PROBABILISTIC LIMIT THEOREMS FOR CHAOTIC DYNAMICAL SYSTEMS, SOME RESULTS FOR DISPERSIVE BILLIARDS AND LORENTZ GASES

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PROBABILISTIC LIMIT THEOREMS FOR CHAOTIC DYNAMICAL SYSTEMS, SOME RESULTS FOR DISPERSIVE BILLIARDS AND LORENTZ GASES

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Abstract. This proceeding is based on a mini-course given at the Summer School "From kinetic equations to statistical mechanics" organized by the Centre Henri Lebesgue at Saint Jean de Monts from the 28th of June to the 2nd of July 2021.
After recalling classical probabilistic limit theorems for sums of independent identically distributed random variables, we consider analogous results in a dynamical context. Motivated by examples coming from statistical mechanics, we are mostly interested in the Sinai billiard and in the $\mathbb{Z}^2$-periodic Lorentz gas. We will also consider the Bunimovich stadium billiard and dispersive billiards with cusps. All these billiards are chaotic, with different behaviours. Additional explanations are given in the four independent appendices.

1. FCLT for sums of i.i.d. random variables and Wiener process

Let $(X_k)_k$ be a sequence of centered $\mathbb{R}$-valued **independent identically distributed (i.i.d.)** random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- The **Strong Law of Large Numbers (SLLN)** ensures that $\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^n X_k = 0$ almost surely (i.e. this convergence holds true with probability one).

The **Central Limit Theorem (CLT)** deals with the convergence in distribution of $(\sum_{k=1}^n X_k/\mathfrak{a}_n)_{n\geq 1}$ to a non-degenerate random variable for some sequence $(\mathfrak{a}_n)_{n\geq 1}$ such that $\lim_{n\to+\infty} \mathfrak{a}_n = +\infty$.
Functional Central Limit Theorems (FCLT) state convergence in distribution of the sequence of processes $\left(t \to \mathfrak{a}_n^{-1}\sum_{k=1}^{\lfloor nt\rfloor} X_k\right)_{n\geq 1}$ to a non degenerate càdlàg[1] process for $\mathfrak{a}_n$ as above. Different normalizations and limit processes may appear. In particular:

- **(Standard FCLT)** If $\mathbb{E}[X_1^2] < \infty$, then $\left(t \mapsto n^{-\frac{1}{2}}\sum_{k=1}^{\lfloor nt\rfloor} X_k\right)_{n\geq 1}$ converges in distribution[2] to $(t \mapsto \|X_1\|_{L^2}\mathcal{W}_t)$ with $\mathcal{W} : t \mapsto \mathcal{W}_t$ is standard Wiener motion
- **(FCLT with non-standard normalization)** If $\lim_{x\to\infty} x^2\mathbb{P}(|X_1| \geq x) = A > 0$, then X_1 is not in $\mathbb{L}^2$ but is in $\mathbb{L}^p$ for any $p \in [1, 2)$ and the family of processes $\left(t \to \frac{1}{\sqrt{n\log n}}\sum_{k=1}^{\lfloor nt\rfloor} X_k\right)_{n\geq 1}$ converges in distribution to $A\mathcal{W}$, with $\mathcal{W}$ as above.
- **(Convergence to Lévy processes)** Let $\alpha \in]1; 2[$. If $\lim_{x\to\infty} x^\alpha\mathbb{P}(\pm X_1 \geq x) = A_\pm$ with $A_+ + A_- > 0$. Then the family of processes $\left(t \mapsto n^{-\frac{1}{\alpha}}\sum_{k=1}^{\lfloor nt\rfloor} X_k\right)_{n\geq 1}$ converges in distribution[3] to a Lévy process $\mathcal{Z}$ of order α.

Date:

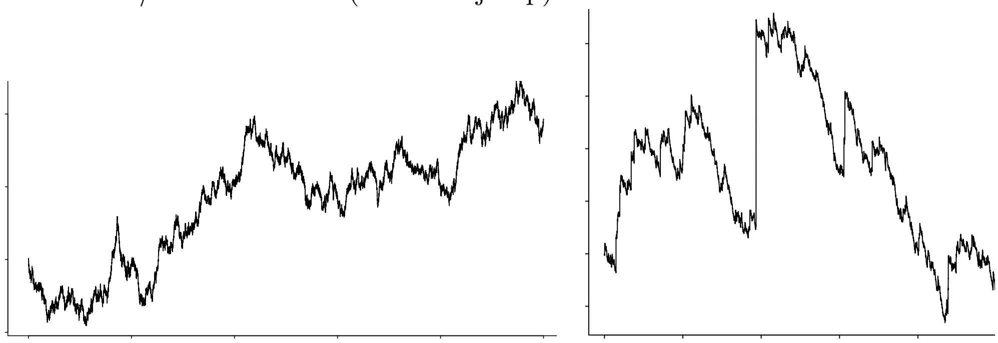
[1]càdlàg means right continuous with left limits, in reference to 'continu à droite et limit''e à gauche' in french.

[2]for the uniform topology on every compact

[3]Here the convergence is in the Skorohorod space of càdlàg functions, with respect to the usual J_1 metric, we won't detail this metric.

Let us say a few words about the limit processes $\mathcal{W}$ and $\mathcal{Z}$.

FIGURE 1. A trajectory of respectively a Wiener process and a Lévy process of order $\alpha = 3/2$ with $A_- = 0$ (no down jump)



The standard **Wiener (or Gaussian) process** $\mathcal{W}$ has continuous trajectories $t \mapsto \mathcal{W}_t$, has independent increments, that $\mathcal{W}_0 = 0$ and that $\mathcal{W}_t - \mathcal{W}_s$ has centered Gaussian distribution with variance $|t-s|$. The **Lévy processes** $\mathcal{Z}$ of order $\alpha \in]1;2[$ are not continuous, but càdlàg with independent centered increments and, for all $s \le t$, $\mathcal{Z}_t - \mathcal{Z}_s$ has same distribution as $|t-s|^{\frac{1}{\alpha}} \mathcal{Z}_1$. Throughout this article, $\mathcal{W} = (t \mapsto \mathcal{W}_t)$ will be a standard Wiener process.

2. Probabilistic limit theorems for dynamical systems

We consider a deterministic dynamics given by the iterations of a map. Thus, the evolution in time is completely determined by the perfect knowledge of the initial state of the system. We assume that this initial state is not perfectly known (e.g. we just know an approximation of it) and that it is chosen randomly. Formally we consider a probability (resp. σ-finite measure) preserving dynamical system, i.e.

- a probability space $(\Omega, \mathcal{F}, \mu)$ (resp. a measurable space $(\Omega, \mathcal{F})$ endowed with a σ-finite non negative measure μ),
- a transformation $T : \Omega \to \Omega$ **preserving the measure** μ, this means that, for all $k \in \mathbb{N}^*$, $\mu(T^{-k}(A)) = \mu(A)$ for any $A \in \mathcal{F}$, i.e. $\int_\Omega f \circ T^k \, d\mu = \int_\Omega f \, d\mu$ for any $f \in \mathbb{L}^1(\mu)$.

Additionally, we consider

- a probability measure $\mathbb{P}$ which admits a density with respect to μ,
- a μ-integrable function $f : \Omega \to \mathbb{R}$ (called **observable**) satisfying $\int_\Omega f \, d\mu = 0$.

We are interested in probabilistic limit theorems for **ergodic sums** (also called **Birkhoff sums**), that is for sums of the form:

$$S_n(f) = \sum_{k=0}^{n-1} f \circ T^k \, .$$

We are interested in limit theorems similar to SLLN and FCLT for $(f \circ T^k)_{k \ge 0}$ instead of $(X_k)_{k \ge 1}$.

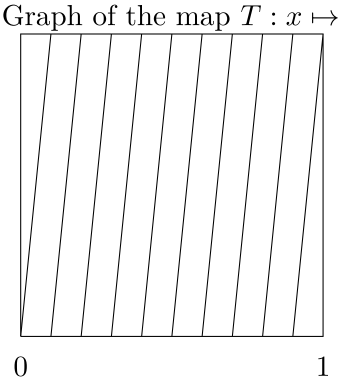
- If μ is a probability measure, the **ergodicity** of the dynamical system $(\Omega, \mathcal{F}, \mu, T)$ means that, for any μ-centered observable $f : \Omega \to \mathbb{R}$, $\left(\frac{S_n(f)}{n}\right)_{n \ge 1}$ converges in distribution to 0 (this follows from the Birkhoff ergodic theorem).
- If μ is σ-finite, the **recurrence ergodicity** of the dynamical system $(\Omega, \mathcal{F}, \mu, T)$ means that, for any μ-centered observable f and any $g \in L^1(\mu)$, $\left(\frac{S_n(f)}{S_n(g)}\right)_{n \ge 1}$ converges in distribution to 0 (this follows from the Hopf ergodic theorem).

- The FCLT cannot hold for any square integrable observable of a same dynamical system (even bounded counterexamples with very various behaviours have been constructed).
- Zweimüller proved in [28] that CLT (or FCLT) for $S_n(f)$ with respect to some probability measure $\mathbb{P}_0$ absolutely continuous with respect to μ implies the CLT (or FCLT) for $S_n(f)$ with respect to any $\mathbb{P}$ absolutely continuous with respect to μ.

3. A simple example of chaotic dynamical system

To enlight how a deterministic map can give rise to chaotic behaviour, we start by presenting a very simple dynamical system modeling independent and identically distributed random variables. The map $T : [0,1[\to [0,1[$ given by $T(x) = 10x \mod 1$ preserves[4] the probability measure μ corresponding to the Lebesgue measure on $[0,1[$: $\int_0^1 f(T(x))\,dx = \int_0^1 f(x)\,dx$.

Figure 2. Graph of the map $T : x \mapsto 10x \mod 1$



- If $f_0(x) = \lfloor 10x \rfloor$ (first decimal digit), the random variables $(X_k := f_0 \circ T^k,\ k \geq 0)$ are i.i.d. with uniform distribution on $\{0,...,9\}$ and so the SLLN and also the following FCLT holds true:
$$\left(t \mapsto \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor -1} (f_0 \circ T^k - 4.5)\right) \xrightarrow[n\to+\infty]{\mathcal{L}} (t \mapsto \|f_0 - 4.5\|_{L^2} \mathcal{W}_t)\ .$$
- More generally this system is ergodic[5], thus the SLLN holds for any integrable function:
$$\forall f \in \mathbb{L}^1(\lambda), \quad \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \xrightarrow[n\to+\infty]{a.s.} \int_0^1 f(x)\,dx\,.$$
Moreover, for any f Hölder continuous and centered with respect to the Lebesgue measure, then the following FCLT holds true[6]:
$$\left(t \mapsto \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor -1} f \circ T^k\right) \xrightarrow[n\to+\infty]{\mathcal{L}} \left(t \mapsto \sqrt{\sigma^2(f)}\mathcal{W}_t\right)\ ,$$
with $\sigma^2(f) := \sum_{n\in\mathbb{Z}} \mathbb{E}_\mu[f.f \circ T^{|n|}] = \lim_{n\to+\infty} \mathbb{E}\left[(S_n(f)/\sqrt{n})^2\right]$. Note that in general $\sigma^2(f) \neq \mathbb{E}[f^2]$ (contrarily to the i.i.d. case). The expression $\sum_{n\in\mathbb{Z}} \mathbb{E}_\mu[f.f \circ T^{|n|}]$ is called

[4] This can be proved by changes of variables, since T is a $\mathcal{C}^1$-diffeomorphism from each $]\frac{k}{10};\frac{k+1}{10}[$ onto $]0;1[$.

[5] The Birkoff theorem ensures the almost sure convergence of $\left(\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k\right)_{n\geq 1}$ to $\mathbb{E}[f|\mathcal{I}]$ where $\mathcal{I}$ is the σ-algebra of invariant measurable sets A, i.e. of measurable sets satisfying $A = T^{-1}(A)$. To prove ergodicity, we may observe that $\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k$ is measurable with respect to the asymptotic σ-algebra $\bigcap_{n\geq 0} \sigma(X_m,\, m \geq k)$ and then conclude using Kolmogorov's 0-1 law.

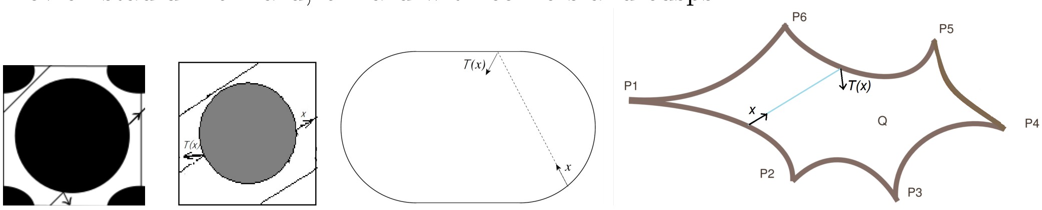
[6] Several strategies of proof exist, one can follow e.g. the strategy roughly presented in Appendix D.

the **Green-Kubo formula**. Moreover, the asymptotic variance $\sigma^2(f)$ (and thus the limit process) is null if and only if f is a **coboundary in** $\mathbb{L}^2(\lambda)$, meaning that there exists $g \in \mathbb{L}^2(\lambda)$ such that $f = g - g \circ T$ almost surely. [7]

4. Limit theorems for smooth observables of chaotic billiards in a bounded domain

Let $Q \subset \mathbb{R}^2$ or $\mathbb{T}^2$. We study the dynamics at collision times of a point particle moving in Q, going straight inside Q and with elastic reflections with ∂Q (reflected angle=incident angle). Let Ω be the set of unit post-collisional vectors. The billiard map $T : \Omega \to \Omega$ maps a post-collisional vector $x = (q, \vec{v})$ to the post-collisional vector $T(x) = (q', \vec{v}')$ at the next collision time. This map T preserves the probability measure μ with density $\rho : (q, \vec{v}) \mapsto \frac{1}{2|\partial Q|} \sin \angle(\mathcal{T}_q \partial Q, \vec{v})$ (with $\mathcal{T}_q \partial Q$ the tangent line to ∂Q at q). We consider the following chaotic models. We refer to the book [8] by Chernov and Markarian for a general reference on chaotic billiards.

Figure 3. Sinai billiard in the torus with finite and with infinite horizon, Bunimovich stadium billiard, billiard with corners and cusps



(1) The domain Q of the **Sinai billiard** (corresponding to the two first pictures of Figure 3) is contained in the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and is given, for some $I \geq 1$, by the complement in the torus of a union of I convex sets $\mathcal{O}_i$ (called **obstacles**) with pairwise disjoint closures, with $\mathcal{C}^3$ boundary and non null curvature. The **horizon of the billiard is said to be finite** if the time between two collisions is uniformly bounded, and is said to be infinite otherwise. The a priori most simple case corresponding to one single obstacle (e.g. second picture of Figure 3) has infinite horizon and is much more complicated to study than the Sinai billiard with finite horizon (e.g. first picture of Figure 3).
(2) The domain Q of the **Bunimovich stadium billiard** has $\mathcal{C}^1$ boundary, is delimited by two semicircles and two segments of positive length (see the third picture of Figure 3).
(3) The domain Q of a **dispersive billiard with corners and cusps** is delimited by a continuous, piecewise $\mathcal{C}^3$, closed curve, the singularity points of which are either corners (i.e. points with two different tangent lines) or cusps, the curvature is assumed to be positive (for the clockwise curvilinear absciss parametrization) except possibly at cusps (see the last picture of Figure 3).

The **ergodicity** has been proved by Sinai in 1970 [22] for the Sinai billiard (rough ideas are given in Appendix A) and by Bunimovich in [4] for the billiard in a stadium. Thus the SLLN holds true. Consider now $f : \Omega \to \mathbb{R}$ Hölder continuous and μ-centered. Different limit behaviour occur depending on the billiard domain.

[7] Additional explanations If $f = g - g \circ T$, then $S_n(f) = g - g \circ T^n$ and $\sigma^2(f) = 0$. Conversely, if $\sigma^2(f) = 0$, we can prove that $(S_n(f))_{n\geq 1}$ is bounded in $\mathbb{L}^2$ and infer that $f = g - g \circ T$ with g a weak limit in $\mathbb{L}^2$ of $\left(\frac{1}{N} \sum_{n=1}^{N} S_n(f)\right)_{n\geq 1}$.

(1) **Standard FCLT for Hölder observables of the Sinai billiard.**
Whether the horizon is finite or not, the sequence of random processes $\left(t \mapsto \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor -1} f \circ T^k\right)_{n\geq 1}$ converges in distribution to $\sigma(f)\mathcal{W}$, with again $\sigma^2(f) = \sum_{m\in\mathbb{Z}} \mathbb{E}_\mu [f.f \circ T^m]$. This result has been proved by Bunimovich and Sinai in 1981 [5] in the finite horizon case (see also [6] and [27]) and by Chernov in 1999 [7] in the infinite horizon case.
We will see in Section 6 that, contrarily to Hölder observables, the asymptotic behaviour of the Birkhoff sums of the free flight function Ψ depends on the finiteness of the horizon.

(2) **Non-standard CLT for Hölder observables of the Bunimovich stadium.**
Bálint and Gouëzel [2] proved that $\left(\frac{\sum_{k=0}^{n-1} f\circ T^k}{\sqrt{n \log n}}\right)_{n\geq 1}$ converges in distribution to a centered Gaussian random variables with standard deviation $\sigma_f = c_Q \int_I f(q,\uparrow)\, dq + \int_J f(q,\downarrow)\, dq$, where I and J are the bottom and top flat segments of the boundary of the stadium.

(3a) **Non-standard CLT for Hölder observables of the dispersing billiard with a single standard cusp.**
In Case (3) with a single cusp at a point P with non vanishing curvature, Bálint, Chernov and Dolgopyat [1] proved also a CLT with non standard normalization and with $\sigma_f := c. \int_{S^1} f(P,\vec{v})\, |\sin\angle(\mathcal{T}_P \partial Q, \vec{v})|^{\frac{1}{2}}\, d\vec{v}$.

(3b) **Convergence to Lévy for Hölder observables of the dispersing billiard with higher order cusps.**
Consider Case (3) with cusps at points P_i with parametrizations $z_{i,\pm}(s) \approx \pm c_{i,\pm} s^{\beta_i}$, with $c_{i,\pm} \geq 0$, $c_{i,+} + c_{i,-} > 0$. Assume $\beta_* := \max_j \beta_j > 2$, set $\alpha := \frac{\beta_*}{\beta_*-1}$. Then the sequence of processes $\left(t \mapsto n^{-\frac{1}{\alpha}} \sum_{k=0}^{\lfloor nt \rfloor -1} f \circ T^k\right)_{n\geq 1}$ converges in distribution[8] to a Lévy process $\mathcal{Z} := \sum_{i\,:\,\beta_i = \max_j \beta_j} \sigma_{f,P_i} \mathcal{Z}^{(i)}$ where $\mathcal{Z}^{(i)}$ are i.i.d. Lévy processes of order α with $\sigma_{f,P_i} := c. \int_{S^1} f(P,\vec{v})\, |\sin\angle(\mathcal{T}_{P_i} \partial Q, \vec{v})|^{\frac{1}{\alpha}}\, d\vec{v}$. The CLT was proved by Jung and Zhang in 2018 in [14], the FCLT by Jung, Zhang and the author [15] and by Melbourne and Varandas [16] (see also [13]).

Heuristic explanations on these CLT or FCLT are given in Appendix B.

5. A billiard in an unbounded domain: the $\mathbb{Z}^2$-periodic Lorentz gas

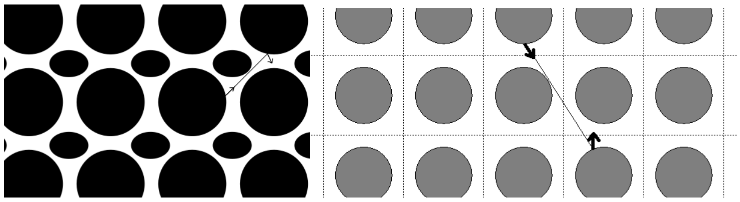
The $\mathbb{Z}^2$-periodic Lorentz gas describes the evolution of a point particle moving between a $\mathbb{Z}^2$-periodic configuration of convex obstacles. More precisely, it corresponds to the billiard system in the unbounded domain $\widetilde{Q} = \mathbb{R}^2 \setminus \bigcup_{i=1}^I \bigcup_{\ell\in\mathbb{Z}^2} \mathcal{O}_i + \ell$. The obstacles $\mathcal{O}_i + \ell$ are assumed to have pairwise disjoint closures, $\mathcal{C}^3$ boundary and non null curvature. The horizon is said to be **finite** if every line of $\mathbb{R}^2$ intersects the boundary of at least one obstacle and is said to be **infinite** otherwise. We will say that the horizon is **2-dimensionally infinite**, if there exist at least two non parallel lines in $\mathbb{R}^2$ meeting no obstacle boundary.

This corresponds to a **dynamical system preserving an infinite measure**:

- We consider $\widetilde{\Omega}$ for the set of all possible post-collisional vectors based on obstacles.
- As previously, we consider the transformation $\widetilde{T} : \widetilde{\Omega} \to \widetilde{\Omega}$ which maps a post-collisional vector to the post-collisional vector corresponding to the next reflection time.

[8] Complements for readers interested in questions of metrics in the Skorohod space: the convergence does not hold with respect to J_1 because when a trajectory enters a cusp where e.g. $f \geq c > 0$ it increases of at least c at each collision in the cusp, and long succession of collisions in the cusp occur. The convergence holds with respect to M_1 if the function has constant sign around cusps and in some other situations, and with respect to M_2 in any case (we refer to [26] for a presentation of the J_1, M_1 and M_2 metrics).

FIGURE 4. $\mathbb{Z}^2$-periodic Lorentz gas with respectively finite and with 2-dimensionally infinite horizon



- The map $\widetilde{T}$ preserves the infinite measure $\widetilde{\mu}$ with density proportional to $(q,\vec{v}) \mapsto \sin\angle(\mathcal{T}_q\partial\widetilde{Q},\vec{v})$, normalized so that $\widetilde{\mu}\left(\bigcup_{i=1}^I \partial O_i\right) = 1$, where $\mathcal{C}_\ell$, called the **ℓ-th cell**, is the set of $\widetilde{x} = (\widetilde{q},\vec{v}) \in \widetilde{\Omega}$ such that $\widetilde{q} \in \bigcup_{i=1}^I \mathcal{O}_i + \ell$.

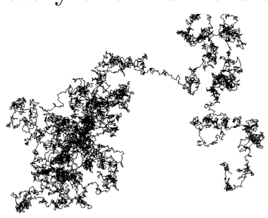
This dynamical system is **strongly related to the Sinai billiard**:

- Let $\mathfrak{p} : \mathbb{R}^2 \to \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the canonical projection. We consider the Sinai billiard (Ω,μ,T) in the domain $Q = \mathbb{T}^2 \setminus \bigcup_{i=1}^I \mathfrak{p}(\mathcal{O}_i)$. The dynamics of this **Sinai billiard corresponds to the dynamics of the Lorentz gas modulo $\mathbb{Z}^2$ for the position**, i.e. $T \circ \mathfrak{P} = \mathfrak{P} \circ \widetilde{T}$, with $\mathfrak{P}$ the projection given by $\mathfrak{P}(q,\vec{v}) = (\mathfrak{p}(q),\vec{v})$.
- Conversely, the Lorentz gas $(\widetilde{\Omega},\widetilde{T},\widetilde{\mu})$ can be modeled using the Sinai billiard, by identifying $(x = (q,\vec{v}),\ell) \in \Omega \times \mathbb{Z}^2$ with $(\widetilde{q},\vec{v}) \in \widetilde{\Omega}$ where $\mathfrak{p}(\widetilde{q}) = q$ and $\widetilde{q} \in \bigcup_{i=1}^\ell \partial\mathcal{O}_i + \ell$. The $\mathbb{Z}^2$-periodicity ensures that, with this representation, $\widetilde{T} : (\widetilde{q},\vec{v}) \mapsto (\widetilde{q}',\vec{v})$ corresponds to:
$$(x,\ell) \mapsto (T(x), \ell + \Phi(x)) , \quad \text{for some } \Phi : \Omega \to \mathbb{Z}^2 .$$
In other terms, the **Lorentz gas** $(\widetilde{\Omega},\widetilde{T},\widetilde{\mu})$ **can be represented by the $\mathbb{Z}^2$-extension over the Sinai billiard** (Ω,T,μ) by some $\Phi : \Omega \to \mathbb{Z}^2$.

6. Periodic Lorentz gas: Limit theorem for the position

We choose the initial position and direction randomly with respect to some probability measure $\mathbb{P}$ absolutely continuous with respect to the Lebesgue measure. We are interested in the asymptotic behaviour of the position q_n at the n-th collision. We observe that, if the initial state is $(x = (q,\vec{v}),\ell) \in \Omega \times \mathbb{Z}^2$, then the corresponding $q_n = \ell + \sum_{k=0}^{n-1} \Psi(T^k(x))$, for some $\Psi : \Omega \to \mathbb{R}^2$ called the free flight. We will see that this quantity is asymptotically Gaussian with either a standard or nonstandard normalization, depending whether the horizon is finite or infinite. More precisely the behaviour is the following:

FIGURE 5. A trajectory of a 2-dimensional Wiener process



- <u>If the horizon is finite</u>, then Ψ is centered, bounded, piecewise $\frac{1}{2}$-Hölder continuous and satisfies a standard FCLT as the Hölder functions studied in Case (1) of Section 3:

$\left(t \mapsto \frac{q_{\lfloor nt \rfloor}}{\sqrt{n}}\right)_{n\ge 1}$ converges in distribution[9] to $\Sigma(\Psi)W$ where W is a standard two-dimensional Wiener process[10] and where $\Sigma(\Psi)$ is the nonnegative symmetric matrix square root of $\Sigma^2(\Psi) := \left(\sum_{m\in\mathbb{Z}} \mathbb{E}[\psi_i.\psi_j \circ T^m]\right)_{i,j=1,2}$ (Green-Kubo formula for Ψ).
- If the horizon is infinite, then Ψ is not in $\mathbb{L}^2(\mu)$, but centered and $\mu(|\Psi| > t) \sim At^2$. Szász and Varjú proved in 2007 [25] that $\left(\frac{q_n}{\sqrt{n\log n}}\right)_{n\ge 1}$ converges in distribution to a Gaussian random variable with variance matric Σ^2 expressed explicitely in terms of the width and the periodical length of the corridors of parallel lines meeting no obstacle.

The same limit theorems holds true for $(S_n\Phi)_{n\ge1}$ since $\sup_{n\ge1} \|S_n(\Phi) - S_n(\Psi)\|_\infty < \infty$.

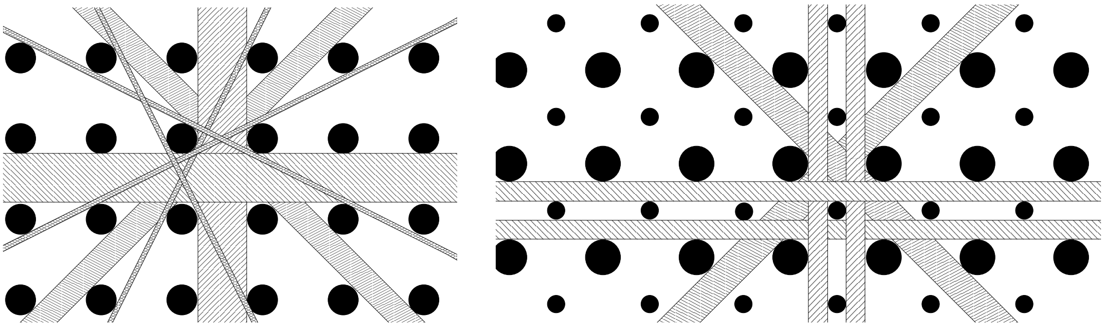


FIGURE 6. Corridors for two different periodic billiard domains (from [19])

7. Periodic Lorentz gas: study of return times

Let τ be the **return time** (i.e. number of collisions before coming back) **to the initial cell**. Let τ_ε be the **return time to the ε-neighbourhood of the initial state** (recall that a state is a couple position-direction).
Note that the function $\tau(q+\ell, \vec{v}) = \tau(q, \vec{v})$ $\tau_\varepsilon(q+\ell, \vec{v}) = \tau_\varepsilon(q, \vec{v})$ for any $\ell \in \mathbb{Z}^2$. So we identify them with their quotient defined on the Sinai billiard (Ω, μ, T).

- Lorentz gas with finite horizon.
  - The fact that $\tau < \infty$ a.e. (i.e. the recurrence of this Lorentz gas) follows from the standard CLT for the cell-change function Φ combined with a general argument (for $\mathbb{Z}^2$-extensions) by Conze [9], extended by Schmidt [23].
    Another proof of the recurrence has been given by Szász and Varjú [24]. This proof uses **Local limit Theorem (LLT) type estimates** of the form $\mu(S_n(\Phi) = 0) \sim c_0/n$ (not summable) and $\mu(S_n(\Phi) = S_{n+m}(\Phi) = 0) \sim c_0^2/(nm)$ combined with a Borel-Cantelli type argument (Lamperti's lemma).
  - Dolgopyat, Szász and Varjú proved in 2008 [10] that $\mu(\tau > N) \sim_{N\to\infty} \frac{1}{c_0 \log N}$.
  - The author and Saussol proved in 2010 [18] that $\mu(\tau_\varepsilon > e^{\frac{t}{4\varepsilon^2\rho(\cdot)}}) \xrightarrow[\varepsilon\to 0]{} \frac{1}{1+c_0 t}$, ensuring the convergence in distribution of $(\varepsilon^2 \log \tau_\varepsilon)_{\varepsilon>0}$ when $\varepsilon \to 0$ to some random variable, showing that the return times to small sets are very long.
- Lorentz gas with 2-dimensionally infinite horizon.

[9]with respect to the infinite norm

[10]A standard two-dimensional Wiener process is a process $t \mapsto (W_t^{(1)}, W_t^{(2)})$ where $W^{(1)}$ and $W^{(2)}$ are two independent standard Wiener processes

- The fact that this Lorentz gas is recurrent has been proved by Szász and Varjú in [25] using again LLT type estimates: $\mu(S_n\Phi=0)\sim c_0/(n\log n)$ and $\mu(S_n(\Phi)=S_{n+m}(\Phi)=0)\sim c_0^2/(n\log(n)m\log(m))$. The argument by Conze or Schmidt does not apply directly to this context. The question whether the CLT with nonstandard normalization for Φ implies or not the recurrence is still open.
- The author and Terhesiu proved in [19] that $\mu(\tau>N)\sim_{N\to\infty}\frac{1}{c_0\log\log N}$.

Note that, both in finite and infinite horizon, the above estimates of $\mu(\tau>N)$ can be rewritten:

$$\mu\left(\tau>N\right)\sim_{N\to+\infty}\frac{1}{\sum_{n=0}^{N}\mu(S_n(\Phi)=0)}=\frac{1}{\mathbb{E}_\mu[\mathcal{N}_N]}\,,$$

where $\mathcal{N}_N$ is the number of visits to $(S_n(\Phi))_{n\ge0}$ to 0, i.e. the number of visits to 0-cell before the N-th collision considering the particle starts from the 0-cell, and

$$\mu(S_n(\Phi)=0)\sim\frac{c_0}{\mathfrak{a}_n^2}\,,$$

where $\mathfrak{a}_n$ is the normalization of $S_n(\Phi)$ in the CLT: $\mathfrak{a}_n=\sqrt{n}$ if the horizon is finite and $\mathfrak{a}_n=\sqrt{n\log n}$ if the horizon is infinite (the presence of a square above in $\mathfrak{a}_n^2$ comes from the fact that Φ is 2-dimensional). See Appendix C for a presentation of the proof of these results.

8. Pinball in finite horizon

In the Lorentz gas with **finite horizon**, we assume that the point particle wins $\beta_{\mathcal{O}}$ each time it hits the obstacle $\mathcal{O}$. Let Z_n be the amount won up to time n. This random variable Z_n is a Birkhoff sum $\sum_{k=0}^{n-1}\widetilde{f}\circ\widetilde{T}^k$ for the Lorentz gas system $\left(\widetilde{\Omega},\widetilde{\mu},\widetilde{T}\right)$. We consider the following cases:

(a) $\mathbb{Z}^2$-periodic values. Assume $\beta_{\mathcal{O}_j+\ell}=\beta_{\mathcal{O}_j}$ for all $\ell\in\mathbb{Z}^2$. Then Z_n can be expressed as a Birkhoff sum $S_n(f)$ for the Sinai billiard and, applying the results of Section 4, we obtain that $(Z_n/n)_{n\ge1}$ converges almost surely to $I_0(\beta):=\sum_{j=1}^{J}\beta_{\mathcal{O}_j}$, and that, if $I_0(\beta)=0$, then $(Z_n/\sqrt{n})_{n\ge1}$ converges in distribution to $\sqrt{\sigma^2(\beta)}\mathcal{W}_1$.

(b) Summable Values: Assume $\sum_{\mathcal{O}}|\beta_{\mathcal{O}}|<\infty$ and set $I(\beta):=\sum_{\mathcal{O}}\beta_{\mathcal{O}}$. Then
  (b1) Dolgopyat, Szász and Varjú proved in [10] that $\left(\frac{Z_n}{\log n}\right)_{n\ge1}$ converges in distribution to $c_0I(\beta)\mathcal{E}$, with $\mathcal{E}$ an exponential random variable with mean 1.
  (b2) The author and Thomine proved in [20, 21] that if $I(\beta)=0$ and if there exists $\eta>0$ such that $\sum_{\mathcal{O}}d(0,\mathcal{O})^\eta|\beta_{\mathcal{O}}|<\infty$, then $\left(\frac{Z_n}{\sqrt{\log n}}\right)_{n\ge1}$ converges in distribution to $\sqrt{\widetilde{\sigma}^2(\beta)c_0\mathcal{E}}\mathcal{W}_1$, with $\mathcal{E}$ as above independent of $\mathcal{W}_1$, where $\widetilde{\sigma}^2(\beta)$ is given by the Green Kubo formula with respect to $\widetilde{T}$.

(c) i.i.d. values (see [17]) If the $\beta_{\mathcal{O}}$ are i.i.d. centered and square integrable and independent of the Lorentz gas, then $\left(\frac{Z_n}{\sqrt{n\log n}}\right)_{n\ge1}$ converges in distribution to $c_4\mathcal{W}_1$, where c_4 depends on the common distribution of $\beta_{\mathcal{O}}$.

Ideas of proofs for (b-c):
We prove the convergence with respect to the measure $\widetilde{\mu}_{|\mathcal{C}_0}$ ($\widetilde{\mu}$ restricted to the 0-cell) and conclude, by a result by Zweimüller [28], the convergence with respect to any probability measure $\mathbb{P}$ absolutely continuous with respect to the Lebesgue measure on $\widetilde{\Omega}$.

- For (b1): Since the Lorentz gas is recurrent ergodic[11], the Hopf ergodic theorem ensures that $Z_n/\mathcal{N}_n\xrightarrow[n\to+\infty]{a.e.}\sum\beta_{\mathcal{O}}$, with $\mathcal{N}_n:=\sum_{k=0}^{n-1}\mathbf{1}_{S_k(\Phi)=0}$ the number of visits to 0-cell before

[11] Recurrence ergodicity follows from recurrence combined with the argument presented in Appendix A.

time n. So it is enough to prove the convergence result for $\mathcal{N}_n$ instead of Z_n. This can be done by proving the convergence of every moment. The moment of order m of $\mathcal{N}_n$ is $\sum_{k_1,...,k_m=0}^{n-1} \mu(S_{k_1}(\Phi) = ... = S_{k_m}(\Phi) = 0)$, which can be estimated by using a multi-time local limit theorem.

- For (b2): we can prove the convergence of every moment of Z_n with respect to $\widetilde{\mu}_{|\mathcal{C}_0}$ (this is doable but more difficult than for (b1) since cancellations happens).
  another argument: Note that $Z_n \approx \sum_{k=0}^{\mathcal{N}_n} Y_k$, Y_k being the amount won between the k-th and $(k+1)$-th visit to the 0- cell. Prove, via coupling, that it behaves as if the Y_k and $\mathcal{N}_n$ were independent and so, roughly speaking using the CLT for $(Y_k)_k$ and (a2) for $\mathcal{N}_n$:
$$Z_n \approx \sum_{k=0}^{\mathcal{N}_n} Y_k \approx \sqrt{\mathcal{N}_n \widetilde{\sigma}^2(\beta)}\mathcal{W}_1 \approx \sqrt{\log n c_0 \mathcal{E} \widetilde{\sigma}^2(\beta)}\mathcal{W}_1 \,.$$
- For (c): Z_n behaves as a random walk in random scenery $\sum_{k=0}^{n-1} \zeta_{X_1+...+X_k}$: we can adapt Bolthausen's proof [3], with the use of the mixing local limit theorem (see Appendix D for a proof of the mixing local limit theorem which appears also in Appendix C).

Appendix A. Scheme of Sinai's proof of the ergodicity of the Sinai billiard via hyperbolicity via Hopf's chains

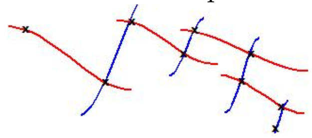
A.1. **Hyperbolicity and Hopf's chains.** Recall that the states space of the billiard is the set Ω of unit post-collisional vectors $x = (q, \vec{v})$ with $q \in \partial Q = \bigcup_{i=1}^{I} \partial \mathcal{O}_i$ and $\vec{v} \in \mathbb{S}^1$ such that $\langle \vec{n}_q, \vec{v} \rangle \geq 0$, where $\vec{n}_q$ is the inward unit normal vector to ∂Q at q. This space is two-dimensional (one dimension for the position in ∂Q, one dimension for the direction). A main difficulty in the study of the Sinai billiard comes from the fact that the billiard map T is discontinuous $x = (q, \vec{v})$ such that $T(x)$ is tangent to an obstacle. But this system enjoys hyperbolicity:

- the action of T (resp. T^{-1}) expands the length of the increasing (resp. decreasing) $\mathcal{C}^1$-curve of Ω, increasing (resp. decreasing) meaning that the angle $\angle(\vec{n}_q, \vec{v})$ increases (resp. decreases) with the counter-clockwise curvilinear absciss.

Luckily, hyperbolicity wins against discontinuities. The map T is **uniformly hyperbolic**: there exist $C > 0$, $\Lambda > 1$ such that, for μ-almost every $x \in \Omega$, there exist two $\mathcal{C}^1$-curves $\gamma^{(s)}(x)$ and $\gamma^{(u)}(x)$, called respectively **stable and unstable manifold**, containing x and satisfying
$$\forall n \in \mathbb{N}, \quad \mathrm{diam}(T^n(\gamma^{(s)}(x)) + \mathrm{diam}(T^{-n}(\gamma^{(u)}(x)) \leq \Lambda^{-n} \,.$$
These stable and unstable manifolds do not exist everywhere, can be arbitrarily small, but satisfy the nice following property enabling the adaptation of Hopf's argument[12] to the billiard context: For any measurable set $\Omega_0 \subset \Omega$ of full measure, there exists a set Ω_0' of full measure such that, for every $i = 1, ..., I$, every $x, y \in \Omega_0'$ belonging to the connected set $\Omega_i := \{(q, \vec{v}) \in \Omega,\ q \in \partial \mathcal{O}_i\}$, there exists a **Hopf's chain** (or zig-zag line), i.e. $x_0, ..., x_n \in \Omega_0'$ such that:

- $x_0 = x$, $x_N = y$,
- $\forall j = 1, ..., N-1$, $x_j \in \Omega_0' \cap \gamma^{(s)}(x_{j-1}) \cap \gamma^{(u)}(x_{j+1})$ or $x_j \in \Omega_0' \cap \gamma^{(u)}(x_{j-1}) \cap \gamma^{(s)}(x_{j+1})$,

Figure 7. A Hopf's chain



[12]coming from Hopf's proof of ergodicity of the geodesic flow on a negatively curved finite volume surface.

A.2. **Scheme of the proof of the ergodicity.**

- Let $f \in L^1(\mu)$ Hölder continuous. By Birkhoff's ergodic theorem, there exists $\Omega_0 \subset \Omega$ so that $\mu(\Omega \setminus \Omega_0) = 0$ and, for every $x \in \Omega_0$, $\frac{\sum_{k=0}^{n-1} f \circ T^k(x)}{n} \underset{|n|\to+\infty}{\longrightarrow} h(x) := \mathbb{E}_\mu [f|\mathcal{I}](x)$.
- Prove that h is constant on stable and unstable manifolds, and T- and T^{-1}-invariant.
- Infer, by the above Hopf argument, that h is μ-a.e. constant on each Ω_i.
- Prove that, for any couple of connected components (Ω_i, Ω_j) of Ω, there exists $m \in \mathbb{Z}$ such that $\mu(\Omega_i \cap T^{-m}\Omega_j) > 0$.
- Conclude that h is constant almost everywhere and so that $h = \mathbb{E}_\mu[f]$.

This proof can be adapted to prove the recurrence ergodicity of the Lorentz gas, up to:

- replacing Birkhoff's theorem by Hopf's ergodic theorem which ensures that, since the Lorentz gas is recurrent, for every $f, g \in L^1(\widetilde{\mu})$, $g > 0$, $\frac{\sum_{k=0}^{n-1} f \circ \widetilde{T}^k}{\sum_{k=0}^{n-1} g \circ \widetilde{T}^k} \underset{|n|\to+\infty}{\overset{a.e.}{\longrightarrow}} h := \mathbb{E}_{g\widetilde{\mu}}\left[\frac{f}{g}\middle|\mathcal{I}\right]$,
- taking f Hölder continuous compactly supported, $g \in L^1_{>0}(\widetilde{\mu})$ constant on each obstacle.

Appendix B. Heuristic explanations of the Limit theorems for smooth observables of billiards in bounded domains

Case (1) (Sinai billiard) is as chaotic as the example of Section 3 and the method explained in Appendix D can be implemented to prove the standard CLT. In cases (2) (Bunimovich stadium) and (3) (billiard with cusps), the non-standard behaviour comes from the following facts:

- Let $\mathcal{A}$ be the set of states $(q, \vec{v})$ with position q belonging: to $I \cup J$ for (2), to a neighbourhood of cusps for (3).
- The dynamics outside $\mathcal{A}$ is in some sense as chaotic as the example of Section 3.
- The number $\mathcal{R}$ of collisions during an excursion in $\mathcal{A}$ is not $\mathbb{L}^2$ and satisfies $\mu(\mathcal{R} > x) \sim \mathfrak{c}x^{-\alpha}$ for some $\mathfrak{c} > 0$ (with α as defined in Case (3b) or with $\alpha = 2$ in Cases (2) and (3a)). This explains heuristically the type of CLT.
- For (2), during a trajectory of length N out of semi-disks, the direction is very close to vertical and the successive positions form a $\mathcal{O}(N^{-1})$-packing of $I \cup J$.
- For (3), during a trajectory of length N in a neighbourhood of a cusp at P, the position is very close to P and the successive directions form essentially a $\mathcal{O}(N^{-1})$-packing of $\mathbb{S}^1$ (with, on each side of the cusp, angular increments of $\vec{v}$ of size $N^{-1} (\sin \angle(\mathcal{T}_P \partial Q, \vec{v}))^{-\frac{1}{\alpha}}$).

The two last points explain heuristically the integral appearing in the limit.

Appendix C. Ideas behind the proofs of quantitative recurrence

The first ingredient is a mixing local limit theorem saying roughly speaking that

$$\mu\left(A \cap \{S_n(\Phi) = 0\} \cap T^{-n}(B)\right) \approx \mu(A) \frac{c_0}{\mathfrak{a}_n^2} \mu(B)\,,$$

with $\mathfrak{a}_n = \sqrt{n}$ if the horizon is finite and $\mathfrak{a}_n = \sqrt{n \log(n)}$ if the horizon is infinite.

Then, the idea consists in adapting the following argument by Dvoretzki and Erdös in 1951 [11] for planar random walks (i.e. sums of i.i.d. random variables in $\mathbb{Z}^2$): considering the last visit time n to the 0-cell before time M and applying the mixing local limit theorem, we obtain

$$1 = \sum_{n=1}^{M} \mu\left(S_n(\Phi) = 0,\ \tau > M - n\right) \approx \sum_{n=1}^{N} \frac{c_0}{\mathfrak{a}_n^2} \mu\left(\tau > M - n\right)\,.$$

Applying this with $M = N$ (first inequality below) and $M = N\lfloor \log N \rfloor$ (second inequality below) and using the decreasingness of $n \mapsto \mu(\tau > n)$, we infer that

$$\sum_{n=1}^{N} \frac{c_0}{\mathfrak{a}_n^2}\mu(\tau > N) \lesssim 1 \lesssim \sum_{n=1}^{N\lfloor \log N\rfloor - N - 1} \frac{c_0}{\mathfrak{a}_n^2}\mu(\tau > N) \approx \sum_{n=1}^{N} \frac{c_0}{\mathfrak{a}_n^2}\mu(\tau > N)\ ,$$

where we used also the fact that $\mathfrak{a}_n^2 = n$ or $\mathfrak{a}_n^2 = n \log n$. The estimate of $\mu(\tau_\varepsilon > ...)$ uses the same idea (we take for A atoms of a finer and finer partition).

Appendix D. Proof of probabilistic limit theorems using operators

We present here an important tool behind most of the results stated in this article: the study of perturbation of quasi-compact operators. For a detailed and rigourous presentation of this method, we refer to [12] and the references therein.

(1) Consider P the dual of $g \mapsto g \circ T$ (P is called the **transfer operator** of (Ω, μ, T)): $\int_\Omega P(h).g\, d\mu = \int_\Omega h.g \circ T\, d\mu$.

(2) Prove that P is **quasi-compact with only and simple dominating eigenvalue 1**: $P^n(h) = \int_\Omega h\, d\mu + \mathcal{O}(e^{-an})$ in $\mathcal{L}(\mathcal{B})$ for some *nice* complex Banach space $\mathcal{B}$ and $a > 0$. If we cannot work directly with (Ω, μ, T), we may use auxiliary dynamical systems (In [27] and [7] Young towers are constructed for the Sinai billiard adapted to this purpose).

(3) Set $P_t(h) = P(e^{i\langle t, \Phi\rangle} h)$ and use characteristic functions. Observe that

$$\mathbb{E}_\mu\left[\mathbf{1}_A e^{i\langle t, \frac{S_n(\Phi)}{a_n}\rangle} \mathbf{1}_B \circ T^n\right] = \mathbb{E}_\mu[\mathbf{1}_B P^n_{t/a_n}(\mathbf{1}_A)]\,.$$

(4) Deduce from the quasi-compactness of P, by **spectral perturbation method**, that $P_t^n = \lambda_t^n \Pi_t(\cdot) + \mathcal{O}(\theta_0^n)$ in $\mathcal{L}(\mathcal{B})$ with $\lim_{t\to 0} \lambda_t = 1$ in $\mathbb{C}$, with either $\lim_{t\to 0} \|\Pi_t - \mathbb{E}_\mu[\cdot]\mathbf{1}\|_{\mathcal{L}(\mathcal{B})}$ (possible if the horizon is finite) or $\lim_{t\to 0} \|\Pi_t - \mathbb{E}_\mu[\cdot]\mathbf{1}\|_{\mathcal{L}(\mathcal{B}, L^1(\mu))}$ (if the horizon is infinite) and conclude that

$$\mathbb{E}_\mu\left[\mathbf{1}_A e^{i\langle t, \frac{S_n(\Phi)}{\mathfrak{a}_n}\rangle} \mathbf{1}_B \circ T^n\right] \approx \lambda^n_{t/\mathfrak{a}_n} \mathbb{E}_\mu\left[\mathbf{1}_B \Pi_{t/\mathfrak{a}_n}(\mathbf{1}_A)\right] \sim \lambda^n_{t/\mathfrak{a}_n}\mu(B)\mu(A)\,.$$

(5) If $\lambda_u \sim_{u\to 0} e^{-\frac{|\Sigma u|_2^2}{2}}$, then $\lambda^n_{t/\sqrt{n}} \sim_{n\to+\infty} e^{-\frac{|\Sigma t|_2^2}{2}}$ and so $\left(\frac{S_n(\Phi)}{\sqrt{n}}\right)_{n\ge 1}$ converges in distribution to ΣW (applies e.g. to Sinai billiard with finite horizon).

(6) If $\lambda_u \sim_{u\to 0} e^{-|\Sigma u|_2^2 |\log(|u|_2)|}$, then $\lambda^n_{t/\sqrt{n\log(n)}} \sim_{n\to+\infty} e^{-\frac{|\Sigma t|_2^2}{2}}$ and so $\left(\frac{S_n(\Phi)}{\sqrt{n\log(n)}}\right)_{n\ge 1}$ converges in distribution to ΣW (applies e.g. to Sinai billiard with infinite horizon).

(7) Proof of the mixing local limit theorem using point (4) above and $\lambda^n_{t/\mathfrak{a}_n} \sim e^{-\frac{|\Sigma t|_2^2}{2}}$:

$$\mu\left(A \cap \{S_n(\Phi) = 0\} \cap T^{-n}(B)\right) = \frac{1}{(2\pi)^2}\int_{[-\pi,\pi[^2} \mathbb{E}_\mu\left[\mathbf{1}_A e^{i\langle u, S_n(\Phi)\rangle} \mathbf{1}_B \circ T^n\right] du$$

$$= \frac{1}{(2\pi\mathfrak{a}_n)^2}\int_{[-\pi\mathfrak{a}_n,\pi\mathfrak{a}_n[^2} \mathbb{E}_\mu\left[\mathbf{1}_A e^{i\langle \frac{t}{\mathfrak{a}_n}, S_n(\Phi)\rangle} \mathbf{1}_B \circ T^n\right] dt \approx \frac{\mu(A)\mu(B)}{(2\pi\mathfrak{a}_n)^2}\int_{\mathbb{R}^2} \lambda^n_{t/\mathfrak{a}_n}\, dt$$

$$\approx \frac{\mu(A)\mu(B)}{(2\pi\mathfrak{a}_n)^2}\int_{\mathbb{R}^2} e^{-\frac{|\Sigma t|_2^2}{2}}\, dt \quad \approx \frac{\mu(A)\mu(B)}{\sqrt{2\pi\det(\Sigma)}\mathfrak{a}_n^2} = \frac{c_0}{\mathfrak{a}_n^2}\mu(A)\mu(B)\,.$$

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