/\mathcal{I} \cong U(\mathfrak{sl}(2))/(\Omega + \lambda),$$

where $\lambda \in \mathbb{C}$ and $\Omega$ is the Casimir element. It is not difficult to see that $\lambda = 3/4$.

## 4. The map $\delta$ in the general case

In this section we sketch the proof of Theorem 2.3 given by G. Schwarz [14]. We continue with the notation of §2[6].

Fix a coordinate basis $\{z_1, \dots, z_\ell\}$ of $\mathfrak{h}^*$ and set $\partial_i = \frac{\partial}{\partial z_i}$. Let $P \in \mathcal{D}(\mathfrak{g})^G$. We have, with the usual conventions,

$$\delta(P) = \sum_m c_m(z)\partial^m,\;\; c_m \in \mathcal{O}(\mathfrak{h}') \text{ for all } m \in \mathbb{N}^\ell.$$

We want to show that $a_m \in \mathcal{O}(\mathfrak{h})$. Since $\mathcal{O}(\mathfrak{h}') = \mathcal{O}(\mathfrak{h})_\pi$, this is equivalent to showing that the $a_m$ have no pole along the reflecting hyperplanes $\mathcal{H}_\gamma = \{h \in \mathfrak{h} : \gamma(h) = 0\}$ for $\gamma \in R^+$.

Fix $\gamma \in R^+$. Choose $b \in \mathcal{H}_\gamma$, $b \notin \mathcal{H}_\beta$ for $\beta \in R^+ \setminus \{\gamma\}$. The idea is to prove that $\delta(P)$ is smooth in a neighborhood of $b$; this will be done by a "Luna's slice type argument". We have

$$\mathfrak{g}^b = \mathfrak{sl}(2)_\gamma \oplus \mathcal{H}_\gamma, \;\text{ where } \mathfrak{sl}(2)_\gamma = \mathbb{C}H_\gamma + \mathbb{C}X_\gamma + \mathbb{C}X_{-\gamma}.$$

The group $G^b$ is reductive and we have a $G^b$-decomposition $\mathfrak{g} = \mathfrak{g}^b \oplus [b, \mathfrak{g}]$. Recall that, since $G.b \equiv G/G^b$ via the adjoint action, $T_b(G.b) = \mathfrak{g}/\mathfrak{g}^b \cong [\mathfrak{g}, b]$ is generated by the tangent vectors $\tau(\xi)_b = [b, \xi]$. Note also that $W(\mathfrak{g}^b, \mathfrak{h}) = W^b = \{1, s = s_\gamma\}$, $R(\mathfrak{g}^b, \mathfrak{h}) = \{\pm\gamma\}$.

Set $p = \dim G.b$ and define

$$U = \{u \in \mathfrak{g} \,:\, \exists\, X_1, \dots, X_p \in \mathfrak{g},\; \mathfrak{g} = \mathfrak{g}^b \oplus \langle \tau(X_1)_u, \dots, \tau(X_p)_u \rangle_{\mathbb{C}}\}.$$

[6] Note that we may, if necessary, assume that $\mathfrak{g}$ is simple and that $G \subset \mathrm{GL}(\mathfrak{g})$ is the adjoint group.

(a) $U$ is an open neighbourhood of $b$. Indeed: Let $u \in U$ and let $X_1, \dots, X_p$ be such that $\mathfrak{g} = \mathfrak{g}^b \oplus \langle \tau(X_1)_u, \dots, \tau(X_p)_u \rangle_{\mathbb{C}}$, then

$$U' = \{u' \in \mathfrak{g} \,:\, \mathfrak{g} = \mathfrak{g}^b \oplus \langle \tau(X_1)_{u'}, \dots, \tau(X_p)_{u'} \rangle_{\mathbb{C}}\}$$

is an affine open neighbourhood of $u$ and $U' \subseteq U$.

(b) $U$ is $G^b$-stable. Let $u \in U$. Note first that, for all $g \in G$,

$$g.\tau(X_i)_u = g.[u, X_i] = [g.u, g.X_i] = \tau(g.X_i)_{g.u}.$$

When $g \in G^b$, we also have $g.\mathfrak{g}^b = \mathfrak{g}^b$. Hence

$$\mathfrak{g} = g.\mathfrak{g} = \mathfrak{g}^b \oplus g.\langle \tau(X_1)_u, \dots, \tau(X_p)_u \rangle_{\mathbb{C}} = \mathfrak{g}^b \oplus \langle \tau(g.X_1)_{g.u}, \dots, \tau(g.X_p)_{g.u} \rangle_{\mathbb{C}}.$$

This shows that $g.u \in U$.

(c) Let $t_1, \dots, t_{\ell-1}$ be coordinate functions on $\mathcal{H}_\gamma$, and let $\{x, y, z\}$ be the dual basis of $\{X_\gamma, X_{-\gamma}, H_\gamma\}$. It follows from (a) and (b) that, on the open subset $U$,

$$\mathcal{D}(U) = \sum_{i,j,k \in \mathbb{N}, \mu \in \mathbb{N}^{\ell-1}} \mathcal{O}(U) \partial_x^i \partial_y^j \partial_z^k \partial_t^\mu + \mathcal{D}(U)\tau(\mathfrak{g}).$$

Therefore we can write $P = \tilde{P} + Q$ (on $U$), with $\tilde{P} \in \sum \mathcal{O}(U)\partial_x^i \partial_y^j \partial_z^k \partial_t^\mu$ and $Q \in \mathcal{D}(U)\tau(\mathfrak{g})$. Since $P \in \mathcal{D}(\mathfrak{g})^G \subset \mathcal{D}(U)^{G^b}$, and since $G^b$ is reductive, we may as well assume that $\tilde{P}$ and $Q$ are $G^b$-invariant.

Set $\tilde{U} = U \cap \mathfrak{g}^b$, $\tilde{\mathfrak{h}} = U \cap \mathfrak{h}$ and $\tilde{\mathfrak{h}}' = U \cap \{h \in \mathfrak{h} \,:\, \gamma(h) \neq 0\}$. Denote by $\tilde{r}$ and $\tilde{\delta} = \gamma \circ \tilde{r} \circ \gamma^{-1}$ the morphisms from $\mathcal{D}(\tilde{U})^{G^b}$ to $\mathcal{D}(\tilde{\mathfrak{h}}')^{W^b}$. From the $\mathfrak{sl}(2)$-case we can deduce that $\operatorname{Im} \tilde{\delta} \subseteq \mathcal{D}(\tilde{\mathfrak{h}})^{W^b}$. Therefore $\tilde{\delta}(\tilde{P}) = \gamma \circ \tilde{r} \circ \gamma^{-1} \in \mathcal{D}(\tilde{\mathfrak{h}})^{W^b}$.

Note that, since $\tau(\mathfrak{g})$ kills the $G$-invariant functions, $P(f) = \tilde{P}(f)$ for all $f \in \mathcal{O}(U)^G$. In particular, since $\mathcal{O}(\tilde{\mathfrak{h}})^W \subset \mathcal{O}(\tilde{\mathfrak{h}})^{W^b}$, we have that $r(P) = \tilde{r}(\tilde{P})$ on $A := \mathcal{O}(\tilde{\mathfrak{h}})^W$. Set $\tilde{\pi} = \prod_{\{\gamma \neq \alpha \in R^+\}} \alpha$; then $\pi = \tilde{\pi}\gamma$ and $\tilde{\pi}^{\pm 1}$ is smooth on a neighbourhood of $b$. Now, write $\delta(P) = \tilde{\pi}\gamma r(P)\gamma^{-1}\tilde{\pi}^{-1}$. From the above we know that, on $A$, $\delta(P) = \tilde{\pi}(\gamma\tilde{r}(\tilde{P})\gamma^{-1})\tilde{\pi}^{-1}$. But, we have seen that $\tilde{\delta}(\tilde{P}) = \gamma\tilde{r}\gamma^{-1} \in \mathcal{D}(\tilde{\mathfrak{h}})^{W^b}$ and $\tilde{\pi}^{\pm 1}$ are smooth on a neighbourhood of $b$. Hence, the same is true of $\delta(P)$.

(d) To complete the proof of Theorem 2.3, it remains to show that $\delta$ coincide with the Chevalley isomorphisms. Recall that this is obvious, by construction, for $\delta$ on $\mathrm{S}(\mathfrak{g}^*)^G$. We thus have to show that $\delta = \phi$ on $\mathrm{S}(\mathfrak{g})^G$; this will be done by "Fourier transform". Without loss of generality we can reduce to the case when $\mathfrak{g}$ is simple.

Choose coordinates on $\mathfrak{g}$ such that $\kappa = -\frac{1}{2}\sum_{i=1}^n x_i^2$ and set

$$\omega = \tfrac{1}{2}\textstyle\sum_{i=1}^n \partial_{x_i}^2, \quad \varepsilon_{\mathfrak{g}} = \sum_{i=1}^n x_i \partial_{x_i}.$$

Then, as in the $\mathfrak{sl}(2)$-case, one checks that

$$[\kappa, \omega] = E_{\mathfrak{g}} := \varepsilon_{\mathfrak{g}} + n/2, \quad [E_{\mathfrak{g}}, \kappa] = 2\kappa, \quad [E_{\mathfrak{g}}, \omega] = -2\omega.$$

Hence, $\mathfrak{k} = \mathbb{C}\kappa + \mathbb{C}\omega + \mathbb{C}E_{\mathfrak{g}} \cong \mathfrak{sl}(2) = \mathbb{C}e + \mathbb{C}f + \mathbb{C}h$. Recall that $\operatorname{gr}\mathcal{D}(\mathfrak{g}) = \mathcal{O}(T^*\mathfrak{g}) \equiv \mathcal{O}(\mathfrak{g} \times \mathfrak{g})$. Since $\mathfrak{g} \times \mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}^2$, there is a natural action of $\mathrm{SL}(2)$ on $\mathfrak{g} \times \mathfrak{g}$, and therefore on $\operatorname{gr}\mathcal{D}(\mathfrak{g}) = \mathcal{O}(T^*\mathfrak{g})$. This action lifts to an $\mathrm{SL}(2)$-action on $\mathcal{D}(\mathfrak{g})$. Tracing the identifications, one sees that $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2)$ acts on $\mathcal{D}(\mathfrak{g})$ in the following way

$$g.x_i = ax_i + c\partial_{x_i}, \qquad g.\partial_{x_j} = bx_j + d\partial_{x_j}.$$

Observe now that $[E_{\mathfrak{g}}, x_i] = x_i$, $[E_{\mathfrak{g}}, \partial_{x_i}] = -\partial_{x_i}$, $[\omega, x_i] = \partial_{x_i}$, $[\omega, \partial_{x_i}] = 0$, $[\kappa, x_i] = 0$, $[\kappa, \partial_{x_i}] = x_i$. It follows that, inside $\mathcal{D}(\mathfrak{g})$,

$$\exp(te) = \exp(t \operatorname{ad}\kappa), \quad \exp(tf) = \exp(t \operatorname{ad}\omega), \quad \exp(th) = \exp(t \operatorname{ad} E_{\mathfrak{g}}).$$

Hence, the adjoint action of $\mathfrak{k}$ integrates to the SL(2)-action that we just described. Observe that, since $\kappa, \omega, E_{\mathfrak{g}}$ are $G$-invariant, the SL(2)-action commutes with the $G$-action. Consider now the "Weyl group element" $w = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right) \in \mathrm{SL}(2)$ (here $i = \sqrt{-1} \in \mathbb{C}$). It acts on $\mathcal{D}(\mathfrak{g})$ by $w.x_j = i\partial_{x_j}$, $w.\partial_{x_j} = ix_j$ for all $j = 1, \dots, n$.

Let $\kappa_{\mathfrak{h}}$, $\omega_{\mathfrak{h}}$ and $\varepsilon_{\mathfrak{h}}$ be the analogous elements of $\mathcal{D}(\mathfrak{h})^W$. We have

$$\delta(\kappa) = \kappa_{\mathfrak{h}}, \quad [\kappa_{\mathfrak{h}}, \omega_{\mathfrak{h}}] = E_{\mathfrak{h}} := \varepsilon_{\mathfrak{h}} + \ell/2.$$

Let $f \in \mathrm{S}^p(\mathfrak{g}^*)^G$. Then, $\delta([\varepsilon_{\mathfrak{g}}, f]) = [\delta(\varepsilon_{\mathfrak{g}}), \phi(f)] = \delta(pf) = p\phi(f)$. This implies that $\delta(\varepsilon_{\mathfrak{g}}) = \varepsilon_{\mathfrak{h}} - c$ for some $c \in \mathbb{C}$. We know that $\delta(\omega) \in \mathcal{D}_2(\mathfrak{h})^W$. Note that

$$\delta([E_{\mathfrak{g}}, \omega]) = [\delta(E_{\mathfrak{g}}), \delta(\omega)] = [\epsilon_{\mathfrak{h}}, \delta(\omega)] = -2\delta(\omega).$$

In the appropriate coordinate basis of $\mathfrak{h}$, this forces

$$\delta(\omega) = \sum_{\{|\mu|-|\nu|=-2, |\nu|\leq 2\}} a_{\mu,\nu} x^\mu \partial_x^\nu, \quad a_{\mu,\nu} \in \mathbb{C},$$

and it follows that

$$\delta(\omega) = \sum_\nu a_\nu \partial_x^\nu \in \mathrm{S}^2(\mathfrak{h})^W = \mathbb{C}\omega_{\mathfrak{h}}.$$

Thus $\delta(\omega) = a\omega_{\mathfrak{h}}$ for some $a \in \mathbb{C}$. Then, $\delta([\kappa, \omega]) = [\kappa_{\mathfrak{h}}, a\omega_{\mathfrak{h}}] = \varepsilon_{\mathfrak{h}} - c + n/2$ implies that $a = 1$ and $c = \frac{1}{2}(n - \ell)$. Hence, we have shown

$$\delta(\kappa) = \kappa_{\mathfrak{h}}, \quad \delta(\omega) = \omega_{\mathfrak{h}}, \quad \delta(E_{\mathfrak{g}}) = E_{\mathfrak{h}}.$$

Recall that $\mathcal{D}(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{h})$ have natural SL(2)-actions, which integrate the adjoint actions of $\mathbb{C}\kappa + \mathbb{C}\omega + \mathbb{C}E_{\mathfrak{g}}$ and $\mathbb{C}\kappa_{\mathfrak{h}} + \mathbb{C}\omega_{\mathfrak{h}} + \mathbb{C}E_{\mathfrak{h}}$ respectively. The above formulas prove that the map $\delta$ is SL(2)-equivariant. Let $P \in \mathrm{S}^m(\mathfrak{g})^G$. By definition of $w$, and the fact that the SL(2)-action commutes with the $G$-action, we obtain that $w.P \in \mathrm{S}^m(\mathfrak{g}^*)^G$. Therefore

$$w.\delta(P) = \delta(w.P) = (w.P)_{|\mathfrak{h}}$$

implies that

$$\delta(P) = w^{-1}.\delta(w.P) = w^{-1}(w.P)_{|\mathfrak{h}}.$$

The definition of $w$ then shows that $w^{-1}(w.P)_{|\mathfrak{h}}$ is the projection of $P$ onto $\mathrm{S}^m(\mathfrak{h})^W$, as required. □

## 5. Surjectivity of $\delta$

We have shown that there exists a homomorphism

$$\delta : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h})^W$$

with kernel

$$\mathcal{I} = \{P \in \mathcal{D}(\mathfrak{g})^G \mid P\big(\mathcal{O}(\mathfrak{g})^G\big) = 0\}.$$

Evidently, $\operatorname{Im}\delta$ contains the images of $\mathrm{S}(\mathfrak{g}^*)^G$ and $\mathrm{S}(\mathfrak{g})^G$ which, by Theorem 2.3 coincide with $\mathrm{S}(\mathfrak{h}^*)^W$ and $\mathrm{S}(\mathfrak{h})^W$. Denote by $B$ the subalgebra of $\mathcal{D}(\mathfrak{h})^W$ generated by $\mathrm{S}(\mathfrak{h}^*)^W$ and $\mathrm{S}(\mathfrak{h})^W$. Two questions naturally arise.

(†) Is $\delta$ surjective?

Recall that $\delta$ is a filtered morphism. The second question is more precise: Is it true that $\delta(\mathcal{D}_m(\mathfrak{g})^G) = \mathcal{D}_m(\mathfrak{h})^W$ for all $m \in \mathbb{N}$? Equivalently:

(††) Is $\mathrm{gr}(\delta) : \mathrm{gr}\,\mathcal{D}(\mathfrak{g})^G \to \mathrm{gr}\,\mathcal{D}(\mathfrak{h})^W$ surjective?

If this is true, we shall say that $\delta$ is *graded-surjective.*

N. Wallach has shown [17] that (††) has a positive answer when $\mathfrak{g}$ has no factor of type $\mathsf{E}_p$, $p = 6, 7, 8$. In [7] it is shown that (†) is true in all cases. This follows from a general result about differential operators invariant under a finite group action:

**Theorem 5.1.** [7] *Let $V$ be a finite dimensional $\mathbb{C}$-vector space and $W$ be a finite subgroup of* $\mathrm{GL}(V)$. *Then $\mathcal{D}(V)^W$ is generated by $\mathrm{S}(V)^W$ and $\mathrm{S}(V^*)^W$.*

The proof of Theorem 5.1 is not difficult. In this section we shall give a proof in the case we are presently interested: $(V, W) = (\mathfrak{h}, W = \text{Weyl group})$. The idea of the proof is exactly the same, but, in this particular case, we will bring a little bit more of information.

We fix a coordinate basis $\{x_1, \dots, x_\ell; \partial_1, \dots, \partial_\ell\}$ of $\mathfrak{h}^* \times \mathfrak{h}$[7]. In this situation we may also suppose that $\{\partial_1, \dots, \partial_\ell\}$ is an orthonormal basis, with respect to $\kappa$, on a real form $\mathfrak{h}_\mathbb{R}$ of $\mathfrak{h}$. Then, each $w \in W$ acts on $\mathfrak{h}$ via an orthogonal matrix: $w.\partial_j = \sum_{i=1}^{\ell} w_{ij} \partial_i$.

Recall that $\pi^2 \in B$ and that, up to a nonzero scalar (that we ignore), we have $\pi = \det \mathrm{Jac}(p)$, where $\mathrm{Jac}(p) = [\frac{\partial p_i}{\partial x_j}] \in \mathrm{M}_\ell(\mathrm{S}(\mathfrak{h}^*))$. Moreover $\mathfrak{h}' = \{h : \pi(h) \neq 0\}$ is the set of points where $p : \mathfrak{h} \to \mathfrak{h}/W$ is étale. Define, as usual, the gradient vector field associated to the invariant function $p_j$ by

$$\nabla(p_j) = \sum_{i=1}^{\ell} \partial_i(p_j)\partial_i, \quad j = 1, \dots, \ell.$$

**Lemma 5.2.** *The following assertions hold:*

(1) $\nabla(p_j) \in [\mathrm{Der}\, \mathcal{O}(\mathfrak{h})]^W \cap B$;

(2) $\mathrm{Der}\, \mathcal{O}(\mathfrak{h}') = \bigoplus_{i=1}^{\ell} \mathcal{O}(\mathfrak{h}')\nabla(p_j)$;

(3) $[\mathrm{Der}\, \mathcal{O}(\mathfrak{h}')]^W = \bigoplus_{i=1}^{\ell} \mathcal{O}(\mathfrak{h}')^W \nabla(p_j)$, and

$$[\mathrm{Der}\, \mathcal{O}(\mathfrak{h})]^W = \bigoplus\nolimits_{i=1}^{\ell} \mathcal{O}(\mathfrak{h})^W \nabla(p_j)$$

is a free $\mathcal{O}(\mathfrak{h})^W$-module.

*Proof.* (1) Note first that

$$w.\partial_j(p_k) = (w.\partial_j)(w.p_k) = (w.\partial_j)(p_k) = \sum\nolimits_i w_{ij}\partial_i(p_k).$$

Therefore

$$\begin{aligned} w.\nabla(p_k) &= \sum_j w.\partial_j(p_k)\, w.\partial_j = \sum_{i,j,s} w_{ij}\partial_i(p_k)\, w_{sj}\partial_s \\ &= \sum_{i,s}(\sum_j w_{ij}w_{sj})\partial_i(p_k)\partial_s = \sum_{i,s} \delta_{is}\partial_i(p_k)\,\partial_s \\ &= \nabla(p_k). \end{aligned}$$

Hence, $\nabla(p_k)$ is $W$-invariant. Recall that $\omega_\mathfrak{h} = \frac{1}{2}\sum_i \partial_i^2 \in \mathrm{S}^2(\mathfrak{h})^W$. Note that

$$[\omega_\mathfrak{h}, p_j] = \frac{1}{2}\sum_i [\partial_i^2, p_j] = \nabla(p_j) + \frac{1}{2}\omega_\mathfrak{h}(p_j).$$

Thus, $\nabla(p_j) = [\omega_\mathfrak{h}, p_j] - \frac{1}{2}\omega_\mathfrak{h}(p_j) \in B$.

[7]The elements of $\mathfrak{h}$ are identified with $\mathbb{C}$-linear derivations with constant coefficients on $\mathrm{S}(\mathfrak{h}^*)$, hence $\partial_i = \frac{\partial}{\partial x_i}$.

(2) Denote by $[a_{ij}] \in \mathrm{M}_\ell(\mathcal{O}(\mathfrak{h})_\pi)$ the inverse matrix of $\mathrm{Jac}(p)$. Then, $\pi[a_{ij}] \in \mathrm{M}_\ell(\mathcal{O}(\mathfrak{h}))$ and

$$\sum_m a_{mk}\nabla(p_m) = \sum_i (\sum_m a_{mk}\partial_i(p_m))\partial_i = \sum_i \delta_{ik}\partial_i = \partial_k.$$

Hence, $\mathrm{Der}\,\mathcal{O}(\mathfrak{h}') = \bigoplus_k \mathcal{O}(\mathfrak{h}')\partial_k = \bigoplus_k \mathcal{O}(\mathfrak{h}')\nabla(p_k)$. Observe that we have also shown that

$$\pi\,\mathrm{Der}\,\mathcal{O}(\mathfrak{h}) = \bigoplus_m \mathcal{O}(\mathfrak{h})\nabla(p_m). \tag{5.1}$$

(3) The first claim is consequence of (2) by taking $W$-invariants. Let $d \in \mathrm{Der}\,\mathcal{O}(\mathfrak{h})^W$. From (5.1), we get that $\pi d = \sum_m \varphi_m \nabla(p_m)$ for some $\varphi_m \in \mathcal{O}(\mathfrak{h})$. Thus, for all $w \in W$,

$$w.(\pi d) = w.\pi\, w.d = \epsilon(w)\pi d = \sum_m w.\varphi_m\nabla(p_m).$$

It follows that $w.\varphi_m = \epsilon(w)\varphi_m$, and therefore $\varphi_m = \pi\gamma_m$ for some $\gamma_m \in \mathcal{O}(\mathfrak{h})^W$. Hence, $d = \sum_j \gamma_j\nabla(p_j) \in \bigoplus_j \mathcal{O}(\mathfrak{h})^W\nabla(p_j)$, as required. □

Recall that, since the elements of $\mathcal{O}(\mathfrak{h})$ act locally nilpotently on $\mathcal{D}(\mathfrak{h})$, we can localize at any Öre subset of $\mathcal{O}(\mathfrak{h})$.

**Proposition 5.3.** *We have:* $B_{\pi^2} = \mathcal{D}(\mathfrak{h})^W_{\pi^2} = \mathcal{O}(\mathfrak{h})^W_{\pi^2}[\nabla(p_1), \dots, \nabla(p_\ell)]$.

*Proof.* Recall that

$$\mathcal{D}(\mathfrak{h})^W_{\pi^2} = \mathcal{D}(\mathfrak{h}'/W) = [\mathcal{D}(\mathfrak{h})_\pi]^W = \mathcal{O}(\mathfrak{h}')^W[\mathrm{Der}\,\mathcal{O}(\mathfrak{h}'/W)].$$

But, since $p : \mathfrak{h}' \twoheadrightarrow \mathfrak{h}'/W$ is étale, we obtain from Lemma 5.2(3) that

$$\mathrm{Der}\,\mathcal{O}(\mathfrak{h}'/W) = [\mathrm{Der}\,\mathcal{O}(\mathfrak{h}')]^W = \bigoplus_{i=1}^{\ell} \mathcal{O}(\mathfrak{h}')^W\nabla(p_j).$$

Hence, using Lemma 5.2(1),

$$\mathcal{D}(\mathfrak{h})^W_{\pi^2} \subseteq \mathcal{O}(\mathfrak{h})^W_{\pi^2}[\nabla(p_1), \dots, \nabla(p_\ell)] \subseteq B_{\pi^2}.$$

The other inclusion being obvious, we have the desired equalities. □

We filter $\mathcal{D}(\mathfrak{h})$ and its subspaces by the order of differential operators. In particular, if $B_m = \mathcal{D}_m(\mathfrak{h}) \cap B$, we obtain

$$\mathrm{gr}\,B = \bigoplus B_m/B_{m-1} \hookrightarrow \mathrm{gr}\,\mathcal{D}(\mathfrak{h})^W = \mathcal{O}(\mathfrak{h} \times \mathfrak{h}^*)^W = \mathrm{S}(\mathfrak{h}^* \times \mathfrak{h})^W \subset \mathrm{S}(\mathfrak{h}^* \times \mathfrak{h})$$

where the group $W$ acts diagonally.

**Lemma 5.4.** *The ring $B$ is a noetherian domain, and $\mathcal{D}(\mathfrak{h})^W$ is a finitely generated (left and right) $B$-module.*

*Proof.* Clearly, $B \supseteq \mathrm{S}(\mathfrak{h}^*)^W \otimes_{\mathbb{C}} \mathrm{S}(\mathfrak{h})^W = \mathrm{S}(\mathfrak{h}^* \times \mathfrak{h})^{W\times W}$. It is well known, since the group $W \times W$ is finite, that $\mathrm{S}(\mathfrak{h}^* \times \mathfrak{h})$ is a finite module over the finitely generated algebra $\mathrm{S}(\mathfrak{h}^* \times \mathfrak{h})^{W\times W}$. It follows easily that $\mathrm{gr}\,B$ is a finitely generated $\mathbb{C}$-algebra and that $\mathrm{S}(\mathfrak{h}^* \times \mathfrak{h})^W$ is a finitely generated $(\mathrm{gr}\,B)$-module. A routine argument then yields the claim. □

**Lemma 5.5.** *Let $B \subseteq A$ be two noetherian domains. Assume that $A$ is simple and finitely generated as a left or right $B$-module. Then, if $A$ and $B$ have the same fraction field, we have $A = B$.*

*Proof.* Set $L = \{b \in B \mid bA \subseteq B\}$. Since $A$ is a finitely generated right $B$-module, and $\mathrm{Frac}(A) = \mathrm{Frac}(B)$, $L$ is nonzero. Similarly, $L' = \{b \in B \mid Ab \subseteq B\} \neq 0$. Since $L'$ and $L$ are, respectively, left and right ideals of $A$, $L'L$ is a two-sided ideal of $A$. But $A$ being a domain, $L'L \neq 0$. Therefore $A = L'L \subseteq B$, and $A = B$ as required. □

**Theorem 5.6.** *The homomorphism $\delta : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h})^W$ is surjective.*

*Proof.* We apply Lemma 5.5 to $B = \mathrm{Im}\,\delta \subseteq A = \mathcal{D}(\mathfrak{h})^W$. Recall [10] that $A$ is simple. The theorem then follows from Proposition 5.3 and Lemma 5.4. □

The previous theorem shows that (†) has a positive answer, but does not give the graded surjectivity of $\delta$. In the next sections we will see that question (††) is closely related to geometric questions about the commuting variety of $\mathfrak{g}$. Before going into this interpretation, we have to remark that the graded surjectivity of $\delta$ is easy once we have localized at the discriminant[8]. Indeed:

**Proposition 5.7.** *The map $\delta : \mathcal{D}(\mathfrak{g})^G_{d_\ell} \to \mathcal{D}(\mathfrak{h})^W_{\pi^2}$ is graded-surjective.*

*Proof.* Fix an orthonormal basis of $\mathfrak{g}$ with respect to $\kappa$ and denote the associated coordinate system on $\mathfrak{g}^* \times \mathfrak{g}$ by $\{x_1, \dots, x_n; \partial_1, \dots, \partial_n\}$. Assume that the numbering is chosen such that $\{x_1, \dots, x_\ell; \partial_1, \dots, \partial_\ell\}$ is the previous coordinate system on $\mathfrak{h}^* \times \mathfrak{h}$.

Define the gradient vector field of $u_j \in \mathcal{O}(\mathfrak{g})^G$, by $\nabla(u_j) = \sum_{k=1}^n \partial(u_k)\partial_k$. Recall that $r : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h}/W)$. It is easily checked that

$$r(\nabla(u_j)) = \nabla(p_j), \quad j = 1, \dots, \ell.$$

We have seen in Proposition 5.3 that $\mathcal{D}(\mathfrak{h})^W_{\pi^2} = \mathcal{O}(\mathfrak{h})^W_{\pi^2}[\nabla(p_1), \dots, \nabla(p_\ell)]$, hence

$$\mathrm{gr}\,\mathcal{D}(\mathfrak{h})^W_{\pi^2} = \mathrm{gr}\,\mathcal{D}(\mathfrak{h}')^W = \mathbb{C}[p_1, \dots, p_\ell, \pi^{-2}, \mathrm{gr}(\nabla(p_1)), \dots, \mathrm{gr}(\nabla(p_\ell))].$$

Therefore, with obvious notation,

$$\mathrm{gr}_m\,\mathcal{D}(\mathfrak{h})^W_{\pi^2} = \textstyle\sum_{|k|=m} \mathbb{C}[p_1, \dots, p_\ell, \pi^{-2}]\nabla(p)^k.$$

Recall that $\delta(P) = \pi r(P)\pi^{-1}$; it follows that $\mathrm{gr}(\delta) = \mathrm{gr}(r)$. Since $\phi(u_j) = p_j$, $\phi(d_\ell) = \pi^2$ and $\mathrm{gr}(\delta)(\nabla(u_j)) = \mathrm{gr}(\nabla(u_j))$, we obtain from the above description of $\mathrm{gr}_m\,\mathcal{D}(\mathfrak{h})^W_{\pi^2}$ that $\mathrm{gr}(\delta) : \mathrm{gr}\,\mathcal{D}(\mathfrak{g})^G_{d_\ell} \to \mathrm{gr}\,\mathcal{D}(\mathfrak{h})^W_{\pi^2}$ is surjective. □

Set $A = \mathcal{D}(\mathfrak{g})^G/\mathcal{J}$. Recall that we can identify $\mathcal{D}(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{g})^G$ with $\mathcal{O}(\mathfrak{g} \times \mathfrak{g})$ and $\mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G$ respectively. Let $\mathbf{q}$ be the kernel of the graded morphism

$$\mathrm{gr}(\delta) : \mathrm{gr}\,\mathcal{D}(\mathfrak{g})^G \to \mathrm{gr}\,\mathcal{D}(\mathfrak{h})^W.$$

Hence, $\mathrm{gr}\,\mathcal{J} \subseteq \mathbf{q}$ and $\mathbf{q}$ is prime. Since $\mathrm{gr}\,\mathcal{J}$ and $\mathbf{q}$ are contained in $\mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G$, they define affine subvarieties $\mathcal{V}(\mathbf{q}) \subseteq \mathcal{V}(\mathrm{gr}\,\mathcal{J}) \subseteq (\mathfrak{g} \times \mathfrak{g})/G$.

**Corollary 5.8.** *One has*[9]*:*

(1) $\mathrm{GKdim}\,\mathcal{D}(\mathfrak{g})^G = \dim(\mathfrak{g} \times \mathfrak{g})/G = n + \ell - k$;
(2) $\mathrm{GKdim}\,A = \mathrm{GKdim}\,\mathrm{gr}\,A = \mathrm{GKdim}\,\mathcal{D}(\mathfrak{h})^W = 2\ell$;
(3) $\mathrm{height}(\mathrm{gr}\,\mathcal{J}) = \mathrm{height}(\mathbf{q}) = n - \ell - k$.

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[8] In the rest of this section we do not assume that the surjectivity of $\delta$ has been proved.

[9] Recall that $\dim\mathfrak{g} = n$, $\ell = \mathrm{rk}\,\mathfrak{g} = \dim\mathfrak{h}$, $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$, $\mathfrak{h} = \mathfrak{z} \oplus \mathfrak{t}$ and $k = \dim\mathfrak{t}$ (hence $\dim\mathfrak{z} = \ell - k$). The heights of the ideals in (3) are computed in $\mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G$.

*Proof.* (1) Clearly, if $S$ is the connected semisimple subgroup of $G$ such that $\mathrm{Lie}(S) = \mathfrak{s}$, we have

$$(\mathfrak{g} \times \mathfrak{g})/G \cong ((\mathfrak{s} \times \mathfrak{s})/S) \times (\mathfrak{z} \times \mathfrak{z}).$$

The maximal dimension of an $S$-orbit in $\mathfrak{s} \times \mathfrak{s}$ is $n - \ell + k$: pick $(x, y) \in \mathfrak{s} \times \mathfrak{s}$, with $x$ generic and $y$ regular nilpotent; then $\mathfrak{s}^x$ is a Cartan subalgebra of $\mathfrak{s}$ and $\mathfrak{s}^y$ is contained in the nilpotent cone of $\mathfrak{s}$. Hence, $\mathfrak{s}^x \cap \mathfrak{s}^y = 0$ and[10] $\dim S.(x, y) = \dim \mathfrak{s} = n - \ell + k$. Therefore, $\dim(\mathfrak{g} \times \mathfrak{g})/G = n - \ell + k + 2(\ell - k) = n + \ell - k$.

(2) From Proposition 5.7, we deduce that there is a filtered isomorphism $A_{d_\ell} \cong \mathcal{D}(\mathfrak{h})^W_{\pi^2}$. The localization at $d_\ell$ commutes with gr, hence

$$\mathrm{gr}\big(\mathcal{D}(\mathfrak{g})^G_{d_\ell}/\mathcal{I}_{d_\ell}\big) = \mathrm{gr}\, A_{d_\ell} = (\mathrm{gr}\, A)_{d_\ell} \xrightarrow{\sim} \mathrm{gr}\, \mathcal{D}(\mathfrak{h})^W_{\pi^2}.$$

From $\mathrm{Ker}\big(\mathrm{gr}(\delta) : \mathrm{gr}\, \mathcal{D}(\mathfrak{g})^G_{d_\ell} \to \mathrm{gr}\, \mathcal{D}(\mathfrak{h})^W_{\pi^2}\big) = \mathbf{q}_{d_\ell}$, it follows that $\mathbf{q}_{d_\ell} = (\mathrm{gr}\, \mathcal{I})_{d_\ell}$ and $\big(\mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G/\mathbf{q}\big)_{d_\ell} \cong \mathcal{O}(\mathfrak{h} \times \mathfrak{h})^W_{\pi^2}$. Observe that, since $\mathcal{O}(\mathfrak{g} \times \mathfrak{g})/\mathbf{q}$ is a domain,

$$\mathrm{GKdim}\,\mathrm{gr}\, A_{d_\ell} = \mathrm{GKdim}\big(\mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G/\mathbf{q}\big)_{d_\ell} = \mathrm{GKdim}\, \mathcal{O}(\mathfrak{g} \times \mathfrak{g})/\mathbf{q} = 2\ell.$$

Note that $d_\ell$ is a nonzero divisor in $A$: $\delta(d_\ell) = \pi^2$ is a nonzero element of the domain $\mathcal{D}(\mathfrak{h})^W$, and $\delta : A \to \mathcal{D}(\mathfrak{h})^W$ is injective by definition of $\mathcal{I}$. Moreover, $d_\ell$ acts locally ad-nilpotently on $A$. Therefore, by [6, Lemma 4.7, page 49], $\mathrm{GKdim}\, A = \mathrm{GKdim}\, A_{d_\ell}$. Hence,

$$\mathrm{GKdim}\, A = \mathrm{GKdim}\, A_{d_\ell} = \mathrm{GKdim}\, \mathcal{D}(\mathfrak{h})^W_{\pi^2} = \mathrm{GKdim}\, \mathcal{D}(\mathfrak{h})^W = 2\ell.$$

Now, by [6, Lemma 6.5, page 75] and the previous remarks,

$$2\ell = \mathrm{GKdim}\, \mathcal{O}(\mathfrak{g} \times \mathfrak{g})/\mathbf{q} \le \mathrm{GKdim}\,\mathrm{gr}\, A \le \mathrm{GKdim}\, A = 2\ell.$$

Thus $\mathrm{GKdim}\, \mathcal{O}(\mathfrak{g} \times \mathfrak{g})/\mathbf{q} = \mathrm{GKdim}\,\mathrm{gr}\, A = \mathrm{GKdim}\, A = 2\ell$.

(3) Since $\mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G$ is a finitely generated domain,

$$\mathrm{height}(\mathrm{gr}\, \mathcal{I}) = \mathrm{GKdim}\, \mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G - \mathrm{GKdim}\,\mathrm{gr}\, A = n + \ell - k - 2\ell = n - \ell - k.$$

Similarly,

$$\mathrm{height}(\mathbf{q}) = \mathrm{GKdim}\, \mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G - \mathrm{GKdim}\, \mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G/\mathbf{q} = n - \ell - k$$

as desired. ☐

**Remark .** Corollary 5.8(3) shows that $\mathbf{q}$ is a minimal prime ideal over $\mathrm{gr}\, \mathcal{I}$, and that $\dim \mathcal{V}(\mathrm{gr}\, \mathcal{I}) = \dim \mathcal{V}(\mathbf{q}) = 2\ell$.

**Corollary 5.9.** *The following are equivalent:*

(a) $\delta$ is graded-surjective;

(b) $\delta$ is surjective and $\mathrm{gr}\, \mathcal{I}$ is a prime ideal.

*Proof.* (a) $\Rightarrow$ (b) The hypothesis says that $\mathrm{gr}(\delta) : \mathrm{gr}\, \mathcal{D}(\mathfrak{g})^G \twoheadrightarrow \mathrm{gr}\, \mathcal{D}(\mathfrak{h})^W$ is surjective. Thus $\delta$ is surjective. We have to show that $\mathrm{Ker}\,\mathrm{gr}(\delta) = \mathrm{gr}\, \mathcal{I}$. Let $a \in \mathcal{D}_m(\mathfrak{g})^G$ be such that $0 = \mathrm{gr}(\delta(a)) \in \mathrm{gr}_m \mathcal{D}(\mathfrak{h})^W$, i.e. $\delta(a) \in \mathcal{D}_{m-1}(\mathfrak{h})^W$. Since $\mathcal{D}_{m-1}(\mathfrak{h})^W = \delta(\mathcal{D}_{m-1}(\mathfrak{g})^G)$, we obtain $a \in \mathcal{D}_{m-1}(\mathfrak{g})^G + \mathcal{I}$. Hence, $\mathrm{gr}(a) \in \mathrm{gr}\, \mathcal{I}$ as required.

(b) $\Rightarrow$ (a) Since $\mathrm{gr}\, \mathcal{I} = \mathbf{q}$, $\mathrm{gr}(\delta)$ yields an injection: $\mathrm{gr}\, \mathcal{D}(\mathfrak{g})^G/\mathrm{gr}\, \mathcal{I} \hookrightarrow \mathrm{gr}\, \mathcal{D}(\mathfrak{h})^W$. Let $b \in \mathcal{D}_m(\mathfrak{h})^W$. Then, $b = \delta(a)$ for some $a \in \mathcal{D}_p(\mathfrak{g})^G$. If $p \le m$ we are done; otherwise, $\mathrm{gr}(\delta)(\mathrm{gr}_p(a)) = \mathrm{gr}_p(b) = 0$. Hence, $\mathrm{gr}_p(a) \in \mathrm{gr}\, \mathcal{I}$ and therefore $a \in \mathcal{I} + \mathcal{D}_{p-1}(\mathfrak{g})^G$. By induction we get that $b = \delta(a')$ for some $a' \in \mathcal{D}_m(\mathfrak{g})^G$, proving the graded surjectivity of $\delta$. ☐

[10] Since $S$ is semisimple, $\dim(\mathfrak{s} \times \mathfrak{s})/S = 2\dim \mathfrak{s} - \max\{\dim S.(x, y)\,;\, x, y \in \mathfrak{s}\}$.

## 6. The commuting variety of $\mathfrak{g}$

The *commuting variety* of $\mathfrak{g}$ is the closed subvariety of $\mathfrak{g} \times \mathfrak{g}$ defined by

$$\mathcal{C}(\mathfrak{g}) = \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid [x, y] = 0\}.$$

Note that $\mathcal{C}(\mathfrak{g})$ is a $G$-subvariety of $\mathfrak{g} \times \mathfrak{g}$ under the diagonal (adjoint) action of $G$.

**Remark .** In general, i.e. for an arbitrary Lie algebra, $\mathcal{C}(\mathfrak{g})$ is not irreducible. Take, for example, the 3-dimensional solvable Lie algebra $\mathfrak{g} = \mathbb{C}u + \mathbb{C}v + \mathbb{C}w$, where the nonzero brackets are

$$[u, v] = v, \qquad [u, w] = w.$$

Let $\{x, y, z\}$ be the dual basis of $\{u, v, w\}$ and set $\mathcal{O}(\mathfrak{g} \times \mathfrak{g}) = \mathbb{C}[x, y, z] \otimes_{\mathbb{C}} \mathbb{C}[x', y', z']$. Then,

$$\mathcal{C}(\mathfrak{g}) = \mathcal{V}(xy' - x'y, xz' - x'z)$$

is 4-dimensional and has two irreducible components

$$\mathcal{V}(x, x') = (\mathbb{C}v + \mathbb{C}w) \times (\mathbb{C}v + \mathbb{C}w), \quad \mathcal{V}(xy' - x'y, xz' - x'z, y'z - yz').$$

But, when $\mathfrak{g}$ is reductive, we have the following result.

**Theorem 6.1.** [12] *The variety* $\mathcal{C}(\mathfrak{g})$ *is irreducible. Indeed,*

$$\mathcal{C}(\mathfrak{g}) = \overline{G.(\mathfrak{h} \times \mathfrak{h})}.$$

**Remark .** The study of $\mathcal{C}(\mathfrak{g})$ reduces easily to the case when $\mathfrak{g}$ is semisimple: Write $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$, where $\mathfrak{s}$ is semisimple and $\mathfrak{z}$ is the centre, then

$$\mathcal{C}(\mathfrak{g}) = \mathcal{C}(\mathfrak{s}) \times (\mathfrak{z} \times \mathfrak{z}),$$

where we have identified $\mathfrak{g} \times \mathfrak{g}$ with $(\mathfrak{s} \times \mathfrak{s}) \times (\mathfrak{z} \times \mathfrak{z})$. Therefore we shall assume in this section that $\mathfrak{g}$ is semisimple, and that $G$ is the adjoint group.

Denote by $\mathbf{p}$ the prime ideal of $\mathcal{O}(\mathfrak{g} \times \mathfrak{g}) = \mathrm{S}(\mathfrak{g}^* \times \mathfrak{g}^*)$ such that $\mathcal{C}(\mathfrak{g}) = \mathcal{V}(\mathbf{p})$. Clearly, $(x, y) \in \mathcal{C}(\mathfrak{g})$ is and only if $\kappa(a, [x, y]) = 0$ for all $a \in \mathfrak{g}$. Let $\sigma_a \in \mathcal{O}(\mathfrak{g} \times \mathfrak{g})$ be the function $(x, y) \mapsto \kappa(a, [x, y])$, and define the ideal

$$\mathbf{a} = (\sigma_a \,;\, a \in \mathfrak{g}).$$

Thus, $\sqrt{\mathbf{a}} = \mathbf{p}$ and $\mathcal{C}(\mathfrak{g})/G = \mathcal{V}(\mathbf{p}^G) = \mathcal{V}(\mathbf{a}^G)$. The main questions concerning $\mathcal{C}(\mathfrak{g})$ are the following:

- Is $\mathbf{a} = \mathbf{p}$? If true, this would imply that $\mathcal{J} = \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$[11].
- Is $\mathcal{C}(\mathfrak{g})$ normal? Cohen-Macaulay?
- Is $\mathcal{C}(\mathfrak{g})/G$ normal? Cohen-Macaulay? We shall relate the normality of $\mathcal{C}(\mathfrak{g})/G$ to the graded-surjectivity of $\delta$ in the next section.

We need to know the dimension of $\mathcal{C}(\mathfrak{g})$; this computation is implicit in [12], for sake of completeness we indicate a proof.

**Lemma 6.2.** $\dim \mathcal{C}(\mathfrak{g}) = \dim \mathfrak{g} + \operatorname{rk} \mathfrak{g}$.

[11] Actually, $\mathcal{J} = \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$ has been proved [8]. The equality $\mathbf{a} = \mathbf{p}$ would imply a stronger result: $\operatorname{gr} \mathcal{J} = \operatorname{gr} \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$.

*Proof.* Let $\eta : \mathfrak{g} \times \mathfrak{g} \twoheadrightarrow \mathfrak{g}$ be the first projection. Since $\eta(\mathcal{C}(\mathfrak{g})) = \mathfrak{g}$, we have a surjective morphism: $\eta : \mathcal{C}(\mathfrak{g}) \twoheadrightarrow \mathfrak{g}$. Note that, for all $u = (x, x') \in \mathcal{C}(\mathfrak{g})$,

$$\eta^{-1}(\eta(u)) = \{(x, y) \,:\, y \in \mathfrak{g}^x\} \cong \mathfrak{g}^x$$

is an irreducible variety.

By a standard result, see [15, Theorem 4.1.6], there exists a non-empty open subset $U \subseteq \mathcal{C}(\mathfrak{g})$ such that, for all $u \in U$,

$$\dim U = \dim \mathcal{C}(\mathfrak{g}) = \dim \mathfrak{g} + \dim \eta^{-1}(\eta(u)).$$

Since $(\mathfrak{g}' \times \mathfrak{g}) \cap \mathcal{C}(\mathfrak{g})$ is a non-empty open subset of $\mathcal{C}(\mathfrak{g})$, we can pick $u = (x, y) \in U$ with $x \in \mathfrak{g}'$. Then $\mathfrak{g}^x$ is a Cartan subalgebra of $\mathfrak{g}$. Hence $\dim \mathcal{C}(\mathfrak{g}) = n + \ell$. □

Again, the situation is easy after localization at the discriminant $d_\ell \in \mathcal{O}(\mathfrak{g}) \equiv \mathcal{O}(\mathfrak{g}) \otimes_{\mathbb{C}} 1 \subset \mathcal{O}(\mathfrak{g} \times \mathfrak{g}) = \mathcal{O}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathcal{O}(\mathfrak{g})$.

**Lemma 6.3.** $\mathbf{a}_{d_\ell} = \mathbf{p}_{d_\ell}$.

*Proof.* Let $(x, y) \in \mathfrak{g} \times \mathfrak{g}$ and $v \in \mathfrak{g}$. The differential of $\sigma_v$ at $(x, y)$, that we denote by $d\sigma_v(x, y) \in T^*_{(x,y)}(\mathfrak{g} \times \mathfrak{g})$, is given by

$$\forall\,(a, b) \in \mathfrak{g} \times \mathfrak{g}, \quad d\sigma_v(x, y)(a, b) = \frac{d}{dt}_{|t=0} \sigma_v(x + ta, y + tb) = \kappa(v, [x, b] + [a, y]).$$

It follows that $d\sigma_v(x, y) = 0$ if, and only if, $v \in ([x, \mathfrak{g}] + [y, \mathfrak{g}])^\perp = \mathfrak{g}^x \cap \mathfrak{g}^y$, where $\perp$ denotes the orthogonal with respect to $\kappa$. Therefore, the linear map

$$\vartheta : \mathfrak{g} \to T^*_{(x,y)}(\mathfrak{g} \times \mathfrak{g}), \quad v \mapsto d\sigma_v(x, y),$$

has rank $n - \dim(\mathfrak{g}^x \cap \mathfrak{g}^y)$.

Now, suppose that $(x, y) \in \mathcal{C}(\mathfrak{g}) \cap (\mathfrak{g}' \times \mathfrak{g})$. Then $y \in \mathfrak{g}^x$ and $\mathfrak{g}^x$ is a Cartan subalgebra of $\mathfrak{g}$. Thus, $\mathfrak{g}^y \supseteq \mathfrak{g}^x$ and $\operatorname{rk} \vartheta = n - \ell$. Let $v_1, \dots, v_{n-\ell} \in \mathfrak{g}$ be such that $d\sigma_{v_1}(x, y), \dots, d\sigma_{v_{n-\ell}}(x, y)$ are linearly independent. Denote by $(A, \mathbf{m})$ the local ring of $\mathfrak{g} \times \mathfrak{g}$ at the point $(x, y)$; recall that $T^*_{(x,y)}(\mathfrak{g} \times \mathfrak{g}) \equiv \mathbf{m}/\mathbf{m}^2$. Since $(A, \mathbf{m})$ is a regular local ring, the functions $\sigma_{v_1}, \dots, \sigma_{v_{n-\ell}} \in \mathbf{m}$ can be included in a regular system of parameters. In particular, they generate an ideal of height $n - \ell$ in $A$. Note that they also belong to $\mathbf{a}_{(x,y)} \subseteq \mathbf{p}_{(x,y)}$, and that $\operatorname{height}(\mathbf{p}_{(x,y)}) = \operatorname{height}(\mathbf{p}) = \operatorname{codim} \mathcal{C}(\mathfrak{g}) = n - \ell$. Hence,

$$(\sigma_{v_1}, \dots, \sigma_{v_{n-\ell}}) = \mathbf{a}_{(x,y)} = \mathbf{p}_{(x,y)}.$$

Since $(x, y) \in \mathcal{C}(\mathfrak{g}) \cap (\mathfrak{g}' \times \mathfrak{g}) = \mathcal{C}(\mathfrak{g})_{d_\ell}$ was arbitrary, we obtain that $\mathbf{a}_{d_\ell} = \mathbf{p}_{d_\ell}$. □

**Remark .** The proof of Lemma 6.3 shows that, if $(x, y) \in \mathcal{C}(\mathfrak{g})$ and $\dim(\mathfrak{g}^x \cap \mathfrak{g}^y) = \operatorname{rk} \mathfrak{g}$, then $(x, y)$ is a smooth point of $\mathcal{C}(\mathfrak{g})$. Hence, $\mathcal{C}(\mathfrak{g}) \cap (\mathfrak{g}' \times \mathfrak{g})$ is a smooth open subset of $\mathcal{C}(\mathfrak{g})$.

Recall the following theorem:

**Theorem 6.4.** [13, Theorem 3.2] *Let $(x, y) \in \mathfrak{g} \times \mathfrak{g}$. Then, the orbit $G.(x, y)$ is closed if and only if the algebraic hull of the Lie subalgebra of $\mathfrak{g}$ generated by $x$ and $y$ is reductive in $\mathfrak{g}$.*

Since we are, here, interested in orbits in $\mathcal{C}(\mathfrak{g})$, we will give a proof of Theorem 6.4 in this particular case.

**Lemma 6.5.** *Let $(x, y) \in \mathcal{C}(\mathfrak{g})$. Then, $G.(x, y)$ is closed if and only if $x$ and $y$ are semisimple.*

*Proof.* Recall the Jordan-Chevalley decomposition of $x \in \mathfrak{g}$: $x = x_s + x_n$, $x_s$ semisimple, $x_n$ nilpotent, $[x_s, x_n] = 0$. Note that $\operatorname{ad} x_s$ and $\operatorname{ad} x_n$ are polynomials in $\operatorname{ad} x$. Thus, $[x, y] = 0$ if and only if $[x_s, y_s] = [x_s, y_n] = [x_n, y_s] = [x_n, y_n] = 0$. Since $x_s, y_s$ are commuting semisimple elements, we may assume (after conjugacy) that $x_s, y_s \in \mathfrak{h}$. Observe that $\mathfrak{k} = \mathfrak{g}^{x_s} \cap \mathfrak{g}^{y_s}$ is a reductive Lie algebra in $\mathfrak{g}$, see §2. Denote by $K \subseteq G$ the adjoint group of $\mathfrak{k}$. Furthermore, $x_n, y_n \in [\mathfrak{k}, \mathfrak{k}]$ are nilpotent, see [4, §3 Remark 9]; since they commute, there exists a maximal nilpotent subalgebra $\mathfrak{u}$ of $\mathfrak{k}$ containing $x_n$ and $y_n$. Then, it easy to show that there is a one-parameter subgroup, $\lambda : \mathbb{C}^* \to K$, such that $\lim_{t\to 0} \lambda(t).z = 0$ for all $z \in \mathfrak{u}$.

Now assume that $G.(x, y)$ is closed. Then $\lim_{t\to 0} \lambda(t).(x, y) = (x_s, y_s)$, and therefore $(x_s, y_s) \in G.(x, y)$. This shows that $x, y$ are semisimple.

Conversely, assume that $x, y \in \mathfrak{g}$ are commuting semisimple elements. We may suppose (after conjugacy) that $x, y \in \mathfrak{h}$. Thus the stabilizer $G^{(x,y)} = G^x \cap G^y$ contains $H$. Then [5, III.2.5, Folgerung 3] gives that $G.(x, y)$ is closed. (The proof goes as follows. Let $B = NH$ be a Borel subgroup. Since $N$ is unipotent, $Z = B.(x, y) = N.(x, y)$ is closed. Recall now the well known fact: Let $P$ be a parabolic subgroup of $G$ and $Z$ be a $P$-stable closed subset of some $G$-variety $V$, then, $G.Z$ is closed. (Set $\varphi : G\times V \xrightarrow{\sim} G\times V$, $\varphi((g, v)) = (g, g.v)$, $\eta : G\times V \twoheadrightarrow G/P\times V$, $\eta((g, v)) = (\bar{g}, v)$, and $\varpi : G\times V \twoheadrightarrow V$, $\varpi((g, v)) = v$. Since $\varphi(G \times Z)$ is closed, $\eta(\varphi(G \times Z))$ is closed if and only if $\varphi(G \times Z) = \eta^{-1}(\eta(\varphi(G \times Z)))$, which is clear. Then, since $G/P$ is complete, $G.Z = \varpi(\varphi(G \times Z))$ is closed.)) □

Set $N = N_G(H)$, so that $W = N/H$. We have a natural surjective morphism

$$\mu : \mathcal{X} = G \times_N (\mathfrak{h} \times \mathfrak{h}) \to G.(\mathfrak{h} \times \mathfrak{h}), \quad [g, (h_1, h_2)] \mapsto (g.h_1, g.h_2).$$

By Theorem 6.1, $\mu$ induces a dominant morphism from $\mathcal{X}$ to $\mathcal{C}(\mathfrak{g})$. Furthermore, $\dim \mathcal{X} = \dim G + 2 \dim \mathfrak{h} - \dim N = n + \ell$.

**Theorem 6.6.** *Set $\mathcal{X}' = G \times_N (\mathfrak{h}' \times \mathfrak{h})$ and $\mathcal{S} = \{(x, y) \in \mathcal{C}(\mathfrak{g}) \mid x \in \mathfrak{g}'\}$. Then,*

(1) $\mu : \mathcal{X}' \to \mathcal{S}$ is an isomorphism;

(2) $\mu$ is a birational morphism from $\mathcal{X}$ to $\mathcal{C}(\mathfrak{g})$.

*Proof.* (1) If $x \in \mathfrak{g}'$, $x$ is conjugate to an element of $\mathfrak{h}'$, say $x = g.x_1$. Let $(x, y) \in \mathcal{S}$ and set $y = g.y_1$. Then $[x, y] = [x_1, y_1] = 0$, hence $y_1 \in \mathfrak{g}^{x_1} = \mathfrak{h}$. It follows that $(x, y) = g.(x_1, y_1) \in \mu(\mathcal{X}')$. Hence, $\mu : \mathcal{X}' \to \mathcal{S}$ is surjective. Suppose that $[g, (h_1, h_2)], [g', (h'_1, h'_2)] \in \mathcal{X}'$ and satisfy $g.h_i = g.h'_i$ for $i = 1, 2$. Then $h_i = g^{-1}g'.h'_i$; in particular, the two generic elements $h_1$, $h'_1$ are $G$-conjugate. This implies that $h_1$ and $h'_1$ are $W$-conjugate. Indeed, there exists $n \in N$ such that $h'_1 = n.h_1$. Therefore $h_1 = g^{-1}g'n.h_1$, forcing $t := g^{-1}g'n \in H$. We obtain that $g^{-1}g' = tn^{-1} \in N$ and

$$[g', (h'_1, h'_2)] = [gtn^{-1}, (h'_1, h'_2)] = [g, tn^{-1}(h'_1, h'_2)] = [g, (h_1, h_2)].$$

This proves that $\mu$ restricted to $\mathcal{X}'$ is bijective. We know that $\mathcal{S}$ is contained in the set of smooth points of $\mathcal{C}(\mathfrak{g})$ (see the remark after Lemma 6.3). Therefore $\mu_{|\mathcal{X}'}$ is an isomorphism.

(2) Since $\mathcal{X}'$ and $\mathcal{S}$ are non-empty open subsets of the irreducible varieties $\mathcal{X}$ and $\mathcal{C}(\mathfrak{g})$ respectively, the result follows from (1). □

The previous theorem says that $\mathfrak{h} \times \mathfrak{h}$ is a *rational section* of the action of $G$ on $\mathcal{C}(\mathfrak{g})$, see [11, II.2.5, II.2.8].

The group $G$ acts on $\mathcal{X}$ by left translation and we have a natural isomorphism

$$\mathcal{X}/G \cong (\mathfrak{h} \times \mathfrak{h})/W.$$

The $G$-equivariant morphism $\mu$ then induces $\mu : (\mathfrak{h} \times \mathfrak{h})/W \to \mathcal{C}(\mathfrak{g})/G$. This morphism $\mu$ will be called the *Chevalley restriction map*; it is easily seen that $\mu$ is given by restriction of functions:

$$\mu : (\mathfrak{h} \times \mathfrak{h})/W \longrightarrow \mathcal{C}(\mathfrak{g})/G; \quad \mu : W.(x,y) \mapsto G.(x,y).$$

The comorphism of $\mu$ is

$$\mu^\sharp : \mathcal{O}(\mathcal{C}(\mathfrak{g}))^G \to \mathcal{O}(\mathfrak{h} \times \mathfrak{h})^W, \quad \mu^\sharp(f) = f_{|\mathfrak{h}\times\mathfrak{h}}.$$

Since $\mathcal{C}(\mathfrak{g}) = \overline{G.(\mathfrak{h} \times \mathfrak{h})}$, it is clear that a function $f$ on $\mathcal{C}(\mathfrak{g})$ is determined by its values on $G.(\mathfrak{h} \times \mathfrak{h})$. If $f$ is $G$-invariant, it is therefore determined by $f_{|\mathfrak{h}\times\mathfrak{h}}$. Hence, $\mu^\sharp : \mathcal{O}(\mathcal{C}(\mathfrak{g}))^G \to \mathcal{O}(\mathfrak{h} \times \mathfrak{h})^W$ is injective, i.e. $\mu$ is dominant.

The open question is to show that $\mathfrak{h} \times \mathfrak{h}$ is a *Chevalley section* [11, II.3.8], i.e. $\mu^\sharp : \mathcal{O}(\mathcal{C}(\mathfrak{g}))^G \xrightarrow{\sim} \mathcal{O}(\mathfrak{h} \times \mathfrak{h})^W$. The next result shows that $\mu$ is at least bijective.

**Theorem 6.7.** *The morphism $\mu : (\mathfrak{h} \times \mathfrak{h})/W \longrightarrow \mathcal{C}(\mathfrak{g})/G$ is bijective and is the normalization of $\mathcal{C}(\mathfrak{g})/G$.*

*Proof.* 1. *$\mu$ is surjective.* Let $(x, y) \in \mathcal{C}(\mathfrak{g})$ be such that $G.(x, y)$ is closed in $\mathcal{C}(\mathfrak{g})$ (hence in $\mathfrak{g} \times \mathfrak{g}$). Then, by Lemma 6.5, $x$ and $y$ are commuting semisimple elements. Therefore they are contained in a Cartan subalgebra $\mathfrak{h}_1$ of $\mathfrak{g}$. By conjugacy of the Cartan subalgebras, we can find $g \in G$ such that $g.\mathfrak{h}_1 = \mathfrak{h}$. Thus, $g.(x, y) = (g.x, g.y) \in \mathfrak{h} \times \mathfrak{h}$. This proves the surjectivity of $\mu$.
2. *$\mu$ is injective.* Recall the following well-known facts, cf. [4] for example.

(1) If $x \in \mathfrak{g}$ is semisimple, then $G^x$ is a connected reductive subgroup of $G$.
(2) If $y \in \mathfrak{h}$, then $G.y \cap \mathfrak{h} = W.y$.

We shall denote by $\dot{w} \in N = N_G(H)$ a representative element of $w \in W$. We have to show that: if $x, x', y, y' \in \mathfrak{h}$ are such that $g.x = x', g.y = y'$ for some $g \in G$, then there exists $\dot{u} \in N$ such that $x' = \dot{u}.x, y' = \dot{u}.y$. Since $x' \in G.x \cap \mathfrak{h}$ and $y' \in G.y \cap \mathfrak{h}$, we know from (2) that $x' = \dot{w}_1.x, y' = \dot{w}_2.y$ for some $w_1, w_2 \in W$. Set $y'' = \dot{w}_1^{-1}.y', g' = \dot{w}_1^{-1}g$. We have $g'.x = x, g'.y = y''$; thus, $G.(x, y) = (x, y'')$. Therefore, it is enough to show that there exists $w \in W^x$ such that $y'' = w.y$. Indeed, $y'' = w.y$ implies $y' = \dot{w}_1\dot{w}.y$ and we have $x' = \dot{w}_1\dot{w}.x$. Thus, the result follows by setting $\dot{u} = \dot{w}_1\dot{w}$.

Therefore we may, and we do, assume that $x = x'$, $g \in G^x$, $y' = g.y \in \mathfrak{h}$. The proof of the injectivity of $\mu$ reduces then to show that

$$x,\, y \in \mathfrak{h} \implies G^x.y \cap \mathfrak{h} = W^x.y.$$

Observe that $H \subset G^x$. Therefore $\mathfrak{h}$ is Cartan subalgebra of $\mathfrak{g}^x$. Futhermore, cf. (1), $G^x$ is a connected reductive subgroup of $G$. Since $H \subseteq N \cap G^x = N_{G^x}(H)$, the Weyl group of $G^x$ is $W^x$ (with respect to the choice of the Cartan $\mathfrak{h}$). Now, use (2) in the connected reductive group $G^x$ to get $G^x.y \cap \mathfrak{h} = W^x.y$.

By [2, Theorem 4.6], $\mu$ is then birational. Since $(\mathfrak{h} \times \mathfrak{h})/W$ is a normal variety, the result follows. □

**Remark .** The fact that $\mu$ is the normalization of $\mathcal{C}(\mathfrak{g})$ is a corollary of [9, Proposition 2.2][12]. Recall [9, Lemme 1.8] that, if $\varphi : \mathcal{X} \to \mathcal{Y}$ is a surjective birational morphism between affine irreducible varieties, and if $\mathcal{Y}$ is normal, then $\varphi$ is an isomorphism. Therefore, the open problem of whether $\mu : (\mathfrak{h} \times \mathfrak{h})/W \to \mathcal{C}(\mathfrak{g})/G$ is an isomorphism, is equivalent to showing that $\mathcal{C}(\mathfrak{g})/G$ is normal, cf. Corollary 7.2.

[12] Apply this proposition to $M = \mathfrak{g} \times \mathfrak{g}$.

## 7. Graded-surjectivity of $\delta$

We begin with a preliminary remark. Recall that the map $\delta : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h})^W$ is equal to $\iota \circ r$, where $r : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h}')^W$ is the "radial component" map. We noticed that, when we restrict to the generic elements,

$$\operatorname{gr}(\delta) = \operatorname{gr}(r) : \mathcal{O}(\mathfrak{g}' \times \mathfrak{g})^G = \mathcal{O}((\mathfrak{g}' \times \mathfrak{g})/G) \longrightarrow \mathcal{O}(\mathfrak{h}' \times \mathfrak{h})^W = \mathcal{O}((\mathfrak{h}' \times \mathfrak{h})/W).$$

From the definition of $r$, it is immediate that $\operatorname{gr}(r)$ is induced by restriction of functions: $\operatorname{gr}(r)(f) = f_{|\mathfrak{h}' \times \mathfrak{h}}$. Since $(\mathfrak{g}' \times \mathfrak{g})/G$ is open and dense in $(\mathfrak{g} \times \mathfrak{g})/G$, it follows that $\operatorname{gr}(\delta)$ is also given by restriction of functions.

**Proposition 7.1.** *With the notation of §6, we have*
(1) $\mathbf{q} = \mathbf{p}^G$ and $\mathcal{C}(\mathfrak{g})/G = \mathcal{V}(\operatorname{gr} \mathcal{J})$;
(2) $\mathbf{p}^G_{d_\ell} = \operatorname{gr} \mathcal{J}_{d_\ell}$ and $\mathcal{J}_{d_\ell} = \big(\mathcal{D}(\mathfrak{g}))\tau(\mathfrak{g})\big)^G_{d_\ell}$.

*Proof.* (1) Note first that, since $\mathcal{V}(\mathbf{a}) = \mathcal{C}(\mathfrak{g})$, $\mathcal{C}(\mathfrak{g})/G = \mathcal{V}(\mathbf{p}^G) = \mathcal{V}(\mathbf{a}^G)$. Moreover, $\mathbf{a} \subseteq \operatorname{gr} \mathcal{J}$ implies $\mathbf{a}^G \subset (\operatorname{gr} \mathcal{J})^G = \operatorname{gr} \mathcal{J}$. Therefore,

$$\mathcal{V}(\mathbf{q}) \subseteq \mathcal{V}(\operatorname{gr} \mathcal{J}) \subseteq \mathcal{V}(\mathbf{a}^G) = \mathcal{C}(\mathfrak{g})/G.$$

By Corollary 5.8 and Theorem 6.7, $\dim \mathcal{C}(\mathfrak{g})/G = \dim \mathcal{V}(\mathbf{q}) = 2\ell$. Hence,

$$\mathcal{V}(\mathbf{q}) = \mathcal{V}(\operatorname{gr} \mathcal{J}) = \mathcal{C}(\mathfrak{g})/G = \mathcal{V}(\mathbf{p}^G).$$

This proves the claims.

(2) We have seen (cf. the proof of Corollary 5.8) that $\mathbf{q}_{d_\ell} = \operatorname{gr} \mathcal{J}_{d_\ell}$. Thus, the first assertion follows from (1). By Lemma 6.3, $\mathbf{a}^G_{d_\ell} = \mathbf{p}^G_{d_\ell}$ (recall that $d_\ell$ is invariant) and therefore, $\mathbf{a}^G_{d_\ell} = \operatorname{gr} \mathcal{J}_{d_\ell}$. Since $\mathbf{a}^G \subseteq \big(\operatorname{gr} \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})\big)^G \subseteq \operatorname{gr} \mathcal{J}$, we obtain the equality $\big(\operatorname{gr} \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})\big)^G_{d_\ell} = \operatorname{gr} \mathcal{J}_{d_\ell}$. Hence $\big(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})\big)^G_{d_\ell} = \mathcal{J}_{d_\ell}$. □

**Corollary 7.2.** *The following are equivalent:*
(a) $\delta$ is graded-surjective;
(b) $\mathcal{C}(\mathfrak{g})/G$ is a normal variety;
(c) the Chevalley restriction map $\mu : (\mathfrak{h} \times \mathfrak{h})/W \to \mathcal{C}(\mathfrak{g})/G$ is an isomorphism, i.e. $\mathcal{O}(\mathcal{C}(\mathfrak{g}))^G \xrightarrow{\sim} \mathcal{O}(\mathfrak{h} \times \mathfrak{h})^W$.

*Proof.* By Proposition 7.1 and the preliminary remark, the comorphism of $\operatorname{gr}(\delta) : \mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G / \operatorname{gr} \mathcal{J} \to \mathcal{O}(\mathfrak{h} \times \mathfrak{h})^W$ is the map, $(\mathfrak{h} \times \mathfrak{h})/W \to \mathcal{V}(\operatorname{gr} \mathcal{J}) = \mathcal{C}(\mathfrak{g})/G$, induced by restriction of functions.

(b) $\Leftrightarrow$ (c) is consequence of Theorem 6.7.

(a) $\Rightarrow$ (c) If $\delta$ is graded-surjective, then $\operatorname{gr} \mathcal{J} = \mathbf{q} = \mathbf{p}^G$ by Corollary 5.9. Hence, $\operatorname{gr}(\delta) : \mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G/\mathbf{p}^G \xrightarrow{\sim} \mathcal{O}(\mathfrak{h} \times \mathfrak{h})^W$ and (c) follows.

(c) $\Rightarrow$ (a) If the Chevalley restriction map is an isomorphism, we deduce that $\operatorname{gr}(\delta)$ gives an isomorphism $\mathcal{O}(\mathfrak{g} \times \mathfrak{g})^G/\mathbf{p}^G \xrightarrow{\sim} \mathcal{O}(\mathfrak{h} \times \mathfrak{h})^W$. In particular, $\operatorname{gr}(\delta)$ is surjective. Hence the result. □

The (equivalent) conditions of Corollary 7.2 hold when $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. As explained in [1], this follows from the fact that, in this case, $\mathcal{O}(\mathfrak{h} \times \mathfrak{h})^W$ is well understood. Recall that when $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, one can choose $\mathfrak{h} = \{\operatorname{diag}(\lambda_1, \dots, \lambda_n) \in \mathrm{M}_n(\mathbb{C})\}$ as a Cartan subalgebra. Then, the Weyl group $W = W(\mathfrak{g}, \mathfrak{h})$ identifies with the symmetric group $\mathfrak{S}_n$ acting on $\mathfrak{h}$ by permutation of the entries:

$$w.\operatorname{diag}(\lambda_1, \dots, \lambda_n) = \operatorname{diag}(\lambda_{w^{-1}(1)}, \dots, \lambda_{w^{-1}(n)}).$$

Set $\mathcal{O}(\mathfrak{h}\times\mathfrak{h}) = \mathbb{C}[X_1,\dots,X_n]\otimes_{\mathbb{C}}\mathbb{C}[Y_1,\dots,Y_n]$. Thus $W$ acts on $\mathcal{O}(\mathfrak{h}\times\mathfrak{h})$ by $w.X_j = X_{w(j)}$, $w.Y_j = Y_{w(j)}$.

For every $r, s \in \mathbb{N}$, define the "polarized power sums" $p_{r,s} \in \mathcal{O}(\mathfrak{h}\times\mathfrak{h})$ by

$$p_{r,s} = \sum_{i=1}^{n} X_i^r Y_i^s.$$

Clearly, $p_{r,s}$ is $W$-invariant. One has the following result, due to H. Weyl:

**Theorem 7.3.** [18] *$\mathcal{O}(\mathfrak{h}\times\mathfrak{h})^W$ is generated by the polynomials $p_{r,s}$.*

**Corollary 7.4.** *Assume that $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$. Then, the Chevalley restriction map $\mu : (\mathfrak{h}\times\mathfrak{h})/W \to \mathcal{C}(\mathfrak{g})/G$ is an isomorphism.*

*Proof.* We have already noticed in §6 that $\mu^\sharp : \mathcal{O}(\mathcal{C}(\mathfrak{g}))^G \to \mathcal{O}(\mathfrak{h}\times\mathfrak{h})^W$ is injective. It remains to show that $\mu^\sharp$ is surjective. By Theorem 7.3, this is equivalent to showing that $p_{r,s} \in \operatorname{Im}\mu^\sharp$. Consider the polynomial function $u_{r,s}$ on $\mathfrak{g}\times\mathfrak{g}$ defined by

$$u_{r,s}(x,y) = \operatorname{tr}(x^r y^s).$$

Then, $u_{r,s}$ is $G$-invariant and induces a function $u_{r,s} \in \mathcal{O}(\mathcal{C}(\mathfrak{g}))^G$. Obviously, $u_{r,s|\mathfrak{h}\times\mathfrak{h}} = p_{r,s}$; hence the result. □

**Remark .** When $\mathfrak{g}$ is of type $\mathsf{B}_n$ or $\mathsf{G}_2$, then Theorem 7.3 has an analog and the same proof yields Corollary 7.4. For $\mathfrak{g}$ of type $\mathsf{D}_n$ and $\mathsf{F}_4$, Theorem 7.3 fails, but Wallach [17] has shown that Corollary 7.4 is true. Therefore, it remains to investigate the types $\mathsf{E}_6, \mathsf{E}_7$ and $\mathsf{E}_8$.

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