Corrigendum to: Semi-simplicity of Invariant Holonomic Systems on a Reductive Lie Algebra

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CORRIGENDUM TO: SEMI-SIMPLICITY OF INVARIANT HOLONOMIC SYSTEMS ON A REDUCTIVE LIE ALGEBRA

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The proof of Proposition 6.5 of [2] is inadequate (the application of the Chinese Remainder Theorem is incorrect) and so the aim of this corrigendum is to provide a correct proof of that result. All the results in [2] are correct as stated.

Recall the notation of [2]: $A = \mathcal{D}(\mathfrak{h})$, $d = \prod_{\alpha>0} \alpha^2$, $\lambda \in \mathfrak{h}^*$, $\mathbf{m} = \operatorname{Ker} \lambda \subset S(\mathfrak{h})$. Set $P = A/A\mathbf{m}$. Then the A-module P identifies with $\mathcal{O}(\mathfrak{h})e^{\lambda}$ endowed with the natural action of A.

Definition 1.1. Let M be an A-module. We say that M has a C-filtration if there exists a finite chain of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ satisfying

$$M_i/M_{i-1} \cong A \otimes_{C_i} (C_i/I_i) \text{ for each } i, \tag{1.1}$$

where $C_i = \mathbb{C}[y_1, \ldots, y_\ell]$ is a commutative polynomial ring such that $\mathcal{D}(C_i) = A$, and I_i is an ideal of C_i of finite codimension.

Remarks 1.2. (1) The A-module P has a C-filtration: $0 = M_0 \subset M_1 = P$.

(2) Suppose that M has a C-filtration and keep the notation of Definition 1.1. Then, M is automatically holonomic. Moreover, classical results (for example, Kashiwara's equivalence) imply that A/AI_i is isomorphic to a finite direct sum of simple modules of the form $A/A\mathbf{y}$, where $\mathbf{y}$ is maximal ideal of C_i. In particular, every subquotient of M has a C-filtration.

(3) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Then, M' and M'' have a C-filtration if and only if M has one.

Lemma 1.3. *Let D be left noetherian ring and B be a (D, A)-bimodule finitely generated as left D-module. Let M be an A-module having a C-filtration. Then* $\operatorname{Tor}_j^A(B, M) = 0$ *for all* $j \geqslant 1$.

Proof. Keep the notation of Definition 1.1. By [2, Lemma 6.4], B is a flat right C_i-module and so $\operatorname{Tor}_j^A(B, M_i/M_{i-1}) \cong \operatorname{Tor}_j^{C_i}(B, C_i/I_i) = 0$, by[3, Theorem 11.53]. The lemma follows easily by induction on t from the short exact sequence

$$\operatorname{Tor}_j^A(B, M_1) \longrightarrow \operatorname{Tor}_j^A(B, M) \longrightarrow \operatorname{Tor}_j(B, M/M_1).$$

□

Let M be an A-module and $f \in \mathcal{O}(\mathfrak{h})$. We denote by $M_{(f)}$ the localization $\mathcal{O}(\mathfrak{h})[f^{-1}] \otimes_{\mathcal{O}(\mathfrak{h})} M$ with respect to $\{f^i\}$. Recall that if M is holonomic, then $M_{(f)}$ is also a holonomic A-module [1, Theorem 3.2.11]. In particular, $M_{(f)}$ has finite

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length. This applies to $P = \mathcal{O}(\mathfrak{h})e^\lambda$ and $P_{(f)} = \sum_{i\in\mathbb{N}} \mathcal{O}(\mathfrak{h})f^{-i}e^\lambda$ (with the natural action of A by differential operators). Since $\{f^{-i}e^\lambda\}_i$ is a system of generators of $P_{(f)}$ this implies that $P_{(f)} = Af^{-m}e^\lambda$ for some $m \in \mathbb{N}$. Notice that in the notation of [2], $P_{(d)} = P_{\mathcal{C}}$.

Lemma 1.4. *The A-module $P_{\mathcal{C}}/P$ has a C-filtration.*

Proof. Set $P_0 = P$. Let $\beta_1, \dots, \beta_k$ be distinct positive roots for some $k \geqslant 1$. Observe that, since $d = \prod_{\alpha>0} \alpha^2$, we have $P_{(\beta_1\cdots\beta_k)^2} = P_{(\beta_1\cdots\beta_k)} \subset P_{\mathcal{C}}$. Define an A-submodule of $P_{\mathcal{C}}$ by

$$P_k = \sum_{\beta_1,\dots,\beta_k} P_{(\beta_1\cdots\beta_k)}$$

where the sum is taken over all possible sets of k distinct, positive roots. Notice that we have

$$P_0 \subset \cdots \subset P_k \subset \cdots \subset P_\nu = P_{\mathcal{C}},$$

where ν is the number of positive roots.

Set $N_\gamma = (P_{(\gamma_1\cdots\gamma_k)} + P_{k-1})/P_{k-1}$, where the γ_i are some distinct positive roots. Clearly, $P_k/P_{k-1} = \sum N_\gamma$ where the sum is over all possible γ. Thus, by Remarks 1.2, in order to prove that $P_{\mathcal{C}}$ has a C-filtration it suffices to prove this for each P_k/P_{k-1} and hence for each N_γ. So, consider $N = N_\gamma$. As noticed above, $P_{(\gamma_1\cdots\gamma_k)} = A\gamma_1^{-2i}\cdots\gamma_k^{-2i}e^\lambda$ for some $i \in \mathbb{N}$; hence N is generated by the class $e = [\gamma_1^{-2i}\cdots\gamma_k^{-2i}e^\lambda]$. The significance behind the definition of the P_ℓ is that $\gamma_j^{2i}e = 0$ for $1 \le j \le k$. Order the γ_j's so that $\gamma_1, \dots, \gamma_h$ are linearly independent while $\gamma_j \in \sum_{i=1}^h \mathbb{C}\gamma_j$ for $j \geqslant h+1$. Pick $x_{h+1}, \dots, x_\ell \in \mathfrak{h}^*$ such that $\{y_1 = \gamma_1, \dots, y_h = \gamma_h, x_{h+1}, \dots, x_\ell\}$ is a basis of $\mathfrak{h}^*$. Then, for $j \geqslant h+1$, one has $y_j = \partial_{x_j} - \alpha_j \in \mathbf{m}$ for some $\alpha_j \in \mathbb{C}$. Then, $[y_j, \gamma_i] = 0$, for all $j \ge h+1$ and $i \le k$ and so:

$$y_1^{2i}.e = y_2^{2i}.e = \cdots = y_h^{2i}.e = y_{h+1}.e = \cdots = y_\ell.e = 0.$$

Notice that $C_i = \mathbb{C}[y_1, \dots, y_\ell]$ is a polynomial ring with $\mathcal{D}(C_i) = A$. By the last displayed equation, $A.e$ is a factor of the module $A \otimes_{C_i} (C_i/I_i)$, where $I_i = (y_1^{2i}, \dots, y_h^{2i}, y_{h+1}, \dots, y_\ell)$. Hence the lemma. □

Corollary 1.5. ([2, Proposition 6.5].) *Let B be the $(\mathcal{D}(\mathfrak{g}), A)$-bimodule defined in* [2, p.1109]. *Let $\mathbf{m}$ be a maximal ideal of $S(\mathfrak{h})$ and write $P = A/A\mathbf{m}$. Then,* $\mathrm{Tor}_1^A(B, P_{\mathcal{C}}/P) = 0$.

Proof. By its construction, B is a finitely generated left $\mathcal{D}(\mathfrak{g})$-module. Thus, the corollary follows from Lemma 1.4 and Lemma 1.3 with $M = P_{\mathcal{C}}/P$. □

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