# LIMIT THEOREMS FOR BIRKHOFF SUMS AND LOCAL TIMES OF THE PERIODIC LORENTZ GAS WITH INFINITE HORIZON

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Abstract. This work is a contribution to the study of the ergodic and stochastic properties of $\mathbb{Z}^d$-periodic dynamical systems preserving an infinite measure. We establish functional limit theorems for natural Birkhoff sums related to local times of the $\mathbb{Z}^d$-periodic Lorentz gas with infinite horizon, for both the collision map and the flow. In particular, our results apply to the difference between the numbers of collisions in two different cells. Because of the $\mathbb{Z}^d$-periodicity of the model we are interested in, these Birkhoff sums can be rewritten as additive functionals of a Birkhoff sum of the Sinaĭ billiard. For completness and in view of future studies, we state a general result of convergence of additive functionals of Birkhoff sums of chaotic probability preserving dynamical systems under general assumptions.

## Introduction

Let $d \in \{1, 2\}$. We are interested in the stochastic behaviour of The $\mathbb{Z}^d$-periodic Lorentz gas. We recall that this model has been introduced in [28] as a naive model to discribe the behaviour of an electron moving in a weakly conductor metal. This model is a particular case of chaotic billiard systems preserving an infinite measure.

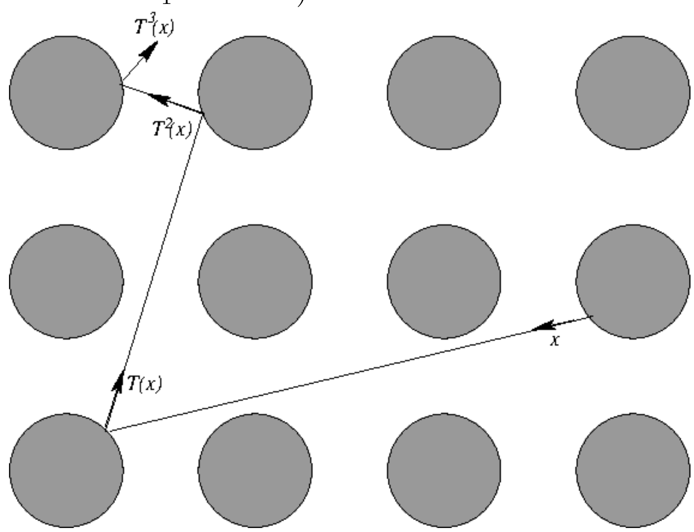
**Billiards.** The billiard systems we are interested in model the displacement of a point particle moving at unit speed in some domain $Q$, going straight inside the domain and enjoying elastic collisions off the boundary of the domain. It is then natural to study both the dynamics in continuous time described by the billiard flow, as well as the dynamics at collision times described by the billiard map. Both for the billiard flow and billiard map, a state is a couple $(q, \vec{v})$ made of a position $q$ and a unit velocity vector $\vec{v} \in \mathbb{S}^1$.
For the billiard flow, the positions $q$ are taken in the domain and we identify, at collision times, pre-collisional and post-collisional vectors. The flow $Y_t$ then maps a state of a particle at time 0 to the state of the same particle at time $t$. The flow $(Y_t)_t$ preserves the Lebesgue measure on the space of states. This measure is finite or infinite if the area of the billiard domain is so.
For the billiard map, the states are the couples of a position on the boundary of an obstacle and of a unit post-collisional vector. The billiard map maps a state at a collision time to the state at the next collision time. This map preserves an explicit measure absolutely continuous with respect to the Lebesgue measure on the space of states. This measure is finite or infinite if the length of the boundary of the billiard domain is so.

**$\mathbb{Z}^d$-periodic Lorentz gas.** The $\mathbb{Z}^2$-periodic Lorentz gas is the billiard system in a domain $Q$ obtained from $\mathbb{R}^2$ by removing a finite number of obstacles $\mathcal{O}_1, ..., \mathcal{O}_I$ (with $I \in \mathbb{N}^*$) and also all their translates $\mathcal{O}_i + \ell$ with $\ell \in \mathbb{Z}^2$.
Analogously, the $\mathbb{Z}$-periodic Lorentz gas is the billiard system in a domain $Q$ obtained from $\mathbb{R} \times \mathbb{T}$ by removing a finite number of obstacles $\mathcal{O}_1, ..., \mathcal{O}_I$ and their translates by $\mathbb{Z}$.

In both cases, for $d \in \{1,2\}$, the obstacles $\mathcal{O}_i + \ell$, $i = 1, ..., I$, $\ell \in \mathbb{Z}^d$ are assumed to be open, convex, with boundary $\mathcal{C}^3$, non null curvature and with pairwise disjoint closures.

The **horizon** of the $\mathbb{Z}^d$-periodic Lorentz gas **is said to be finite** if every line touches at least an obstacle, then the horizon is bounded, meaning that the distance of a trajectory between two collisions is uniformly bounded. **The horizon is said to be infinite** if there exists at least a line touching no obstacle. Whereas some results are now known to be true both in finite and infinite horizon, some quantities have very different behaviours depending on the finiteness or infiniteness of the horizon. Lots of questions, such as the one investigated in the present article, are even more challenging in the infinite horizon case.

FIGURE 1. Beginning of a trajectory of the $\mathbb{Z}^2$-periodic Lorentz gas map, with infinite horizon and with circular obstacles centered at points with integer coordinates (here $I = 1$ and $\mathcal{O}_1$ is a disk).




We will write $(Y_t)_{t\in\mathbb{R}}$ for the $\mathbb{Z}^d$-periodic Lorentz gas flow, and $T$ for the $\mathbb{Z}^d$-periodic gas map. We write $\mathcal{M}$ and $M$ for the set of states respectively for the flow and for the map. Finally, we write $\mathfrak{m}$ for the Lebesgue measure on $\mathcal{M}$, and $\mu$ for the $T$-invariant measure absolutely continuous with respect to the Lebesgue measure on $M$ normalized so that the measure of the set of states $(q, \vec{v}) \in M$ with position $q \in [0;1[^2$ is equal to 1.

**Main results: Limit Theorem for Birkhoff sums in the infinite horizon case.** The goal of the present article is to investigate the behaviour of ergodic sums for integrable observables of the $\mathbb{Z}^d$-periodic Lorentz gas in infinite horizon. We assume that the horizon is $d$-dimensionally infinite in the sense that there exist at least $d$ non parallel unbounded lines touching no obstacle. For any $\ell \in \mathbb{Z}^d$, we write $\mathcal{C}_\ell$ and call $\ell$-th cell the set of states with position belonging to $\bigcup_{i=1}^I (\mathcal{O}_i + \ell)$. We will restrict our study to the case of observables depending only on the cell label. We consider a couple $(g, f)$ of such integrable observables of $M$ such that the second coordinate satisfies

$$\int_M f \, d\mu = 0 \,,$$

and some extra integrability assumptions. Let us notice that the Birkhoff sums of observables depending only on the cell label are related to local time (also called occupation time) in the cells. If $g = \mathbf{1}_{\mathcal{C}_0}$, then the Birkhoff sum

$$\sum_{k=0}^{n-1} g \circ T^k$$

corresponds to the local time up to time $n$ in the 0-cell (i.e. the time spent in the 0-cell until the $n$-th collision). If e.g. $f = \mathbf{1}_{\mathcal{C}_b} - \mathbf{1}_{\mathcal{C}_a}$, then the Birkhoff sum

$$\sum_{k=0}^{n-1} f \circ T^k$$

corresponds to the difference between the local time in the $b$-th cell and in the $a$-th cell. We prove the convergence in distribution of families of the following quantities

$$\left(\frac{\sum_{k=0}^{n-1} g \circ T^k}{\mathfrak{A}_n}, \frac{\sum_{k=0}^{n-1} f \circ T^k}{\sqrt{\mathfrak{A}_n}}\right)_{n\in\mathbb{N}^*}$$

where $\mathfrak{A}_n := \sum_{k=1}^{n} \mathfrak{a}_k^{-d}$ with $\mathfrak{a}_k := \max(1, \sqrt{k \log k})$. Observe that

$$\mathfrak{A}_n \sim 2\sqrt{\frac{n}{\log n}} \quad \text{if} \quad d = 1\,, \quad \text{and} \quad \mathfrak{A}_n \sim \log\log n \quad \text{if} \quad d = 2\,.$$

This question is related with the asymptotic behaviour of additive functional of a Birkhoff sum of the Sinaĭ billiard. Indeed, in the general case, let us consider the sequence $(\beta_\ell)_{\ell\in\mathbb{Z}^d}$ such that $f = \beta_\ell$ on $\mathcal{C}_\ell$. Then, on the set $\mathcal{C}_0$, the Birkhoff sum $\sum_{k=0}^{n-1} f \circ T^k$ corresponds to

$$\sum_{k=0}^{n-1} \beta_{\overline{S}_k}\,,$$

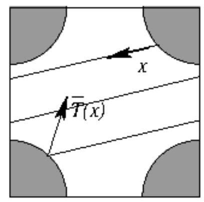
where $\overline{S}_k$ is the label of the cell $\mathcal{C}_{\overline{S}_k}$ containing $T^k(\cdot)$ for a particle starting from the 0-cell. As a consequence of a functional version of the above mentioned distributional convergence, we also obtain analogous results for the $\mathbb{Z}^2$-periodic Lorentz gas flow. More precisely, denoting $\mathcal{N}_t(\ell)$ for the number of collisions in the $\ell$-cell until time $t$ (for the flow), we also prove the convergence in distribution of families of random variables of the following form :

$$\left(\frac{\mathcal{N}_t(0)}{\mathfrak{A}_{\lfloor t\rfloor}}, \frac{\mathcal{N}_t(b) - \mathcal{N}_t(a)}{\sqrt{\mathfrak{A}_{\lfloor t\rfloor}}}\right)_{t\in[0;+\infty)}.$$

**Link with the Sinaĭ billiard and Central Limit Theorem for the position.** Because of the $\mathbb{Z}^d$-periodicity of the model, it is natural to consider the billiard dynamics obtained by quotienting the positions by $\mathbb{Z}^d$. This quotient dynamics corresponds also to a billiard dynamics in a domain $\overline{Q} = \mathbb{T}^2 \setminus \bigcup_{i=1}^{I} \bar{\mathcal{O}}_i$ contained in the two dimensional torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$.

This quotient billiard system is the Sinaĭ billiard, which preserves a probability measure and enjoys the following nice chaotic properties : ergodicity and mixing by Sinaĭ in [49], exponential decorrelation of Hölder observables (both in finite horizon by Young in [54] and in infinite horizon by Chernov in [8]), the standard Central Limit Theorem for Hölder observables (both

FIGURE 2. The Sinaĭ billiard map in the two-dimensional torus with one circular obstacle.




in finite horizon [6, 7, 54] and in infinite horizon [8]).
Conversely, a crucial fact is that the $\mathbb{Z}^d$-periodic Lorentz gas can be represented by a $\mathbb{Z}^d$-extension of the Sinaĭ billiard. This means that the $\mathbb{Z}^d$-periodic Lorentz gas $T^k$ can be represented by the couple made by the Sinaĭ billiard $\bar{T}^k$ and by the cell label. This representation corresponds to the decomposition of the dynamics at microscopic scale (Sinaĭ billiard) and at macroscopic scale (cell label).
This representation allows the study of the ergodic properties of the $\mathbb{Z}^d$-periodic Lorentz gas via the stochastic properties of the corresponding Sinaĭ billiard.
When the horizon is finite, then the displacement function (between two consecutive collision) is a Hölder observable of the Sinaĭ billiard, and so the above mentioned Central Limit Theorem applies and ensures that the position $q_n$ in the $\mathbb{Z}^d$-periodic Lorentz gas at the $n$-th collision time satisfies a standard Central Limit Theorem. This means that the following convergence holds in distribution

$$\frac{q_n}{\sqrt{n}} \stackrel{distrib.}{\longrightarrow} Z \quad \text{as } n \to +\infty\,, \tag{1}$$

where $Z$ is a centered Gaussian random variable with a variance given by an infinite sum.
When the horizon is infinite and not degenerate, in the sense that there exist at least $d$ non parallel unbounded lines touching no obstacle, then the displacement function is not Hölder, and the above mentioned Central Limit Theorem does not hold; but a non-standard Central Limit Theorem for the position $q_n$ has been proved by Szász and Varjú in [51]. More precisely, they proved that

$$\frac{q_n}{\sqrt{n\log(n)}} \stackrel{distrib.}{\longrightarrow} Z' \quad \text{as } n \to +\infty\,, \tag{2}$$

where $Z'$ is a centered Gaussian random variable, with a very explicit variance expressed in terms of the geometry of the obstacles.
Let us indicate that the degenerate infinite horizon case has been studied in [16]. In particular, when $d = 2$ and when there exists only one unbounded line touching no obstacle, then a central limit theorem for $q_n$ has been established, with two different normalizations, standard and non-standard. And, when $d = 1$ and when the only lines touching no obstacle are vertical and so are bounded, then it is proved in [16] that $(q_n)_{n\in\mathbb{N}^*}$ satisfies the standard central limit theorem (1).

**Recurrent ergodicity.** The recurrence property ensures that the trajectory of almost every state belonging to some measurable set will return in this set. The recurrence property holds true for any probability preserving dynamical system (Poincaré recurrence theorem). But this is not true anymore for dynamical system preserving an infinite measure. So the recurrence property is not automatic for the $\mathbb{Z}^d$-periodic Lorentz gas.
When $d = 1$, the recurrence of the $\mathbb{Z}$-periodic Lorentz gas appears as a consequence of the following convergence

$$\lim_{n\to+\infty} \frac{q_n}{n} = 0 \quad \text{almost everywhere.}$$

which follows from the ergodicity of the Sinaĭ billiard.
When $d = 2$, the recurrence of the $\mathbb{Z}^2$-periodic Lorentz gas has been proved first in the finite horizon case using two different arguments : an argument based on a Central Limit type Theorem (related to (1)) by Conze [10] (see also the work of Schmidt [46] for an analogous argument) and an argument by Szász and Varjú in [50] based on some local limit theorem estimates. This last argument was also adapted by Szász and Varjú in [51] to prove the recurrence of the $\mathbb{Z}^2$-periodic Lorentz gas in the infinite horizon case. The ergodicity then follows as for the Sinaĭ billiard (see the work of Simanyi [48], and also [31]).
Due to the recurrent ergodicity of the periodic Lorentz gas, it follows from the Hopf ratio ergodic theorem that, for any integrable observables, the following limits holds true Lebesgue-almost everywhere

$$\lim_{n\to+\infty} \frac{\sum_{k=0}^{n-1} f\circ T^k}{\sum_{k=0}^{n-1} g\circ T^k} = \frac{\int_M f\, d\mu}{\int_M g\, d\mu} \quad \text{and} \quad \lim_{t\to+\infty} \frac{\int_0^t F\circ Y_s\, ds}{\int_0^t F\circ Y_s\, ds} = \frac{\int_{\mathcal{M}} F\, d\mathfrak{m}}{\int_{\mathcal{M}} G\, d\mathfrak{m}}, \tag{3}$$

as soon as the limits are well defined. This result implies that, if $g : M \to \mathbb{R}$ is an integrable function with non null integral such that $(\sum_{k=0}^{n-1} g\circ T^k/\mathfrak{A}_n)_{n\in\mathbb{N}}$ converges in distribution to a random variable $Z$, then $(\sum_{k=0}^{n-1} f\circ T^k/\mathfrak{A}_n)_{n\in\mathbb{N}}$ converges in distribution to $\frac{\int_M f\,d\mu}{\int_M g\,d\mu} Z$.

**Limit theorems for the periodic Lorentz gas with finite horizon.** Among the results proved for the $\mathbb{Z}^d$-periodic Lorentz gas in finite horizon, let us mention mixing rate in infinite measure [50, 33, 34, 17] including expansions of any order (both for the map [34] and for the flow [17]). More precisely, for the flow, these results state that for any smooth enough observables bounded and integrable,

$$\int_M f.g\circ T^n\, d\mu = \sum_{k=0}^{N} \frac{c_k(f,g)}{n^{\frac{d}{2}+k}} + o\left(n^{-\frac{d}{2}-N}\right) \quad \text{as } n\to+\infty\,, \tag{4}$$

and

$$\int_{\mathcal{M}} F.G\circ Y_t\, d\mathfrak{m} = \sum_{k=0}^{N} \frac{C_k(F,G)}{t^{\frac{d}{2}+k}} + o\left(t^{-\frac{d}{2}-N}\right) \quad \text{as } t\to+\infty\,. \tag{5}$$

Limit theorems for Birkhoff sums both for integrable observables have been obtained in [18]. More precisely, it follows from [18] that for any integrable observables, the following families of random variables converges in distribution as the parameter goes to infinity

$$\left(\frac{\sum_{k=0}^{n-1} f\circ T^k}{\mathfrak{A}'_n}\right)_{n\in\mathbb{N}^*} \quad \text{and} \quad \left(\frac{\int_0^t F\circ Y_s\, ds}{\mathfrak{A}'_{\lfloor t\rfloor}}\right)_{t\in[0;+\infty)} \tag{6}$$

to some random variable times the integral of the observable, with

$$\mathfrak{A}'_n := \sum_{k=1}^{n-1} k^{-\frac{d}{2}}$$

which has order $\sqrt{n}$ if $d=1$ or $\log(n)$ if $d=2$.
Furthermore, limit theorems for Birkhoff sums for smooth integrable observables with **null integral** have been established in [40, 41] (let us indicate that analogous results for null integral observables have been obtained in other contexts by Thomine in [52, 53]). These results state the convergence in distribution of the following families of random variables as the parameter goes to infinity :

$$\left(\frac{\sum_{k=0}^{n-1} f\circ T^k}{\sqrt{\mathfrak{A}'_n}}\right)_{n\in\mathbb{N}^*} \quad \text{and} \quad \left(\frac{\int_0^t F\circ Y_s\, ds}{\sqrt{\mathfrak{A}'_{\lfloor t\rfloor}}}\right)_{t\in[0;+\infty)} . \tag{7}$$

Further limit theorems have been established in this context, including quantitative recurrence estimates (estimates for the tail probability of the first return time in the initial cell [18], limit theorem for the return time in a small neighbourhood of the initial state or of its initial position [36]), limit theorem for the self-intersections number [32, 42], study of differential equations perturbed by the Lorentz gas [43, 44], etc.

**Previous results for the periodic Lorentz gas with infinite horizon.** In the present article, we focus on the case when the horizon is $d$-dimensionally infinite. We recall that this means that there exist $d$ non parallel unbounded lines touching no obstacle. In this case, the time between two consecutive collisions is not bounded anymore, and even worth it is not square integrable with respect to the invariant probability measure $\bar\mu$ of the Sinaĭ billiard map. It is still possible to apply operator techniques as in the finite horizon case, but with a loss of important nice properties, and the study requires much more delicate study.

Nevertheless some results have been established in this infinite horizon context, overcoming these difficulties by creative ideas combined with additional technicality. A first specific result is the non-standard Central Limit Theorem satisfied by the position at the $n$-th collision time established in [51] and recalled in (2). This result was established together with a non-standard local limit theorem for the cell label [51], leading to a mixing rate in $(n\log n)^{-\frac{d}{2}}$ for the Lorentz gas map of the following form :

$$\int_M f.g\circ T^n\, d\mu \sim c_0 \frac{\int_M f\, d\mu\, \int_M g\, d\mu}{(n\log n)^{\frac{d}{2}}} \quad \text{as } n\to+\infty\,, \tag{8}$$

for any smooth enough observables bounded and integrable $f$ and $g$. Whereas a mixing expansion of any order has been established in the finite horizon case (as recalled in (4)), this does not seem reachable in the infinite horizon case because of the weak smoothness properties in $t$ of some operators family $(P_t)_t$. Nevertheless, further mixing estimates, including an error term and also different mixing rates for some null integral smooth observables have been established in [38].
Among the recent results in infinite horizon, let us mention an estimate on the tail probability of the first return time of the map $T$ to the initial cell [38], a Local Large Deviation (LLD)

estimate [29], and also a mixing rate for the flow of the following form

$$\int_{\mathcal{M}} F.G\circ Y_t\, d\mathfrak{m} \sim C_0 \frac{\int_{\mathcal{M}} F\, d\mathfrak{m} \int_{\mathcal{M}} G\, d\mathfrak{m}}{(t\log t)^{\frac{d}{2}}} \quad \text{as } t\to+\infty\,. \tag{9}$$

obtained in [39] for natural observables (such as indicator functions of balls). Going from (8) to (9) is neither direct not easy. The proof of (9) required a coupled version of the above mentioned LLD, combined with several new tricks such as a large deviation estimate on the time of the $n$-th collision, a joint mixing local limit theorem, a new tightness-type criteria, etc.

**About the technical difficulties in the infinite horizon case.** We consider some tranfer operator $P$ related to the Sinaĭ billiard map. We work with the Fourier-perturbed operators family $\left(P_t = P(e^{i\langle t,\Psi\rangle}\cdot)\right)_{t\in\mathbb{R}^d}$, where $\Psi$ represents the cell change function. When the horizon is finite, $t\mapsto P_t = P(e^{i\langle t,\Psi\rangle}\cdot)$ is $\mathcal{C}^\infty$ from $\mathbb{R}$ to $\mathcal{L}(\mathcal{B})$ for some nice Banach space $\mathcal{B}$. This plays an important role in the proof of the expansion of any order given by (4) and and (5), and also in the proof of the convergence in distribution of normalizing Birkhoff sums or integrals (7) of observables with null expectation. When the horizon is infinite, $t\mapsto P_t$ is only smooth (and not even $\mathcal{C}^2$) from $\mathbb{R}$ to $\mathcal{L}(\mathcal{B}\to L^1)$. This complicates seriously the use of this operator family, especially when working with iterates or expansion this operator. For this reason, the general study of [40] does not apply to this context and it is a challenge to adapt in the infinite horizon case, for this result as for others, proofs valid in the finite horizon case. Nevertheless, we find a way to implement the moment method used in [40] and to overcome these difficulties.

**Outline.** The present article is organized as follows. In Section 1 we present our main results for the periodic Lorentz gas flow in infinite horizon. These results will appear as an application of general results stated in a general framework in Section 2. In Section 3, we present a general strategy to prove our general assumptions of Section 2 via Fourier type operator perturbation techniques and we use this approach to prove our main results stated in Section 1. We prove in Section 4 the general results of Section 2.

## 1. Main results for the periodic Lorentz gas in infinite horizon

1.1. **Limit theorem for Birkhoff sums for the map.** Recall that the $\mathbb{Z}^d$-periodoc Lorentz gas is the billiard system in the domain $Q=\mathbb{R}^2\setminus\bigcup_{\ell\in\mathbb{Z}^d}\bigcup_{i=1}^I(\mathcal{O}_i+\ell)$, where the obstacles are given by $\mathcal{O}_i+\ell$ with $\ell\in\mathbb{Z}^d$ and with $i=1,...,I$ for some $I\in\mathbb{N}^*$ (up to identifying $\mathbb{Z}^1$ with $\mathbb{Z}\times\{0\}$ when $d=1$). We write $\mathcal{C}_\ell$ and call $\ell$-cell the set of states $(q,\vec{v})\in M$ such that the position $q$ is in $\bigcup_{i=1}^I\mathcal{O}_i+\ell$. We recall that we set $\mu$ for the only $T$-invariant measure equivalent to the Lebesgue measure and so that $\mu(\mathcal{C}_0)=1$. The density of this measure at $(q,\vec{v})$ is given by $\langle\vec{n}(q),\vec{v}\rangle$, where $\vec{n}(q)$ is the normal vector to $\partial Q$ at $q$. The measure $\mu$ is thus infinite and invariant by translation.
We identify $\bar{M}$ with $\mathcal{C}_0$. With this identification, the Sinaĭ billiard map is identified with the transformation $\bar{T}$ defined on $\bar{M}=\mathcal{C}_0$ corresponding to $T$ modulo $\mathbb{Z}^d$ for the position. We also consider $\bar\Psi:\bar{M}\to\mathbb{Z}^d$ to be the cell change function satisfying, for all $\bar{x}\in\bar{M}=\mathcal{C}_0$, $T(\bar{x})\in\mathcal{C}_{\bar\Psi(\bar{x})}$. Thus, for any $\bar{x}\in\bar{M}=\mathcal{C}_0$, $\left(\bar{T}(\bar{x}),\bar\Psi(\bar{x})\right)$ is the unique element of $\overline{M}\times\mathbb{Z}^d$ such that

$$\bar{T}(\bar{x})=(\bar{q}_1,\vec{v}_1)\quad\text{and}\quad T(\bar{x})=\left(\bar{q}_1+\bar\Psi(\bar{x}),\vec{v}_1\right)\,.$$

More generally, by $\mathbb{Z}^d$-periodicity,

$$\forall \bar{x}=(\bar{q},\vec{v})\in\bar{M},\ \forall \ell\in\mathbb{Z}^d,\quad \bar{T}(\bar{x})=(\bar{q}_1,\vec{v}_1)\ \Rightarrow\ T((\bar{q}+\ell,\vec{v}))=\left(\bar{q}_1+\ell+\bar{\Psi}(\bar{x}),\vec{v}_1\right)\,.$$

It then follows by a direct induction that, for all $n\in\mathbb{Z}$,

$$\forall \bar{x}=(\bar{q},\vec{v})\in\bar{M},\ \forall \ell\in\mathbb{Z}^d,\quad \bar{T}^n(\bar{x})=(\bar{q}_n,\vec{v}_n)\ \Rightarrow\ T^n((\bar{q}+\ell,\vec{v}))=\left(\bar{q}_n+\ell+\bar{S}_n(\bar{x}),\vec{v}_n\right)\,,$$

where $\bar{S}_n:=\sum_{k=0}^{n-1}\bar{\Psi}\circ\bar{T}^k$ and $\bar{S}_{-n}:=-\sum_{k=1}^{n}\bar{\Psi}\circ\bar{T}^{-k}$ for all $n\in\mathbb{N}^*$. This means that the dynamics of the Lorentz gas is totally determined by the joint dynamics of the Sinaĭ billiard and of the Birkhoff sum $\bar{S}_n$. In other words, $(M,T,\mu)$ can be represented as the $\mathbb{Z}^d$-extension of $(\bar{M},\bar{T},\bar{\mu})$ by $\bar{\Psi}$, with $\bar{\mu}:=\mu_{|\bar{M}}$. We assume that the horizon is $d$-dimensionally infinite. Recall that Szász and Varjú proved in [51] that $(\bar{S}_n/\mathfrak{a}_n)_n$ converges in distribution to a non degenerate Gaussian distribution, with $\mathfrak{a}_n:=\max(1,\sqrt{n\log n})$. Let us write $\Phi$ for the density function of this limit Gaussian random variable. We recall that we set $\mathfrak{A}_n:=\sum_{k=1}^n\mathfrak{a}_k^{-d}$ and that

$$\mathfrak{A}_n\sim 2\sqrt{\frac{n}{\log n}}\quad\text{if}\quad d=1\,,\quad\text{and}\quad \mathfrak{A}_n\sim\log\log n\quad\text{if}\quad d=2\,.$$

**Theorem 1.1.** *Let $f$ be a $\mu$-integrable function constant on the cells $\mathcal{C}_\ell$.*

- *If $f=\mathbf{1}_{\mathcal{C}_0}$, then*

$$\left(\sum_{k=0}^{n-1}f\circ T^k/\mathfrak{A}_n\right)_{n\in\mathbb{N}^*}$$

*converges in distribution (with respect to any probability measure absolutely continuous with respect to the Lebesgue measure on $M$) to $\Phi(0)|\mathcal{Z}|$, where*
  - *$\mathcal{Z}$ is a standard gaussian distribution if $d=1$,*
  - *$\mathcal{Z}$ is a random variable with standard exponential distribution if $d=2$.*
- *If $\int_M f\,d\mu=0$ and $\int_M(1+d(0,\cdot))^{\frac{2+\varepsilon-d}{2}}|f|\,d\mu<\infty$ for some $\varepsilon\in(0,1/2)$, then*

$$\left(\sum_{k=0}^{n-1}f\circ T^k/\sqrt{\mathfrak{A}_n}\right)_{n\in\mathbb{N}^*}$$

*converges in distribution (with respect to any probability measure absolutely continuous with respect to the Lebesgue measure on $M$) to $\sqrt{\sigma_f^2\Phi(0)|\mathcal{Z}|}\mathcal{N}$ where $\mathcal{N}$ is a standard Gaussian random variable, independent of $\mathcal{Z}$ and where*

$$\sigma_f^2=\sum_{k\in\mathbb{Z}}\int_M f.f\circ T^k\,d\mu\,.$$

The first part of Theorem 1.1 will appear as a consequence of [51] via moment estimates, in the spirit of [18] in the finite horizon case. As a consequence of the first part of Theorem 1.1 and of the recurrent ergodicity of $(M,\mu,T)$, we get the following result.

**Corollary 1.2** (Weak Law of Large Numbers for the infinite measure preserving dynamical system $(M,T,\mu)$)**.** *For any $\mu$-integrable $h:M\to\mathbb{R}$, the sequence of random variables*

$$\left(\sum_{k=0}^{n-1}h\circ T^k/\mathfrak{A}_n\right)_{n\in\mathbb{N}^*}$$

*converges in distribution to $\Phi(0)\int_M h\,d\mu\,|\mathcal{Z}|$.*

*Proof.* We use the first part of Theorem 1.1. Since $(M,\mu,T)$ is recurrent ergodic, the Hopf ratio ergodic theorem states that the sequence of ergodic ratios

$$\left(\frac{\sum_{k=0}^{n-1} h\circ T^k}{\sum_{k=0}^{n-1} f\circ T^k}\right)_{n\ge 1}$$

converges almost surely to $\int_M h\, d\mu/\mu(\mathcal{C}_0)=\int_M h\, d\mu$. Due to the Slutzky lemma, this combined with Theorem 1.1 implies that

$$\left(\frac{\sum_{k=0}^{n-1} h\circ T^k}{\mathfrak{A}_n}\right)_{n\in\mathbb{N}^*}=\left(\frac{\sum_{k=0}^{n-1} h\circ T^k}{\sum_{k=0}^{n-1} f\circ T^k}\frac{\sum_{k=0}^{n-1} f\circ T^k}{\mathfrak{A}_n}\right)_{n\in\mathbb{N}^*}$$

converges in distribution to $\int_M h\, d\mu\, \Phi(0)\int_M f\, d\mu\, |\mathcal{Z}|$. □

We recall that

$$\forall x\in\bar M,\quad f\left(T^n(\bar x,0)\right)=\left(\bar T^n(\bar x),\bar S_n(\bar x)\right)\, .$$

Thus, the first part of Theorem 1.1 applied with the probability measure $\overline{\mu}$ ensures the convergence in distribution of

$$\left(\mathfrak{A}_n^{-1}\sum_{k=0}^{n-1}\mathbf{1}_{\{\overline{S}_k=0\}}\right)_{n\in\mathbb{N}^*}$$

to $\Phi(0)|\mathcal{Z}|$. Such results of convergence in distribution in the case where $\overline{S}_n$ is a random walk have been established by Lévy in [27], and extended to Markov processes by Darling and Kac [14, 1]. Let us indicate that the distribution of $|\mathcal{Z}|$ is a Mittag-Leffler distribution of index $1-\frac{d}{2}$, i.e.

$$\mathbb{E}[|\mathcal{Z}|^N]:=N!\frac{\Gamma\left(\frac{3-d}{2}\right)^N}{\Gamma\left(1+N\frac{2-d}{2}\right)}\, .$$

Even in the case of the finite horizon, the second part of Theorem 1.1 (study of Birkhoff sums of null integral) is more delicate to establish. Considering $\beta$ such that $f(x,\ell)=\beta_\ell$, the second part of Theorem 1.1 deals with the convergence in distribution of the following sequences of random variables

$$\left(\mathfrak{A}_n^{-\frac12}\sum_{k=0}^{n-1}\beta_{\overline{S}_k}\right)_{n\in\mathbb{N}^*}$$

when $\sum_{\ell\in\mathbb{Z}}\beta_\ell=0$. Such results have been established by Dobrushin in [15] in the case where where $\overline{S}_n$ is a random walk. This has been extended to Markov processes by Kesten [26, 22, 24] (see also [23, 4, 5]), and by Csáki, M. Csörgő, A. Földes and P. Révész in [11, 12, 13]. The first results of this type in the context of dynamical systems have been proved by Thomine [52, 53], and then by Thomine and the author in [40, 41]. For the $\mathbb{Z}^d$-periodic Lorentz gas with finite horizon, the proof using the method of moment via spectral properties used in [40, 41] requires a delicate care of cancellations in order to identify the main order terms in compositions of perturbed operators using their expansions (see [40]). When the horizon is infinite, taking care of these cancellations becomes even more challenging since the perturbed operators do not admit expansion as a family of operators, but only low order expansions as a family of linear maps from $\mathcal{B}$ to $L^1$, forbidding direct compositions of these expansions. This additional difficulty is related to the fact that the cell change function $\bar\Psi$ is **not square integrable** with respect to $\bar\mu$

(whereas it is bounded and so finite-valued when the horizon of the Lorentz process is finite).

### 1.2. **Functional limit theorem for the map and for the flow.**

**Theorem 1.3.** *Let $f,g:M\to\mathbb{R}$ be two integrable functions, with $f$ constant on each cell $\mathcal{C}_\ell$. Assume furthermore that $\int_M (1+d(0,\cdot))^{\frac{2+\varepsilon-d}{2}}|f|\,d\mu<\infty$ for some $\varepsilon\in(0,\frac12)$ and $\int_M f\,d\mu=0$, then the following family of couple of processes*

$$\left(\left(\sum_{k=0}^{\lfloor nt\rfloor-1} g\circ T^k/\mathfrak{A}_n\right)_t, \left(\sum_{k=0}^{\lfloor nt\rfloor-1} f\circ T^k/\sqrt{\mathfrak{A}_n}\right)_t\right)_{n\ge1} \tag{10}$$

*converges in distribution (with respect to any probability measure absolutely continuous with respect to the Lebesgue measure on $M$) to $((\int_M g\,d\mu\mathcal{L}_t)_t,(B_{\sigma_f^2\mathcal{L}_t})_t)$ (in $(\mathcal{D}([0;T]))^2$ for all $T>0$ if $d=1$ and in $(\mathcal{D}([T_0;T]))^2$ for all $0<T_0<T$ if $d=2$), where $\sigma_f^2$ is the quantity introduced in Theorem 1.1, where $B$ is a standard brownian motion independent of the process $\mathcal{L}_t$ and where*

- *if $d=1$, $\mathcal{L}_t$ is the local time at 0 in the time interval $[0;t]$ of the Brownian motion $W$ limit in distribution of $(\bar S_{\lfloor nt\rfloor}/\sqrt{n\log(n)})_t$ as $n\to+\infty$,*
- *if $d=2$, for all $t>0$ $\mathcal{L}_t=\mathcal{L}_1$ is a random variable with exponential distribution with mean $\Phi(0)$.*

.

Our proofs of Theorems 1.1 and 1.3 are given in Section 3. They rely on the general results (in a general framework) stated in the Section 2.

As an immediate consequence of Theorem 1.3 combined with the classical random time change result (see e.g. [2, Chapter 14]), we obtain the following result valid for the periodic Lorentz gas flow. Recall that we write $\mathcal{M}$ for the set of states of the Lorentz gas flow, and $\mathfrak{m}$ for the Lebesgue measure on $\mathcal{M}$. We denote $\mathcal{N}_t(\ell)$ for the number of collisions of the flow in the cell $\mathcal{C}_\ell$ up to time $t$.

**Theorem 1.4.** *Let*

$$\bar c:=\frac{\pi\,\mathrm{Area}(\overline{Q})}{\sum_{i=1}^I|\partial O_i|}\,. \tag{11}$$

*For any real valued sequence $(\beta_\ell)_{\ell\in\mathbb{Z}^d}$ such that $\sum_{\ell\in\mathbb{Z}^d}(1+|\ell|)^{\frac{2+\varepsilon-d}{2}}|\beta_\ell|$ for some $\varepsilon>0$ and $\sum_{\ell\in\mathbb{Z}^d}\beta_\ell=0$, the family of processes*

$$\left(\left(\mathfrak{A}_n^{-1}\mathcal{N}_{nt}(0),\mathfrak{A}_n^{-\frac12}\sum_{\ell\in\mathbb{Z}^d}\beta_\ell\mathcal{N}_{nt}(\ell)\right)_t\right)_{n\ge1} \tag{12}$$

*converges in distribution (with respect to any probability measure absolutely continuous with respect to $\mathfrak{m}$) to*

$$((\bar c^{d-2}\mathcal{L}'_t)_t,(B_{(\sigma^2\bar c^{d-2}\mathcal{L}'_t})_t)$$

*(in $(\mathcal{D}([0;T]))^2$ for all $T>0$ if $d=1$ and in $(\mathcal{D}([T_0;T]))^2$ for all $0<T_0<T$ if $d=2$), where $\sigma^2$ is the quantity $\sigma_f^2$ introduced in Theorem 1.1 for the function $f$ such that $f_{|\mathcal{C}_\ell}\equiv\beta_\ell$ for all*

*$\ell \in \mathbb{Z}^d$, or equivalently, with*

$$\sigma^2 := \sum_{k\in\mathbb{Z}} \sum_{a,b\in\mathbb{Z}^d} \beta_a\beta_b\bar\mu\left(\bar S_k = b-a\right)$$
$$= \sum_{k\in\mathbb{Z}} \sum_{a,b\in\mathbb{Z}^d} \beta_a\beta_b\left(\bar\mu\left(\bar S_k = b-a\right) - \bar\mu\left(\bar S_k = b\right) - \bar\mu\left(\bar S_k = -a\right) + \bar\mu\left(\bar S_k = 0\right)\right)\,,$$

*where $B$ is a standard brownian motion independent of the process $\mathcal{L}'_t$ where*

- *if $d=1$, $\mathcal{L}'_t$ is the local time at 0 in the time interval $[0;t]$ of the Brownian motion $W'$ limit in distribution, as $n\to+\infty$, of $((q'_{nt}/\sqrt{n\log(n)})_t)_n$ where $q'_{nt}$ is the first coordinate of the position of $Y_{nt}(\cdot)$, i.e. the first coordinate of the position at time $nt$ of the particle,*
- *if $d=2$, for all $t>0$ $\mathcal{L}'_t = \mathcal{L}'_1$ is a random variable with exponential distribution with mean $\Phi(0)$.*

.

*Proof of Theorem 1.4.* Recall that the Sinaĭ billiard flow (at unit speed) endowed with the Lebesgue measure $\mathfrak{m}$ can naturally be represented by the suspension flow over $(\bar M, \bar T, 2\sum_{i=1}^I |\partial O_i|\bar\mu)$ with roof function $\tau$, the time before the next collision. Indeed, this representation consists in identifying each $y\in\mathcal{M}$ with the unique couple $(x,s)$ such that $x\in M$, $s\in[0;\tau(x))$ and $y=Y_s(x)$ ($x$ corresponds to the state at the previous collision time and $s$ to the time spent since this previous collision time). This representation ensures in particular that

$$2\sum_{i=1}^I|\partial O_i|\mathbb{E}_{\bar\mu}[\tau] = \mathfrak{m}\left(Y_{[0;\tau(\cdot)]}(\bar M)\right)$$
$$= 2\pi \,\mathrm{Area}\left(Q\cap[0;1[^2\right)$$
$$= 2\pi\,\mathrm{Area}(\overline{Q})\,.$$

Thus we have proved that $\mathbb{E}_{\bar\mu}[\tau]=\bar c$.
Writing $T_n$ for the time of the $n$-th collision, we conclude by applying Theorem 1.3 to $g=\mathbf{1}_{\mathcal{C}_0}$ and to $f=\sum_{\ell\in\mathbb{Z}^d}\beta_\ell\mathbf{1}_{\mathcal{C}_\ell}$ that

$$\left(\left(\mathfrak{A}_n^{-1}\mathcal{N}_{T_{\lfloor nt\rfloor}}(0), \mathfrak{A}_n^{-\frac12}\sum_{\ell\in\mathbb{Z}^d}\beta_\ell\mathcal{N}_{T_{\lfloor nt\rfloor}}(\ell)\right)_t\right)_{n\in\mathbb{N}}$$

converges in distribution with respect e.g. to the conditional probability measure $\mathfrak{m}(.|Y_{[0;\min\tau]}(\mathcal{C}_0))$ to

$$\left(\mathcal{L}_t, B_{\sigma_f^2\mathcal{L}_t}\right)_t\,.$$

We observe that $\mathcal{N}_{nt}(\ell)=\mathcal{N}_{T_{\mathfrak{n}_{nt}}}(\ell)$, where we set $\mathfrak{n}_{nt}$ for the number of collisions of the billiard flow (or equivalently for the $\mathbb{Z}^d$-periodic Lorentz gas flow) in the time interval $[0;nt]$. Since the Sinaĭ billiard system is ergodic, it follows from the Birkhoff ergodic theorem that $\mathfrak{n}_{nt}/n$ is almost surely equivalent to $t/\bar c$ as $n\to+\infty$. Combining this with the change time argument of [2, Chapter 14], we conclude that the joint process (12) converges in distribution to the joint process

$$\left(\mathcal{L}_{t/\bar c}, B_{\sigma_f^2\mathcal{L}_{t/\bar c}}\right)_t\,.$$

We recall that $(\overline{S}_{nt}/\mathfrak{a}_n)_{n\in\mathbb{N}^*}$ converges to a Brownian motion $W$. Furthermore, for every $x = (q_0,\vec{v}_0)\in M$ with representant $\bar{x}=(\bar{q}_0,\vec{v}_0)\in\bar{M}$, setting $T(q_0,\vec{v}_0)=(q_1,\vec{v}_1)$ and $\bar{T}(\bar{x})=(\bar{q}_1,\vec{v}_1)$ its representant in $\bar{M}$, we observe that the displacement $q_1-q_0$ is cohomologous to $\bar{\Psi}(\bar{x})$:

$$q_1-q_0=\bar{\Psi}(\bar{q}_0,\vec{v}_0)+\bar{q}_1-\bar{q}_0\,.$$

Thus $(q_{\lfloor nt\rfloor}/\mathfrak{a}_n)_{n\in\mathbb{N}^*}$ converges also in distribution to $W$. Hence, $(q'_{T_{\lfloor nt\rfloor}}/\mathfrak{a}_n)_{n\in\mathbb{N}^*}$ converges in distribution to $W$. And, using the same random time change argument as above, we conclude that $(q'_{nt}/\mathfrak{a}_n)_{n\in\mathbb{N}^*}$ converges in distribution to $(W'_t=W_{t/\bar{c}})_t$. In particular, when $d=1$, the local time $\mathcal{L}'_t$ of $W'$ at 0 is equal to $\bar{c}\mathcal{L}_{t/\bar{c}}$. Indeed, $\mathcal{L}_t$ (resp. $\mathcal{L}'_t$) is the value at 0 of the density of the image measure of the Lebesgue measure on $[0;t]$ by $s\mapsto W_s$ (resp. by $s\mapsto W'_s$) and so, writing $\mathcal{L}_t(x)$ (resp. $\mathcal{L}'_t(x)$) for the local time of $W$ (resp. $W'$) at time $t$ and at position $x$, we have

$$\begin{aligned}\int_{\mathbb{R}}h(x)\mathcal{L}'_t(x)\,dx&=\int_{[0;t]}h(W'_s)\,ds=\int_{[0;t]}h(W_{s/\bar{c}})\,ds\\&=\bar{c}\int_{[0;t/\bar{c}]}h(W_u)\,du=\bar{c}\int_{\mathbb{R}}h(y)\mathcal{L}_{t/\bar{c}}(y)\,dy\,,\end{aligned}$$

which implies that $\mathcal{L}'_t(x)=\bar{c}\mathcal{L}_{t/\bar{c}}(x)$ and so that $\mathcal{L}_{t/\bar{c}}=\mathcal{L}'_t/\bar{c}$. When $d=2$, then $\mathcal{L}'_t=\mathcal{L}_{t/\bar{c}}=\mathcal{L}_t$. This ends the proof of the corollary. □

**Corollary 1.5.** *Assume the assumptions of Theorem 1.4. Let $G:\mathcal{M}\to\mathbb{R}$ be an integrable function with respect to the Lebesgue measure $\mathfrak{m}$ on $\mathcal{M}$ (velocity vectors $\vec{v}\in\mathbb{S}^1$ being identified with an angle in $\mathbb{R}/\mathbb{Z}$). Then the family of processes*

$$\left(\left(\mathfrak{A}_n^{-1}\int_0^{nt}G\circ Y_s\,ds,\mathfrak{A}_n^{-\frac{1}{2}}\sum_{\ell\in\mathbb{Z}^d}\beta_\ell\mathcal{N}_{nt}(\ell)\right)_t\right)_n \tag{13}$$

*converges in distribution (with respect to any probability measure absolutely continuous with respect to the Lebesgue measure on $Q\times\mathbb{S}^1$) to*

$$\left(\frac{\int_M G\,d\mathfrak{m}}{2\pi\,\mathrm{Area}(\overline{Q})}\widetilde{\mathcal{L}}_t,B_{(\sigma^2/\bar{c})\widetilde{\mathcal{L}}_t}\right)_t$$

*in the same sense as in Theorem 1.4, with $\sigma^2$ the quantity appearing in Theorem 1.4 and where*

- *if $d=1$, $\widetilde{\mathcal{L}}_t$ is the local time at 0 in the time interval $[0;t]$ of the Brownian motion $W'$ limit in distribution, as $n\to+\infty$, of $((q'_{nt}/\sqrt{n\log(n)})_t)_n$ where $q'_{nt}$ is the first coordinate of the position of $Y_{nt}(\cdot)$,*
- *if $d=2$, for all $t>0$, $\widetilde{\mathcal{L}}_t=\widetilde{\mathcal{L}}_1$ is a random variable with exponential distribution with mean $\widetilde{\Phi}(0)$, where $\widetilde{\Phi}$ is the density function of the Gaussian random variable $W'_1$ limit of $(q'_n/\sqrt{n\log(n)})_{n\in\mathbb{N}^*}$, where $q'_n$ is the position of $Y_n$.*

*In particular, if $G=\mathbf{1}_{(Q\cap[0;1[^2)\times\mathbb{S}^1}$, then (13) converges in distribution to*

$$\left(\widetilde{\mathcal{L}}_t,B_{(\sigma^2/\bar{c})\widetilde{\mathcal{L}}_t}\right)_t\,.$$

*Proof.* To prove this corollary, we set $\widetilde{\mathcal{L}}_t = \bar{c}^{d-1}\mathcal{L}'_t$ where $\mathcal{L}'_t$ has been defined in Theorem 1.4 and prove the convergence of (13) to

$$\left(\frac{\bar{c}^{d-2}}{2\sum_{i=1}^I |\partial O_i|}\int_M G\, d\mathfrak{m}\, \mathcal{L}'_t, B_{\sigma^2\bar{c}^{d-2}\mathcal{L}'_t}\right)_t \tag{14}$$

and conclude by noticing that $\sigma^2\bar{c}^{d-2}\mathcal{L}'_t = (\sigma^2/\bar{c})\widetilde{\mathcal{L}}_t$ and by using (11) which ensures that

$$\frac{\bar{c}^{d-2}}{2\sum_{i=1}^I |\partial O_i|} = \frac{\bar{c}^{d-1}}{2\pi\,\mathrm{Area}(\bar{Q})}\,.$$

Let us prove that $\widetilde{\mathcal{L}}_t$ satisfies the properties announced in Corollary 1.5. When $d=1$, we observe that $\widetilde{\mathcal{L}}_t = \mathcal{L}'_t$. When $d=2$, we observe that $W'_1 = W_{1/\bar{c}}$ and so that

$$\begin{aligned}\widetilde{\Phi}(0) &= \left(2\pi\sqrt{\det(Var(W'_1))}\right)^{-1}\\ &= \bar{c}\left(2\pi\sqrt{\det(Var(W'_1))}\right)^{-1} = \bar{c}\Phi(0)\,.\end{aligned}$$

And so, when $d=1$,

$$\widetilde{\mathcal{L}}_t = \bar{c}\mathcal{L}_t = \bar{c}\mathcal{L}_1 = \widetilde{\mathcal{L}}_1$$

has exponential distribution of mean $\bar{c}\Phi(0) = \widetilde{\Phi}(0)$.

To prove the convergence of (13) to (14), a first possibility is to adapt the proof of Theorem 1.3, by considering the function $g$ defined on $M$ by $g(x) := \int_0^{\tau(x)} G(Y_s(x))\, ds$ where $\tau(x)$ is the time before the next collision for a point particle starting with state $x$, and by noticing that

$$\int_M g\, d\mu = \frac{\int_{\mathcal{M}} G\, d\mathfrak{m}}{2\sum_{i=1}^I |\partial O_i|}\,.$$

An alternative proof consists in rewriting (13) as follows :

$$\left(\left(\frac{\int_0^{nt} G\circ Y_s\, ds}{\mathcal{N}_{nt}(0)}\frac{\mathcal{N}_{nt}(0)}{\mathfrak{A}_n}, \mathfrak{A}_n^{-\frac{1}{2}}\sum_{\ell\in\mathbb{Z}^d}\beta_\ell\mathcal{N}_{nt}(\ell)\right)_t\right)_n\,.$$

Since the $\mathbb{Z}^d$-periodic Lorentz gas is recurrent ergodic, the the Hopf ergodic theorem which ensures that

$$\begin{aligned}\lim_{t\to+\infty}\frac{\int_0^t G\circ Y_s\, ds}{\mathcal{N}_t(0)} &= \lim_{t\to+\infty}\frac{\int_0^t G\circ Y_s\, ds}{\int_0^t \mathbf{1}_{Y_{[0;1]}(\mathcal{C}_0)}\circ Y_s\, ds}\\ &= \frac{\int_{\mathcal{M}} G\, d\mathfrak{m}}{\int_{\mathcal{M}}\mathbf{1}_{Y_{[0;1]}(\mathcal{C}_0)}\, d\mathfrak{m}}\\ &= \frac{\int_{\mathcal{M}} G\, d\mathfrak{m}}{2\sum_{i=1}^I|\partial O_i|}\quad \mathfrak{m}-a.e.\,.\end{aligned}$$

It follows from Theorem 1.3 combined with the above almost sure convergence and with the Slutsky lemma that (13) converges in distribution to (14). □

## 2. General results

We recall that, considering $\beta_\ell$ such that $f(x,\ell)=\beta_\ell$, on the set $\bar M=\mathcal{C}_0$, the Birkhoff sums (in infinite measure) $\sum_{k=0}^{n-1} f\circ T^k$ considered in Theorem 1.1 can be rewritten as the additive functional $\sum_{k=0}^{n-1}\beta_{\bar S_k}$ of the Birkhoff sums $(\bar S_n)_n$ with respect to the Sinaǐ billiard system $(\bar M,\bar T,\bar\mu)$. We keep this formulation in the present section and state limit theorems for additive functionals of Birkhoff sums of a probability preserving dynamical systems under general assumptions expressed in terms of operators. We will see in Section 3 how these assumptions can be proved using Fourier-perturbations of the transfer operator and how this result can be used to prove Theorem 1.1.

### 2.1. General assumptions.

**Hypothesis 2.1.** *Let $d\in\{1,2\}$ and $\alpha\in[d;2]$. Let $(\Delta,F,\nu)$ be a probability preserving dynamical system with transfer operator $P$. Let $\Psi:\Delta\to\mathbb{Z}^d$. For any $a\in\mathbb{Z}^d$ and any non-negative integer $n$, we set $S_n:=\sum_{k=0}^{n-1}\Psi\circ F^k$ and we set $Q_{n,a}$ for the operator given by*

$$Q_{n,a}:=P^n(\mathbf{1}_{\{S_n=a\}}\cdot)\,.$$

*Let $(\mathfrak{a}_n)_{n\ge0}$ be a $(1/\alpha)$-regularly varying sequence such that $\mathfrak{A}_n:=\sum_{k=0}^n\mathfrak{a}_n^{-d}\to+\infty$ as $n\to+\infty$. Let $(\mathcal{B},\|\cdot\|_{\mathcal{B}})$ be a Banach space preserved by the operators $Q_{n,a}$ such that*

$$\mathbf{1}_\Delta\in\mathcal{B}\hookrightarrow L^1(\nu)\,, \tag{15}$$

*where the notation $\hookrightarrow$ means a continuous inclusion. We assume furthermore that*

$$\|Q_{n,0}\|_{\mathcal{B}}=\mathcal{O}(\mathfrak{a}_n^{-d}) \tag{16}$$

*and that there exists $\Phi(0)>0$ such that* [1]

$$Q_{n,0}=\Phi(0)\mathfrak{a}_n^{-d}\mathbb{E}_\nu[\cdot]+o(\mathfrak{a}_n^{-d})\quad \textit{in } \mathcal{L}(\mathcal{B}\to L^1(\nu))\,. \tag{17}$$

In [40], to study Birkhoff sums of the periodic Lorentz gas with finite horizon, we used the following condition

$$Q_{n,a}=\Phi(a/\mathfrak{a}_n)\mathfrak{a}_n^{-d}\mathbb{E}_\nu[\cdot]+o(\mathfrak{a}_n^{-d})\quad\text{in }\mathcal{L}(\mathcal{B})\,.$$

A crucial difference between this condition and the assumptions of the present article is that (17) is much weaker since it holds in $\mathcal{L}(\mathcal{B}\to L^1(\nu))$ instead of $\mathcal{L}(\mathcal{B})$. In practice, this weaker condition comes from the fact that the family of perturbed operators $t\mapsto P_t\in\mathcal{L}(\mathcal{B})$ behind (see Section 3) is not continuous, but that $t\mapsto P_t\in\mathcal{L}(\mathcal{B}\to\mathcal{L}^1(\nu))$ is continuous.
To study additive functionals $\sum_{k=0}^{n-1}\beta_{S_k}$ with $\sum_{a\in\mathbb{Z}^d}\beta_a=0$, we will reinforce the previous assumption as follows.

**Hypothesis 2.2.** *Assume Hypothesis 2.1 and that*

$$\sup_{a\in\mathbb{Z}^d}\|Q_{k,a}\|_{\mathcal{B}}=\mathcal{O}(\mathfrak{a}_n^{-d})\,, \tag{18}$$

*and that, for all $\kappa\in[0;1]$,*

$$\left\|Q'_{k,a,b}:=Q_{k,b}-Q_{k,a}\right\|_{\mathcal{B}}=\mathcal{O}\left(|b-a|^\kappa\mathfrak{a}_k^{-d-\kappa}\right)\,, \tag{19}$$

[1] $\Phi(0)$ will appear to be the value at 0 of the density function $\Phi$ of the limit in distribution of $(S_n/\mathfrak{a}_n)_n$ (see Section 3).

*and*

$$\left\|Q''_{k,a,b} := Q_{k,b-a} - Q_{k,b} - Q_{k,-a} + Q_{k,0}\right\|_{\mathcal{B}} = \mathcal{O}\left((|a||b|)^{\kappa}\mathfrak{a}_k^{-d-2\kappa}\right)\,, \tag{20}$$

*uniformly in* $a, b \in \mathbb{Z}^d$.

## 2.2. Limit theorem for additive functionals of Birkhoff sums.

**Theorem 2.3.** *Assume Hypothesis 2.1. Then* $\left(\sum_{k=0}^{n-1}\mathbf{1}_{\{S_k=0\}}/\mathfrak{A}_n\right)_{n\ge1}$ *converges in distribution (and in the sense of moments), with respect to* $\nu$, *to* $\Phi(0)\mathcal{Y}$, *where* $\mathcal{Y}$ *is a Mittag-Leffler distribution of index* $\frac{\alpha-d}{\alpha}$, *i.e.*

$$\mathbb{E}[\mathcal{Y}^N] := N!\frac{\Gamma(1+\frac{\alpha-d}{\alpha})^N}{\Gamma(1+N\frac{\alpha-d}{\alpha})}\,.$$

*If furthermore* $\sum_{\ell\in\mathbb{Z}^d}|1+|\ell||^{\eta}|\beta_\ell| < \infty$ *with* $\eta := \frac{\alpha+\varepsilon-d}{2}$ *for some* $\varepsilon \in (0,1/2)$ *and* $\sum_{\ell\in\mathbb{Z}^d}\beta_\ell = 0$, *and if Hypothesis 2.2 holds true, then* $\left(\sum_{k=0}^{n-1}\beta_{S_k}/\sqrt{\mathfrak{A}_n}\right)_{n\ge1}$ *converges in distribution (and in the sense of moments), with respect to* $\nu$, *to* $\sqrt{\sigma_\beta^2\Phi(0)\mathcal{Y}}\mathcal{N}$ *where* $\mathcal{N}$ *is a standard Gaussian random variable, independent of* $\mathcal{Y}$ *and where*

$$\sigma_\beta^2 := \sum_{k\in\mathbb{Z}}\sum_{a,b\in\mathbb{Z}^d}\beta_a\beta_b\nu\left(S_{|k|}=b-a\right) \tag{21}$$

$$= \sum_{k\in\mathbb{Z}}\sum_{a,b\in\mathbb{Z}^d}\beta_a\beta_b\left(\nu\left(S_{|k|}=b-a\right)-\nu\left(S_{|k|}=b\right)-\nu\left(S_{|k|}=-a\right)+\nu\left(S_{|k|}=0\right)\right)\,. \tag{22}$$

Let us notice that, if the dynamical system $(\Delta, F, \nu)$ is invertible, then $-S_{-k}$ has the same distribution with respect to $\nu$ as $S_k$, and so the quantity $\sigma_\beta^2$ defined in Theorem 2.3 coincide with the quantity $\sigma^2$ of Theorem 1.4.

**Remark 2.4.** *The summability assumption of* $\beta_\ell$ *appearing in Theorem 2.3 is to our knowledge the optimal one even in the case of additive observables of random walks with i.i.d. increments.*

**Remark 2.5.** *It follows from our assumptions that, if* $\beta$ *is not identically null, only the second sum* (22) *defining* $\sigma_\beta^2$ *is absolutely convergent in* $k, a, b$. *Indeed,* (20) *with* $\kappa = \eta$ *ensures that*

$$\nu(S_k=b-a)-\nu(S_k=b)-\nu(S_k=-a)+\nu(S_k=0) = \mathbb{E}_\nu[Q''_{k,a,b}(\mathbf{1})]$$

*is summable in* $(k,a,b) \in \mathbb{N}\times\mathbb{Z}^d\times\mathbb{Z}^d$, *whereas* $\nu(S_k=0) = \mathbb{E}_\nu[Q_{k,0}(\mathbf{1})] \sim \Phi(0)\mathfrak{a}_k^{-d}$ *is not summable. The summability of* (22) *combined with the fact that*

$$\forall k\ge0,\qquad \sum_{a,b\in\mathbb{Z}^d}\beta_a\beta_b\nu(S_k=b-a) = \sum_{a,b\in\mathbb{Z}^d}\beta_a\beta_b\mathbb{E}_\nu[Q''_{k,a,b}(\mathbf{1})]$$

*implies the absolute convergence in* $k$ *of the sum appearing in the right hand side of* (21).

## 2.3. Joint Limit theorem for additive functional of Birkhoff sums.

**Theorem 2.6.** *Assume Assumptions 2.1 and 2.2. Let* $\eta := \frac{\alpha+\varepsilon-d}{2}$ *for some* $\varepsilon\in(0,1/2)$. *Let* $(\beta_a^{(0)})_{a\in\mathbb{Z}^d}$ *and* $(\beta_a^{(1)})_{a\in\mathbb{Z}^d}$ *be two families of real numbers such that* $\sum_{a\in\mathbb{Z}^d}(1+|a|)^\eta|\beta_a^{(j)}| < \infty$

*and* $\sum_{a\in\mathbb{Z}^d}\beta^{(1)}_a=0$*. Then the following family of couples of processes*

$$\left(\left(\sum_{k=0}^{\lfloor nt\rfloor-1}\beta^{(0)}_{S_k}/\mathfrak{A}_n\right)_t,\left(\sum_{k=0}^{\lfloor nt\rfloor-1}\beta^{(1)}_{S_k}/\sqrt{\mathfrak{A}_n}\right)_t\right)_{n\ge 1} \tag{23}$$

*converges in distribution, with respect to* $\nu$*, to* $((\sum_{a\in\mathbb{Z}^d}\beta^{(0)}_a\mathcal{L}_t)_t,(\sigma_{\beta^{(1)}}B_{\mathcal{L}_t})_t)$*, (in* $(\mathcal{D}([0;T]))^2$ *for all* $T>0$ *if* $d=1$ *and in* $(\mathcal{D}([T_0;T]))^2$ *for all* $0<T_0<T$ *if* $d=2$*), where* $\sigma^2_{\beta^{(1)}}$ *is defined in Formula* (21) *of Theorem 2.3 taking* $\beta=\beta^{(1)}$*, where* $B$ *is a standard Brownian motion independent of the process* $(\mathcal{L}_t)_t$ *which is the following process*

- *if* $\alpha>d=1$*,* $\mathcal{L}_t$ *is the local time at 0 in the time interval* $[0;t]$ *of a symmetric* $\alpha$*-stable process* $W$ *with independent increments, independent of* $B$*, such that* $W_1$ *has density probability* $\Phi$ *with* $\Phi(0)$ *satisfying* (17)*,*
- *if* $\alpha=d$*,* $\mathcal{L}_t=\mathbf{1}_{\{t>0\}}\mathcal{L}_1$*, where* $\mathcal{L}_1$ *is a random variable with exponential distribution with mean* $\Phi(0)$*.*

## 3. Proof of Theorems 1.1 and 1.3 via Fourier perturbations

A strategy to prove Hypotheses 2.1 and 2.2 consists in proceding as follows:

- We first define the Fourier-perturbed operator $P_t$ of the transfer operator $P$ associated to $\Psi$ as follows:
$$P_t(h):=P(e^{i\langle t,\Psi\rangle})\,.$$
- Using the fact that $P(g.h\circ T)=hP(g)$, we notice that the $k$-th iterate $P_t^k$ of this operator is given by the following formula :
$$P_t^k(h)=P^n(e^{i\langle t,S_k\rangle})\,.$$
- Using the orthonality of the trigonometric monomials combined with the previous formula, we write
$$\begin{aligned}Q_{k,a}&=P^k\left(\mathbf{1}_{\{S_k=a\}}\right)\\&=P^k\left(\frac{1}{(2\pi)^d}\int_{[-\pi;\pi]^d}e^{i\langle t,S_k-a\rangle}\right)\,dt\\&=\frac{1}{(2\pi)^d}\int_{[-\pi;\pi]^d}e^{-i\langle t,a\rangle}P_t^k(\cdot)\,dt\,.\end{aligned}\tag{24}$$
- We establish nice properties for $P_t$ as the one listed in the assumptions of the next result.

**Proposition 3.1.** *Assume that* $\mathcal{B}$ *is a Banach space satisfying* (15) *and that there exist two constants* $b\in(0,\pi)$ *and* $\alpha_0>0$ *such that :*

$$\forall t\in[-b,b]^d,\ P_t^k=\lambda_t^k\Pi_t+\mathcal{O}(e^{-\alpha_0k})\quad\text{and}\quad\sup_{b<|u|_\infty<\pi}\|P_u^k\|_{\mathcal{B}}=\mathcal{O}(e^{-\alpha_0k})\,,\tag{25}$$

*with* $(\lambda^k_{t/\mathfrak{a}_k})_{k\ge0}$ *converging to the characteristic function* $\varphi$ *of an* $\alpha$*-stable distribution, with* $\Pi_t=\mathbb{E}_\nu[\cdot]+o(1)$ *in* $\mathcal{L}(\mathcal{B}\to L^1(\nu))$ *as* $t\to0$*, and with*

$$\sup_{t\in[-b;b]^d}\|\Pi_t\|_{\mathcal{B}}<\infty\quad\text{and}\quad\int_{\mathbb{R}}(1+|t|^2)\left(\sup_{k\ge1}|\lambda^k_{t/\mathfrak{a}_k}|\mathbf{1}_{\{|t|<b\mathfrak{a}_k\}}\right)\,dt<\infty\,.\tag{26}$$

*Then Hypotheses 2.2 (and so 2.1) hold true with $\Phi(0)$ the value at $0$ of the density function $\Phi$ of the $\alpha$-stable distribution with characteristic function $\varphi$.*

*Proof of Proposition 3.1.* It follows from (24) and from our assumptions that

$$\begin{aligned} Q_{k,0} &= \frac{1}{(2\pi)^d}\int_{[-\pi;\pi]^d} P_t^k\,dt \\ &= \frac{\mathfrak{a}_k^{-d}}{(2\pi)^d}\int_{[-b\mathfrak{a}_k;b\mathfrak{a}_k]^d} \lambda_{t/\mathfrak{a}_k}^k \Pi_{t/\mathfrak{a}_k}\,dt + \mathcal{O}(e^{-\alpha_0 k}) \\ &= \Phi(0)\mathfrak{a}_k^{-d} + o\left(\mathfrak{a}_k^{-d}\right)\,, \end{aligned}$$

in $\mathcal{L}(\mathcal{B}\to L^1(\nu))$, via the dominated convergence theorem since $\lim_{n\to+\infty}\lambda_{t/\mathfrak{a}_n}^n\Pi_{t/\mathfrak{a}_n} = \varphi(t)\mathbb{E}_\nu[\cdot]$ in $\mathcal{L}(\mathcal{B}\to L^1(\nu))$ and since $\Phi(0) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}}\varphi(t)\,dt$ (the domination comes from (26)). Thus (17) holds true.

Furthermore, for all $\eta\in[0;2]$ and $a\in\mathbb{Z}^d$, in $\mathcal{L}(\mathcal{B})$,

$$\begin{aligned} \int_{[-\pi;\pi]^d} |t|^\eta \left\|P_t^k\right\|_{\mathcal{B}}\,dt &= \mathfrak{a}_k^{-d-\eta}\int_{[-b\mathfrak{a}_k;b\mathfrak{a}_k]^d} |t|^\eta \left|\lambda_{t/\mathfrak{a}_k}^k\right| \left\|\Pi_{t/\mathfrak{a}_k}\right\|_{\mathcal{B}}\,dt + \mathcal{O}\left(e^{-\alpha_0 k}\right) \\ &= \mathcal{O}\left(\mathfrak{a}_k^{-d-\eta}\right)\,, \end{aligned} \tag{27}$$

where we used (25) and (26).
Using the expression (24) of $Q_{k,a}$ combined with (27) with $\eta=0$, we obtain (16) and (18). Fix $\kappa\in[0;1]$. Then

$$\begin{aligned} Q'_{k,a,b} = Q_{k,b} - Q_{k,a} &= \frac{1}{(2\pi)^d}\int_{[-\pi;\pi]^d} (e^{i\langle t,b\rangle} - e^{i\langle t,a\rangle})P_t^k(\cdot)\,dt \\ &= \frac{1}{(2\pi)^d}\int_{[-\pi;\pi]^d} \mathcal{O}(\langle t,b-a\rangle)^\kappa P_t^k(\cdot)\,dt \\ &= \mathcal{O}\left(|b-a|^\kappa\mathfrak{a}_k^{-d-\kappa}\right) \quad \text{in } \mathcal{L}(\mathcal{B})\,, \end{aligned}$$

where we used again (27) combined with the bound $|e^{ix}-e^{iy}|\le\min(2,|x-y|)\le 2^{1-\kappa}|x-y|^\kappa$, and so we have proved (19). For (20), in the same way, we obtain

$$\begin{aligned} Q''_{k,a,b} &= Q_{k,b-a} - Q_{k,b} - Q_{k,-a} + Q_{k,0} \\ &= \frac{1}{(2\pi)^d}\int_{[-\pi;\pi]^d} (e^{i\langle t,b-a\rangle} - e^{i\langle t,b\rangle} - e^{-i\langle t,a\rangle} + 1)P_t^k(\cdot)\,dt \\ &= \frac{1}{(2\pi)^d}\int_{[-\pi;\pi]^d} (e^{i\langle t,b\rangle} - 1)(e^{-i\langle t,a\rangle} - 1)P_t^k(\cdot)\,dt \\ &= \frac{1}{(2\pi)^d}\int_{[-\pi;\pi]^d} \mathcal{O}(\langle t,b\rangle\langle t,a\rangle)^\kappa P_t^k(\cdot)\,dt \\ &= \mathcal{O}\left((|a||b|)^\kappa\mathfrak{a}_k^{-d-2\kappa}\right) \quad \text{in } \mathcal{L}(\mathcal{B})\,, \end{aligned} \tag{28}$$

and so (20). □

*Proof of Theorems 1.1 and 1.3.* Let us write $\beta_\ell$ for the constant to which $f$ is equal on the $\ell$-cell $\mathcal{C}_\ell$. The integrability assumption means that $\sum_{\ell\in\mathbb{Z}}|\beta_\ell|<\infty$. Due to [55], since $\mu$ is equivalent to the Lebesgue measure on $M$, it is enough to prove the results with respect to the measure

$\mu_{\mathcal{C}_0}$ (the restriction of $\mu$ to $\mathcal{C}_0$). Thus, we consider this reference measure and establish the convergence of every moment with respect to this probability measure. We observe that, with the identification of $\bar{M}$ to $\mathcal{C}_0$, $\mu_{\mathcal{C}_0}$ is identified with $\bar{\mu}$ and $f\circ T^k$ is identified with $\beta_{\bar{S}_k}$.
As in [51, 38], we use the two Young towers [54, 8]. We write $(\hat{\Delta},\hat{F},\hat{\nu})$ for the hyperbolic tower which is an extension of $(\bar{M},\bar{T},\bar{\mu})$, and write $(\Delta,F,\nu)$ for the expanding tower obtained by quotienting $(\hat{\Delta},\hat{F},\hat{\nu})$ along stable curves. We write $\bar{\pi}:\hat{\Delta}\to\bar{M}$ and $\pi:\hat{\Delta}\to\Delta$ for the two measurable maps such that $\bar{\pi}_*\hat{\nu}=\mu$, $\pi_*\hat{\nu}=\nu$, $\bar{T}\circ\bar{\pi}=\bar{\pi}\circ\hat{F}$ and $F\circ\pi=\pi\circ\hat{F}$. Since $\bar{\Psi}$ is constant on stable curves, there exists a function $\Psi:\Delta\to\mathbb{Z}^d$ such that $\Psi\circ\pi=\bar{\Psi}\circ\bar{\pi}$. Setting $S_n:=\sum_{k=0}^{n-1}\Psi\circ F^k$, it follows that $S_n\circ\pi=\bar{S}_n\circ\bar{\pi}$. For the first part of Theorem 1.1, we take $\beta_\ell=\mathbf{1}_{\{\ell=0\}}$. We will conclude by Theorem 2.3. To prove the assumptions of Theorem 2.3, we show that the criterion given in Proposition 3.1 is satisfied here with our choice of $\mathfrak{a}_n$, with $\alpha=2$ and with $\Phi$ the characteristic function of the Gaussian limit distribution of $(\bar{S}_n/\mathfrak{a}_n)_n$. The assumptions of Proposition 3.1 have been proved in [51] with the use of the Banach spaces introduced in [54] combined with the use of the Nagaev-Guivarch perturbation method [30, 19, 20] via the Keller and Liverani theorem [25] (see also [35] for a general reference on this method). The fact that $(\lambda^k_{t/\mathfrak{a}_k})_{k\ge1}$ converges pointwise to the characteristic function of a Gaussian random variable follows from the existence of a positive symmetric matrix $A$ such that $1-\lambda_t\sim\langle At,t\rangle|\log|t||$ as $t\to0$ (this was proved in [51]). For the second part of (26), one can e.g. use the fact that $|\lambda^k_{t/\mathfrak{a}_k}|\mathbf{1}_{\{|t|<b\mathfrak{a}_k\}}\le e^{-c_0\min(|t|^{2-\varepsilon},|t|^{2+\varepsilon})}$ for $|t|$ small enough. Thus Proposition 3.1 holds true and Theorems 2.3 and 2.6 apply. Finally, we identify the formulas of the asymptotic variances $\sigma_f^2$ and $\sigma_\beta^2$ by noticing that

$$\sum_{a,b\in\mathbb{Z}^2}\beta_a\beta_b\nu(S_{|k|}=b-a)=\int_M f.f\circ T^{|k|}\,d\mu=\int_M f.f\circ T^k\,d\mu\,.$$

For Theorem 1.3, we deduce the result for general $g$ using the fact that $(M,T,\mu)$ is recurrent ergodic, together with the Hopf ergodic ratio theorem. Indeed, with the notations of Theorem 1.3, Theorem 2.6 ensures that

$$\left(\left(\sum_{k=0}^{\lfloor nt\rfloor-1}\mathbf{1}_{\{\overline{S}_k=0\}}/\mathfrak{A}_n\right)_t,\left(\sum_{k=0}^{\lfloor nt\rfloor-1}f\circ T^k/\sqrt{\mathfrak{A}_n}\right)_t\right)_{n\ge1} \tag{29}$$

converges in distribution, with respect to $\bar{\mu}$, to $((\mathcal{L}_t)_t,(B_{\sigma_f^2\mathcal{L}_t})_t)$, with $\mathcal{L}$ as in Theorem 2.6 and $\sigma_f^2$ as in Theorem 1.1. But, it follows from the Hopf ratio ergodic theorem that

$$\lim_{n\to+\infty}\frac{\sum_{k=0}^m g\circ T^k}{\sum_{k=0}^m\mathbf{1}_{\{\overline{S}_k=0\}}}=\int_M g\,d\mu\quad\mu\text{-almost everywhere}\,.$$

Thus, we conclude that

$$\left(\left(\frac{\sum_{k=0}^{\lfloor nt\rfloor-1}g\circ T^k}{\sum_{k=0}^m\mathbf{1}_{\{\overline{S}_k=0\}}}\frac{\sum_{k=0}^m\mathbf{1}_{\{\overline{S}_k=0\}}}{\mathfrak{A}_n}\right)_t,\left(\sum_{k=0}^{\lfloor nt\rfloor-1}f\circ T^k/\sqrt{\mathfrak{A}_n}\right)_t\right)_{n\ge1} \tag{30}$$

converges in distribution, with respect to $\bar{\mu}$, to $\left(\int_M g\,d\mu(\mathcal{L}_t)_t,(B_{\sigma_f^2\mathcal{L}_t})_t\right)$. □

## 4. Proofs of Theorems 2.3 and 2.6

4.1. **Proof of Theorem 2.3.** Let $(\beta_\ell)_{\ell\in\mathbb Z^d}$ be a summable sequence of real numbers. We start by writing

$$\mathbb E_\nu\left[\left(\sum_{k=0}^{n-1}\beta_{S_k}\right)^N\right]=\sum_{k_1,...,k_N=0}^{n-1}\mathbb E_\nu\left[\prod_{j=1}^N\beta_{S_{k_j}}\right]$$

$$=\sum_{0\le k_1\le...\le k_N\le n-1}c_{(k_1,...,k_N)}\sum_{a_1,...,a_N\in\mathbb Z^d}\mathbb E_\nu\left[\prod_{j=1}^N\left(\beta_{a_j}\mathbf 1_{\{S_{k_j}=a_j\}}\right)\right]\,,\tag{31}$$

where we denote by $c_{(k_1,...,k_N)}$ the number of $N$-tuples $(k'_1,...,k'_N)\in\{0,...,n-1\}^N$ such that there exists a permutation $\sigma\in\mathfrak S_N$ such that $k'_i=k_{\sigma(i)}$ for all $i=1,...,N$. We observe that

$$\mathbb E_\nu\left[\prod_{j=1}^N\left(\beta_{a_j}\mathbf 1_{\{S_{k_j}=a_j\}}\right)\right]=\mathbb E_\nu\left[\prod_{j=1}^N\left(\beta_{a_j}\mathbf 1_{\{S_{k_j}-S_{k_{j-1}}=a_j-a_{j-1}\}}\right)\right]$$

$$=\mathbb E_\nu\left[\prod_{j=1}^N\left(\beta_{a_j}\mathbf 1_{\{S_{k_j-k_{j-1}}=a_j-a_{j-1}\}}\circ F^{k_{j-1}}\right)\right]\,,$$

with the conventions $k_0=0$ and $a_0=0$. We recall that $P$ is the transfer operator of $F$ with respect to $\nu$, which means that

$$\mathbb E_\nu[P(g).h]=\mathbb E_\nu[g.h\circ T]\,.$$

Using the fact that $\mathbb E_\nu[\cdot]=\mathbb E_\nu[P^{k_N}(\cdot)]$, we obtain

$$\mathbb E_\nu\left[\prod_{j=1}^N\left(\beta_{a_j}\mathbf 1_{\{S_{k_j}=a_j\}}\right)\right]=\mathbb E_\nu\left[P^{k_N}\left(\prod_{j=1}^N\left(\beta_{a_j}\mathbf 1_{\{S_{k_j-k_{j-1}}=a_j-a_{j-1}\}}\circ F^{k_{j-1}}\right)\right)\right]\,.\tag{32}$$

Since $P^k(f.g\circ F^k)=gP^k(f)$, we observe that, for any $j=1,...,N$, for any $k_1\le...\le k_j$ and any $b_1,...,b_j\in\mathbb Z^d$,

$$P^{k_j}\left(\prod_{i=1}^j\mathbf 1_{\{S_{k_i-k_{i-1}}=b_i\}}\circ F^{k_{i-1}}\right)=P^{k_j-k_{j-1}}\left(\mathbf 1_{\{S_{k_j-k_{j-1}}=b_j\}}P^{k_{j-1}}\left(\prod_{i=1}^{j-1}\left(\mathbf 1_{\{S_{k_i-k_{i-1}}=b_i\}}\circ F^{k_{i-1}}\right)\right)\right)$$

$$=Q_{k_j-k_{j-1},b_j}\left(\prod_{i=1}^{j-1}\left(\mathbf 1_{\{S_{k_i-k_{i-1}}=b_i\}}\circ F^{k_{i-1}}\right)\right)\,.\tag{33}$$

Starting from from (32) and applying iteratively (33) $N$ times, we obtain

$$\mathbb E_\nu\left[\prod_{j=1}^N\left(\beta_{a_j}\mathbf 1_{\{S_{k_j}=a_j\}}\right)\right]=\mathbb E_\nu\left[\beta_{a_N}Q_{k_N-k_{N-1},a_N-a_{N-1}}(\cdots(\beta_{a_2}Q_{k_2-k_1,a_2-a_1}(\beta_{a_1}Q_{k_1,a_1}(\mathbf 1)))\cdots)\right]\,.\tag{34}$$

For the first part of Theorem 2.3, we apply (31) with $\beta_\ell=\mathbf 1_{\{\ell=0\}}$ with (34). In that case, applying repeatedly (17), combined with (16) and (15), the right hand side of (34) becomes

$$\mathbb E_\nu\left[Q_{k_N-k_{N-1},0}(...(Q_{k_1,0}(\mathbf 1)))\right]=(\Phi(0))^N\prod_{j=1}^N\mathfrak a^{-d}_{k_j-k_{j-1}}+\sum_{j=1}^N o(\mathfrak a^{-d}_{k_j-k_{j-1}}))\prod_{i\neq j}\mathcal O(\mathfrak a^{-d}_{k_i-k_{i-1}})\,.$$

This leads to

$$\mathbb{E}_\nu\left[\left(\sum_{k=0}^{n-1}\mathbf{1}_{\{S_k=0\}}\right)^N\right] = o(\mathfrak{A}_n^N) + N!\,(\Phi(0))^N \sum_{1<k_1<\ldots<k_N\le n-1}\prod_{j=1}^N \mathfrak{a}_{k_j-k_{j-1}}^{-d}$$

$$= o(\mathfrak{A}_n^N) + N!\mathfrak{A}_n^N\left[(\Phi(0))^N\frac{\Gamma(1+\frac{2-d}{2})^N}{\Gamma(1+N\frac{2-d}{2})} + o(1)\right],$$

using [40, Lemma 2.7] for the last estimate. This ends the proof of the first part of Theorem 2.3.

Now let us prove the second part. We assume from now on that $\sum_{a\in\mathbb{Z}^d}\beta_a = 0$ and that $\sum_{a\in\mathbb{Z}^d}(1+|a|^\eta)|\beta_a| < \infty$ with $\eta := \frac{\alpha-d+\varepsilon}{2}$ for some $\varepsilon\in(0;\frac{1}{2})$. Recall that it follows from (31) combined with (34) that

$$\mathbb{E}_\nu\left[\left(\sum_{k=0}^{n-1}\beta_{S_k}\right)^N\right] = \sum_{0\le k_1\le\ldots\le k_N\le n-1} c_{(k_1,\ldots,k_N)}\sum_{a_1,\ldots,a_N\in\mathbb{Z}^d}$$

$$\mathbb{E}_\nu\left[\beta_{a_N}Q_{k_N-k_{N-1},a_N-a_{N-1}}(\cdots(\beta_{a_2}Q_{k_2-k_1,a_2-a_1}(\beta_{a_1}Q_{k_1,a_1}(\mathbf{1})))\cdots)\right]. \tag{35}$$

In Formula (35), we decompose each $Q_{k,a}$ in $Q_{k,a}^{(0)}+Q_{k,a}^{(1)}$, with $Q_{k,a}^{(0)} := Q_{k,0}$ and $Q_{k,a}^{(1)} := Q'_{k,a} := Q'_{k,0,a} = Q_{k,a}-Q_{k,0}$. Thus

$$\mathbb{E}_\nu\left[\left(\sum_{k=0}^{n-1}\beta_{S_k}\right)^N\right] = \sum_{\varepsilon_1,\ldots,\varepsilon_N\in\{0,1\}} H_{\varepsilon_1,\ldots,\varepsilon_N}^{(n,N)}, \tag{36}$$

with

$$H_{\varepsilon_1,\ldots,\varepsilon_N}^{(n,N)} = \sum_{0\le k_1\le\ldots\le k_N\le n-1} c_{(k_1,\ldots,k_N)}H_{\mathbf{k},\boldsymbol{\varepsilon}}^{n,N}$$

setting $\mathbf{k}=(k_1,\ldots,k_N)$, $\boldsymbol{\varepsilon}=(\varepsilon_1,\ldots,\varepsilon_N)\in\{0,1\}^N$ and

$$H_{\mathbf{k},\boldsymbol{\varepsilon}}^{n,N} := \sum_{a_1,\ldots,a_N\in\mathbb{Z}^d}\mathbb{E}_\nu\left[\beta_{a_N}Q^{(\varepsilon_N)}_{k_N-k_{N-1},a_N-a_{N-1}}(\cdots(\beta_{a_2}Q^{(\varepsilon_1)}_{k_2-k_1,a_2-a_1}(\beta_{a_1}Q_{k_1,a_1}(\mathbf{1})))\cdots)\right]$$

$$= \sum_{a_N\in\mathbb{Z}^d}\mathbb{E}_\nu\left[\beta_{a_N}\left(\sum_{a_{N-1}\in\mathbb{Z}^d}\beta_{a_{N-1}}Q^{(\varepsilon_N)}_{k_N-k_{N-1},a_N-a_{N-1}}\left(\cdots\left(\sum_{a_1\in\mathbb{Z}^d}\beta_{a_1}Q^{(\varepsilon_2)}_{k_2-k_1,a_2-a_1}\left(Q^{(\varepsilon_1)}_{k_1,a_1}(\mathbf{1})\right)\right)\cdots\right)\right)\right].$$

Observe that $H_{\varepsilon_1,\ldots,\varepsilon_N}^{(n,N)}=0$ if at least one of the following conditions is satisfied :

- if $\varepsilon_N=0$, since then the only quantity depending on $a_N$ in $H_{\mathbf{k},\boldsymbol{\varepsilon}}^{n,N}$ is $\beta_{a_N}$ and since $\sum_{a_N\in\mathbb{Z}^d}\beta_{a_N}=0$;
- or, if there exists $j_0=1,\ldots,N-1$ such that $\varepsilon_{j_0}=\varepsilon_{j_0+1}=0$, since then the only quantity depending on $a_{j_0}$ in $H_{\mathbf{k},\boldsymbol{\varepsilon}}^{n,N}$ is $\beta_{a_{j_0}}$ and since $\sum_{a_{j_0}\in\mathbb{Z}^d}\beta_{a_{j_0}}=0$.

Thus, we restrict our study to the case of the $\varepsilon_j's$ for which the $j_0$'s such that $\varepsilon_{j_0}=0$ are isolated and do not include $N$. Let such an $\boldsymbol{\varepsilon}=(\varepsilon_1,\ldots,\varepsilon_N)$. Observe that there are at most

$N/2$ indices $j$'s such that $\varepsilon_j = 0$. We set, by convention, $\varepsilon_{N+1} = \varepsilon_0 = 0$. We will prove that

$$H^{(n,N)}_{\varepsilon_1,\dots,\varepsilon_N} = o\left(\mathfrak{A}_n^{\frac{N}{2}}\right) \quad \text{unless } \#\{j : \varepsilon_j = 0\} = \frac{N}{2}\,, \tag{37}$$

i.e. $H^{(n,N)}_{\varepsilon_1,\dots,\varepsilon_N} = o\left(\mathfrak{A}_n^{\frac{N}{2}}\right)$ unless $N$ is even and $(\varepsilon_1,...,\varepsilon_N) = (0,1,...,0,1)$. For reader's convenience, we first give a short proof of this estimate in a particular case. A proof of this estimate in the general case $\sum_{a\in\mathbb{Z}^d} |a|^{\frac{\alpha-d+\varepsilon}{2}}|\beta_a| < \infty$ with $\eta = \frac{\alpha-d+\varepsilon}{2}$, with $\varepsilon \in (0;\frac12)$ is given in Section 4.2.

- A short proof of (37) in a particular case. Assume in this item only that $\alpha < d+1$ and $\sum_{a\in\mathbb{Z}^d}(1+|a|)^{2\bar\eta}|\beta_a| < \infty$ with $\alpha - d < \bar\eta < 1$. It follows from Hypothesis 2.2 that

$$\begin{aligned} Q^{(\varepsilon_j)}_{k_j-k_{j-1},a_j-a_{j-1}} &= \mathcal{O}\left(|a_j-a_{j-1}|^{\varepsilon_j\bar\eta}\mathfrak{a}_{k_j-k_{j-1}}^{-d-\varepsilon_j\bar\eta}\right)\\ &= \mathcal{O}\left(((1+|a_j|)(1+|a_{j-1}|))^{\varepsilon_j\bar\eta}\mathfrak{a}_{k_j-k_{j-1}}^{-d-\varepsilon_j\bar\eta}\right)\,,\end{aligned}$$

and so that

$$\begin{aligned}\left|H^{(n,N)}_{\varepsilon_1,\dots,\varepsilon_N}\right| &= \mathcal{O}\left(\left(\sum_{a\in\mathbb{Z}^d}(1+|a|)^{2\bar\eta}|\beta_a|\right)^N\prod_{j=1}^N\sum_{k_j=0}^{n-1}\mathfrak{a}_{k_j}^{-d-\varepsilon_j\bar\eta}\right)\\ &= \mathcal{O}\left(\mathfrak{A}_n^{\#\{j:\varepsilon_j=0\}}\right)\,,\end{aligned}$$

where we used the fact that $\sum_{a\in\mathbb{Z}^d}(1+|a|)^{2\bar\eta}|\beta_a| < \infty$ combined with $\sum_{k\ge0}\mathfrak{a}_k^{-d-\bar\eta} < +\infty$, since $d+\bar\eta > \alpha$ and since $(\mathfrak{a}_n)_n$ is $\frac{1}{\alpha}$-regularly varying. This concludes the proof of (37) when $\alpha < d+1$ and $\sum_{a\in\mathbb{Z}^d}(1+|a|)^{2\bar\eta}|\beta_a| < \infty$ with $\alpha-d<\bar\eta<1$..

In particular, Formula (37) ensures that

$$\mathbb{E}_\nu\left[\left(\sum_{k=0}^{n-1}\beta_{S_k}\right)^N\right] = o\left(\mathfrak{A}_n^{\frac{N}{2}}\right) \quad \text{if } N \text{ is odd}\,, \tag{38}$$

and that

$$\mathbb{E}_\nu\left[\left(\sum_{k=0}^{n-1}\beta_{S_k}\right)^N\right] = H^{(n,N)}_{0,1,\dots,0,1} + o\left(\mathfrak{A}_n^{\frac{N}{2}}\right) \quad \text{if } N \text{ is even}\,. \tag{39}$$

Assume from now on that $N$ is even, then

$$\begin{aligned} H^{(n,N)}_{0,1,\dots,0,1} = &\sum_{0\le k_1\le\dots\le k_N\le n-1} c_{(k_1,\dots,k_N)}\sum_{a_1,\dots,a_N\in\mathbb{Z}^d}\left(\prod_{j=1}^N\beta_{a_j}\right)\times\\ &\mathbb{E}_\nu[Q'_{k_N-k_{N-1},a_N-a_{N-1}}(Q_{k_{N-1}-k_{N-2},0}(\dots(Q'_{k_2-k_1,a_2-a_1}(Q_{k_1,0}(\mathbf{1})))\dots))]\,.\end{aligned}$$

Hence

$$H^{(n,N)}_{0,1,\dots,0,1} = \sum_{0\le k_1\le\dots\le k_N\le n-1} c_{(k_1,\dots,k_N)}\mathbb{E}_\nu[\bar Q_{k_N-k_{N-1}}(Q_{k_{N-1}-k_{N-2},0}(\dots(\bar Q_{k_2-k_1}(Q_{k_1,0}(\mathbf{1})))\dots))]\,, \tag{40}$$

with

$$\bar Q_k := \sum_{a,b\in\mathbb{Z}^d}\beta_a\beta_b Q'_{k,b-a} = \sum_{a,b\in\mathbb{Z}^d}\beta_a\beta_b Q_{k,b-a} = \sum_{a,b\in\mathbb{Z}^d}\beta_a\beta_b Q''_{k,a,b}\,, \tag{41}$$

where we used the fact that $\sum_{a\in\mathbb{Z}^d}\beta_a=0$ and the notation $Q''_{k,a,b}=Q_{k,b-a}-Q_{k,b}-Q_{k,-a}+Q_{k,0}$ introduced in (20). Combining (20) with $\kappa=\eta=\frac{\alpha+\varepsilon-d}{2}$ with (41), since $\sum_{a\in\mathbb{Z}^d}(1+|a|)^\eta|\beta_a|<\infty$, we infer that

$$\bar{Q}_k=\mathcal{O}\left(\mathfrak{a}_k^{-\alpha-\varepsilon}\right)\quad\text{in }\mathcal{L}(\mathcal{B})\,, \tag{42}$$

since $d+2\eta=\alpha+\varepsilon$, which ensures the summability of $\|\bar{Q}_k\|_{\mathcal{B}}$. The study of (40) leads us to the question of estimating $\mathbb{E}_\nu[\bar{Q}_{k'}(Q_{k,0}(h))]$. Unfortunately we cannot compose directly (20) and (17) since this last estimate is in $\mathcal{L}(\mathcal{B}\to L^1(\nu))$ and not in $\mathcal{L}(\mathcal{B})$. But, proceeding in two steps, we will prove that

$$\mathbb{E}_\nu[\bar{Q}_{k'}(Q_{k,0}(h))]-\Phi(0)\mathfrak{a}_k^{-d}\mathbb{E}_\nu\left[\bar{Q}_{k'}(\mathbf{1})\right]\mathbb{E}_\nu[h]=\mathcal{O}\left(\mathfrak{a}_{k'}^{-\alpha-\varepsilon'}\|h\|_{\mathcal{B}}\right)o(\mathfrak{a}_k^{-d})\,, \tag{43}$$

where $\varepsilon'\in(0,\frac12)$ is small enough so that

$$\frac{(\alpha-d+2\varepsilon')(\alpha+\varepsilon')}{\alpha+2\varepsilon'}\le\alpha-d+\varepsilon=2\eta\,. \tag{44}$$

First, dominating separately both terms, it follows from (20) with $\kappa=\eta'=\frac{\alpha+2\varepsilon'-d}{2}\in(0,1)$ and from (15) and (16) that

$$\mathbb{E}_\nu[Q''_{k',a,b}(Q_{k,0}(h))]-\Phi(0)\mathfrak{a}_k^{-d}\mathbb{E}_\nu\left[Q''_{k',a,b}(\mathbf{1})\right]\mathbb{E}_\nu[h]=\mathcal{O}\left(|a|^{\eta'}|b|^{\eta'}\mathfrak{a}_{k'}^{-\alpha-2\varepsilon'}\mathfrak{a}_k^{-d}\|h\|_{\mathcal{B}}\right)\,, \tag{45}$$

since $-d-2\eta'=-\alpha-2\varepsilon'$. Second, it follows from the definition of $Q''_{k',a,b}$ and of $Q_{k,a}$ that

$$\mathbb{E}_\nu[Q''_{k',a,b}(h_0)]=\mathbb{E}_\nu\left[P^{k'}\left((\mathbf{1}_{\{S_{k'}=b-a\}}-\mathbf{1}_{\{S_{k'}=b\}}-\mathbf{1}_{\{S_{k'}=-a\}}+\mathbf{1}_{\{S_{k'}=0\}}).h_0\right)\right]\,.$$

Since $\mathbb{E}_\nu[P^{k'}(h)]=\mathbb{E}_\nu[h]$, it follows that

$$\begin{aligned}\mathbb{E}_\nu[Q''_{k',a,b}(h_0)]&=\mathbb{E}_\nu\left[(\mathbf{1}_{\{S_{k'}=b-a\}}-\mathbf{1}_{\{S_{k'}=b\}}-\mathbf{1}_{\{S_{k'}=-a\}}+\mathbf{1}_{\{S_{k'}=0\}}).h_0\right]\\&=\mathcal{O}\left(\|h_0\|_{L^1(\nu)}\right)\,.\end{aligned}$$

This combined with (17) ensures that

$$\begin{aligned}&\mathbb{E}_\nu[Q''_{k',a,b}(Q_{k,0}(h))]-\Phi(0)\mathfrak{a}_k^{-d}\mathbb{E}_\nu\left[Q''_{k',a,b}(\mathbf{1})\right]\mathbb{E}_\nu[h]\\&\quad=\mathbb{E}_\nu\left[Q''_{k',a,b}(Q_{k,0}(h)-\Phi(0)\mathfrak{a}_k^{-d}\mathbb{E}_\nu[h])\right]\\&\quad=\mathcal{O}\left(\left\|Q_{k,0}(h)-\Phi(0)\mathfrak{a}_k^{-d}\mathbb{E}_\nu[h]\right\|_{L^1(\nu)}\right)\\&\quad=\mathcal{O}(\|h\|_{\mathcal{B}})o(\mathfrak{a}_k^{-d})\,.\end{aligned} \tag{46}$$

Thus, combining (45) and (46), we obtain that

$$\begin{aligned}&\mathbb{E}_\nu[Q''_{k',a,b}(Q_{k,0}(h))]-\Phi(0)\mathfrak{a}_k^{-d}\mathbb{E}_\nu\left[Q''_{k',a,b}(\mathbf{1})\right]\mathbb{E}_\nu[h]\\&=\left(\mathcal{O}(|a|^{\eta'}|b|^{\eta'}\mathfrak{a}_{k'}^{-\alpha-2\varepsilon'}\mathfrak{a}_k^{-d}\|h\|_{\mathcal{B}})\right)^{\frac{\alpha+\varepsilon'}{\alpha+2\varepsilon'}}\left(\mathcal{O}(\|h\|_{\mathcal{B}})o(\mathfrak{a}_k^{-d})\right)^{\frac{\varepsilon'}{\alpha+2\varepsilon'}}\\&=\mathcal{O}(|a|^{\frac{\eta'(\alpha+\varepsilon')}{\alpha+2\varepsilon'}}|b|^{\frac{\eta'(\alpha+\varepsilon')}{\alpha+2\varepsilon'}}\mathfrak{a}_{k'}^{-\alpha-\varepsilon'}\|h\|_{\mathcal{B}})o(\mathfrak{a}_k^{-d})\,.\end{aligned}$$

After summation over $a,b\in\mathbb{Z}^d$, we obtain (43), since $\frac{\eta'(\alpha+\varepsilon')}{\alpha+2\varepsilon'}\le\eta$ (due to (44)) and since $\sum_{a\in\mathbb{Z}^d}(1+|a|^\eta)|\beta_a|<\infty$.

Using (43) inductively in (40) (combined with the fact that $\|Q_{k,0}\|_{\mathcal{B}} = \mathcal{O}(\mathfrak{a}_k^{-d})$ and that $\|\bar{Q}_k\|_{\mathcal{B}}$ is summable which follows from (42)), we conclude that, when $N$ is even

$$H^{(n,N)}_{0,1,\dots,0,1} = \sum_{0\le k_1\le\dots\le k_N} c_{(k_1,\dots,k_N)} \prod_{j=1}^{N/2}\left(\Phi(0)\left(\mathfrak{a}^{-d}_{k_{2j-1}-k_{2j-2}}\mathbb{E}_\nu[\bar{Q}_{k_{2j}-k_{2j-1}}(\mathbf{1})]\right)\right)$$
$$+\,\mathcal{O}\left(\left(\sum_{k=1}^n \mathfrak{a}_k^{-d}\right)^{N/2-1}\sum_{k=1}^n o(\mathfrak{a}_k^{-d})\right)\,.$$

Recall that

$$\sigma_\beta^2 = \sum_{k\in\mathbb{Z}}\mathbb{E}_\nu\left[\bar{Q}_{|k|}(\mathbf{1})\right] = \mathbb{E}_\nu\left[\bar{Q}_0(\mathbf{1})\right] + 2\sum_{k\ge1}\mathbb{E}_\nu\left[\bar{Q}_k(\mathbf{1})\right]\,.$$

Therefore, proceeding exactly as in [40, p. 1918-1919], we obtain that

$$H^{(n,N)}_{0,1,\dots,0,1} = \frac{\Gamma(1+\frac{\alpha-d}{\alpha})^{N/2}}{\Gamma(1+\frac{N}{2}\frac{\alpha-d}{\alpha})}\frac{N!}{2^{N/2}}\left(\Phi(0)\sigma_\beta^2\mathfrak{A}_n\right)^{N/2} + o\left(\mathfrak{A}_n^{N/2}\right)$$
$$= \mathfrak{A}_n^{N/2}\mathbb{E}\left[\left(\sqrt{\sigma_\beta^2\Phi(0)\mathcal{Y}}\mathcal{N}\right)^N\right] + o\left(\mathfrak{A}_n^{N/2}\right)\,.$$

This, combined with (38) and (39), ends the proof of the convergence of every moments. We conclude the convergence in distribution by the Carleman criterion [47]. This ends the proof of Theorem 2.3.

4.2. **Proof of** (37) **in the general case.** We assume here that $\sum_{a\in\mathbb{Z}^d}(1+|a|^\eta)|\beta_a|<\infty$ with $\eta := \frac{\alpha-d+\varepsilon}{2}$ for some $\varepsilon\in(0,1/2)$.

In Formula (35), we decompose $Q_{k,b-a}$ using the operators $Q''_{k,a,b}$ and $Q'_{k,c} := Q'_{k,0,c} = Q_{k,c} - Q_{k,0}$ as follows

$$Q_{k,b-a} = Q''_{k,a,b} + Q_{k,b} + Q_{k,-a} - Q_{k,0} = Q''_{k,a,b} + Q'_{k,b} + Q'_{k,-a} + Q_{k,0}\,.$$

In (35), we replace each $Q_{k_j-k_{j-1},a_j-a_{j-1}}$ by this decomposition, we develop and obtain

$$\mathbb{E}_\nu\left[\left(\sum_{k=0}^{n-1}\beta_{S_k}\right)^N\right] = \sum_{\varepsilon_1,\dots,\varepsilon_N} D^{(n,N)}_{\varepsilon_1,\dots,\varepsilon_N}\,, \tag{47}$$

summing a priori over $(\varepsilon_1,\dots,\varepsilon_N)\in(\{0,1\}^2)^N$ such that $\varepsilon_1\in\{(0,0),(0,1)\}$ the following quantity

$$D^{(n,N)}_{\varepsilon_1,\dots,\varepsilon_N} = \sum_{0\le k_1\le\dots\le k_N\le n-1} c_{(k_1,\dots,k_N)}\sum_{a_1,\dots,a_N\in\mathbb{Z}^d}\left(\prod_{j=1}^N\beta_{a_j}\right)\times$$
$$\times\,\mathbb{E}_\nu[\widetilde{Q}^{(\varepsilon_N)}_{k_N-k_{N-1},a_{N-1},a_N}\dots\widetilde{Q}^{(\varepsilon_2)}_{k_2-k_1,a_1,a_2}\widetilde{Q}^{(\varepsilon_1)}_{k_1,a_0,a_1}(\mathbf{1})]\,,$$

with $a_0=0$ and where we set

$$\widetilde{Q}^{(0,0)}_{k,a,b} = Q_{k,0},\ \widetilde{Q}^{(1,0)}_{k,a,b} = Q'_{k,-a},\ \widetilde{Q}^{(0,1)}_{k,a,b} = Q'_{k,b},\ \widetilde{Q}^{(1,1)}_{k,a,b} = Q''_{k,a,b}\,.$$

We assume that $\varepsilon_1\in\{(0,0),(0,1)\}$ since

$$\widetilde{Q}^{(1,0)}_{k,a_0,a_1} = Q'_{k,0} = 0 \quad\text{and}\quad \widetilde{Q}^{(1,1)}_{k,a_0,a_1} = Q''_{k,0,a_1} = 0\,.$$

We will restrict the sum over $\varepsilon_1,...,\varepsilon_N$. Let us write $\varepsilon_j=(\varepsilon_{j,1},\varepsilon_{j,2})$. We observe that, since $\sum_{a_j\in\mathbb Z^d}\beta_{a_j}=0$, $D^{(n,N)}_{\varepsilon_1,...,\varepsilon_N}=0$ if there exists $j=1,...,N$ such that $\varepsilon_{j,2}+\varepsilon_{j+1,1}=0$ (with convention $\varepsilon_{N+1,1}=0$). Thus we restrict the sum in (47) to the sum over the $\varepsilon_1,...,\varepsilon_N$ such that for all $j=1,...,N$, such that $\varepsilon_{j,2}+\varepsilon_{j+1,1}\ge 1$. We call **admissible** any such sequence $\boldsymbol\varepsilon:=(\varepsilon_1,...,\varepsilon_N)$. Let $\boldsymbol\varepsilon:=(\varepsilon_1,...,\varepsilon_N)$ be an admissible sequence.

- We observe that $\#\{j\,:\,\varepsilon_j=(0,0)\}\le N/2$.
- The contribution to (47) of an admissible sequence $\boldsymbol\varepsilon=(\varepsilon_1,...,\varepsilon_N)$ is

$$D^{(n,N)}_{\varepsilon_1,...,\varepsilon_N}=\mathcal O\left(\sum_{a_1,...,a_N\in\mathbb Z^d}\prod_{j=1}^N\left(|\beta_{a_j}|\sum_{k_j=0}^n\|\widetilde Q^{(\varepsilon_j)}_{k_j,a_{j-1},a_j}\|_{\mathcal B}\right)\right).$$

- We observe that there exists $u_0>0$ such that

$$\sum_{k=0}^n\mathfrak a_k^{-d}=\mathfrak A_n\,,\quad\sum_{k=0}^n\mathfrak a_k^{-d-2\eta}=\mathcal O(1)\,,\quad\sum_{k=0}^n\mathfrak a_k^{-d-\eta}=\mathcal O\left(\mathfrak A_n^{\frac{1-u_0}{2}}\right).\tag{48}$$

  Indeed $d+2\eta=\alpha+\varepsilon>\alpha$. For the last estimate, we use the fact that $(\mathfrak a_k)_k$ is $\frac1\alpha$-regularly varying, and infer that $(\sum_{k=0}^n\mathfrak a_k^{-d-\eta})^2$ is either bounded or $2-\frac{2d+2\eta}{\alpha}$-regularly varying whereas $(\mathfrak A_n)_n$ is $(1-\frac d\alpha)$-regularly varying and diverges to infinity (and $2-\frac{2d+2\eta}{\alpha}=1-\frac{d+\varepsilon}{\alpha}<1-\frac d\alpha$).
- If, for all $j=1,...,N$, $\varepsilon_{j,2}+\varepsilon_{j+1,1}=1$, then, it follows from Hypothesis 2.2 that

$$D^{(n,N)}_{\varepsilon_1,...,\varepsilon_N}=\mathcal O\left(d^{(n,N)}_{\varepsilon_1,...,\varepsilon_N}\right),$$

  with

$$d^{(n,N)}_{\varepsilon_1,...,\varepsilon_N}:=\sum_{a_1,...,a_N\in\mathbb Z^d}\prod_{j=1}^N\left(|\beta_{a_j}|\,|a_j|^{\eta(\varepsilon_{j,2}+\varepsilon_{j+1,1})}\sum_{k_j=0}^n\mathfrak a_{k_j}^{-d-\eta\varepsilon_{j,1}-\eta\varepsilon_{j,2}}\right),\tag{49}$$

  and so, using (48), that

$$D^{(n,N)}_{\varepsilon_1,...,\varepsilon_N}=\mathcal O\left(d^{(n,N)}_{\varepsilon_1,...,\varepsilon_N}\right)=\mathcal O\left(\mathfrak A_n^{N_0+N_1\frac{1-u_0}{2}}\right),$$

  where $N_k:=\#\{j\,:\,\varepsilon_{j,1}+\varepsilon_{j,2}=k\}$, since $\sum_{a\in\mathbb Z^d}(1+|a|)^\eta|\beta_a|<\infty$. Observe that $N_0+N_1+N_2=N$ and that $N=\sum_{k=1}^2\sum_{j=1}^N\varepsilon_{j,k}=N_1+2N_2$ and so $N_2=N_0$ and $N_0=\frac{N-N_1}{2}$. Therefore, in this case,

$$D^{(n,N)}_{\varepsilon_1,...,\varepsilon_N}=\mathcal O\left(d^{(n,N)}_{\varepsilon_1,...,\varepsilon_N}\right)=\mathcal O\left(\mathfrak A_n^{\frac{N-u_0N_1}{2}}\right)=o\left(\mathfrak A_n^{N/2}\right)$$

  unless $N_1=0$, i.e. unless $N$ is even and $\varepsilon_1,...,\varepsilon_N$ is the alternate sequence

$$(0,0),(1,1),...,(0,0),(1,1)\,.$$

- Assume now that there exists some $j_0\in\{1,...,N\}$ such that $\varepsilon_{j_0,2}+\varepsilon_{j_0+1,1}=2$. Recall that it follows from Hypothesis 2.2 that, for all $\eta'_{j,1},\eta'_{j,2}\in\{0,\eta\}$,

$$\left\|\widetilde Q^{(\varepsilon_j)}_{k,a_j,a_{j-1}}\right\|_{\mathcal B}=\mathcal O\left(|a_{j-1}/\mathfrak a_k|^{\eta'_{j,1}\varepsilon_{j,1}}|a_j/\mathfrak a_k|^{\eta'_{j,2}\varepsilon_{j,2}}\mathfrak a_k^{-d}\right).\tag{50}$$

Indeed this follows from (18), (19) and (20) combined with the two following facts

$$\forall\eta\in[0;1],\quad \left\|Q''_{k,a,b}\right\|_{\mathcal B}=\left\|Q'_{k,-a,b-a}-Q'_{k,0,b}\right\|_{\mathcal B}=\mathcal O(|b|^\eta)\mathfrak a_k^{-d-\eta}$$

and

$$\forall\eta\in[0;1],\quad \left\|Q''_{k,a,b}\right\|_{\mathcal B}=\left\|Q'_{k,b,b-a}-Q'_{k,0,-a}\right\|_{\mathcal B}=\mathcal O(|a|^\eta)\mathfrak a_k^{-d-\eta}\,.$$

We choose a family $(\eta'_{j,i})_{j=1,...,N;i=1,2}$ of $\{0,\eta\}$ such that, for all $j=1,...,N$, $\eta'_{j,2}\varepsilon_{j,2}+\eta'_{j+1,1}\varepsilon_{j+1,1}=\eta=\frac{\alpha+\varepsilon-d}{2}$. To do so, we can take e.g.

$$\eta'_{j,2}=\eta'_{j+1,1}=\eta\quad\text{if } \varepsilon_{j,2}+\varepsilon_{j+1,1}=1\,,$$
$$\eta'_{j,2}=\eta,\ \eta_{j+1,1}=0\quad\text{if } \varepsilon_{j,2}+\varepsilon_{j+1,1}=2\,.$$

Therefore

$$D^{(n,N)}_{\varepsilon_1,...,\varepsilon_N}=\mathcal O\left(\sum_{a_1,...,a_N\in\mathbb Z^d}\prod_{j=1}^N\left(|\beta_{a_j}|\,|a_j|^\eta\sum_{k_j=0}^n\mathfrak a_{k_j}^{-d-\eta'_{j,1}\varepsilon_{j,1}-\eta'_{j,2}\varepsilon_{j,2}}\right)\right)=\mathcal O\left(d^{(n,N)}_{\varepsilon'_1,...,\varepsilon'_N}\right)\,,$$

where we set $\varepsilon'_j:=(\eta'_{j,1}\varepsilon_{j,1},\eta'_{j,2}\varepsilon_{j,2})/\eta$. We also consider the sequence $\varepsilon''_1,...,\varepsilon''_N$ obtained from $\boldsymbol\varepsilon'$ by permuting the values of $\varepsilon'_{j_0,2}$ and $\varepsilon'_{j_0+1,1}$. Both sequences $\boldsymbol\varepsilon'$ and $\boldsymbol\varepsilon''$ are admissible and satisfy $\varepsilon'_{j,2}+\varepsilon'_{j+1,1}=\varepsilon''_{j,2}+\varepsilon''_{j+1,1}=1$ for all $j=1,...,N$. Thus it follows from the previous item that

$$D^{(n,N)}_{\varepsilon_1,...,\varepsilon_N}=\mathcal O\left(\min\left(d^{(n,N)}_{\varepsilon'_1,...,\varepsilon'_N},d^{(n,N)}_{\varepsilon''_1,...,\varepsilon''_N}\right)\right)=o\left(\mathfrak A_n^{N/2}\right)\,,$$

since $\boldsymbol\varepsilon'$ and $\boldsymbol\varepsilon''$ cannot both coincide with the alternate sequence $(0,0),(1,1),...,(0,0),(1,1)$.

Thus we have prove that

$$D^{(n,N)}_{\varepsilon_1,...,\varepsilon_N}=o\left(\mathfrak A_n^{N/2}\right)$$

unless $N$ is even and $\varepsilon_1,...,\varepsilon_N$ is the alternate sequence $(0,0),(1,1),...,(0,0),(1,1)$. Estimate (37) follows from this fact since

$$Q^{(1)}_{k_j-k_{j-1},a_j-a_{j-1}}=\widetilde Q^{(1,1)}_{k_j-k_{j-1},a_j-a_{j-1}}+\widetilde Q^{(1,0)}_{k_j-k_{j-1},a_j-a_{j-1}}+\widetilde Q^{(0,1)}_{k_j-k_{j-1},a_j-a_{j-1}}$$

and

$$Q^{(0)}_{k_j-k_{j-1},a_j-a_{j-1}}=\widetilde Q^{(0,0)}_{k_j-k_{j-1},a_j-a_{j-1}}\,.$$

4.3. **Proof of Theorem 2.6.** We start by proving the convergence of the finite distributions and we will then prove the tightness. For the convergence of the finite distributions, we use again the convergence of moments. It is enough to study the asymptotic behaviour as $n$ goes to infinity of every moments of the following form

$$E_n:=\mathbb E_\nu\left[\prod_{j=1}^M\left(\sum_{k_j=\lfloor nt_{j-1}\rfloor}^{\lfloor nt_j\rfloor-1}\beta^{(0)}_{S_k}\right)^{N_j^{(0)}}\left(\sum_{k'_j=\lfloor nt_{j-1}\rfloor}^{\lfloor nt_j\rfloor-1}\beta^{(1)}_{S_k}\right)^{N_j^{(1)}}\right]\,,$$

for any $M\in\mathbb N^*$, any $N_j^{(0)},N_j^{(1)}\in\mathbb N$, any $t_0=0<t_1<...<t_M$. We set $\Gamma_k:=\sum_{j=1}^M N_j^{(k)}$ for $k\in\{0,1\}$ and will prove that

$$\lim_{n\to+\infty}\mathfrak A_n^{-\Gamma_0-\frac{\Gamma_1}{2}}E_n=(\Phi(0))^{\Gamma_0+\frac{\Gamma_1}{2}}\left(\sum_{a\in\mathbb Z^d}\beta^{(0)}_a\right)^{\Gamma_0}\sigma^{\Gamma_1}_{\beta^{(1)}}\mathbb E\left[\prod_{j=1}^M\left(\mathcal L_{t_j}-\mathcal L_{t_{j-1}}\right)^{N_j^{(0)}+\frac{N_j^{(1)}}{2}}\right]\prod_{j=1}^M\mathbb E[\mathcal N^{N_j^{(1)}}]\,.$$

We set $m_i := \sum_{r=1}^i (N_r^{(0)} + N_r^{(1)})$ and $N := m_M$. Proceeding as in the proof of Theorem 1.1, we observe that $E_n$ can be rewritten as follows

$$\sum_{\gamma_1,\dots,\gamma_N} \sum_{(k_1,\dots,k_N)\in\mathcal{K}_n} \left(\prod_{i=1}^N c^{(\gamma_{m_{i-1}+1},\dots,\gamma_{m_i})}_{(k_{m_{i-1}+1},\dots,k_{m_i})}\right) \sum_{a_1,\dots,a_N\in\mathbb{Z}^d} \left(\prod_{j=1}^N \beta_{a_j}^{(\gamma_j)}\right) \nu\left(\forall i=1,\dots,N,\ S_{k_i}=a_i\right)\,, \tag{51}$$

where $\mathcal{K}_n$ is the set of increasing $N$-tuples $(k_1,\dots,k_N)$ such that $\lfloor nt_{j-1}\rfloor \le k_m \le \lfloor nt_j\rfloor - 1$ for all $m$ such that $m_{j-1} < m \le m_j$, where the first sum is taken over $\gamma_j \in \{0,1\}$ such that, for all $\gamma \in \{0,1\}$ and all $i = 1,\dots,M$, $\#\{j = m_{i-1}+1,\dots,m_i\ :\ \gamma_j = \gamma\} = N_i^{(\gamma)}$ and where $c^{(\gamma_{m_{i-1}+1},\dots,\gamma_{m_i})}_{(k_{m_{i-1}+1},\dots,k_{m_i})}$ is the number of $(\gamma'_{m_{i-1}+1},\dots,\gamma'_{m_i},k'_{m_{i-1}+1},\dots,k'_{m_i}) \in \{0,1\}^{N_i^{(1)}+N_i^{(2)}} \times \{\lfloor nt_{i-1}\rfloor,\dots,\lfloor nt_i\rfloor-1\}^{N_i^{(0)}+N_i^{(1)}}$ such that there exists a permutation $\sigma$ of $\{m_{i-1}+1,\dots,m_i\}$ such that $\gamma'_{\sigma(r)} = \gamma_r$ and $k'_{\sigma(r)} = k_r$ for all $r \in \{m_{i-1}+1,\dots,m_i\}$. Furthermore we use (34) to express $\nu(\forall i=1,\dots,N,\ S_{k_i}=a_i)$ using a composition of operators $Q_{k_j-k_{j-1},a_j-a_{j-1}}$ and, as in Section 4.2, for each $j$, we decompose each $Q_{k_j-k_{j-1},a_j-a_{j-1}}$ in a sum of $\widetilde{Q}^{(\varepsilon)}$. This leads to

$$E_n := \sum_{\boldsymbol{\gamma}=(\gamma_1,\dots,\gamma_N)} \sum_{\boldsymbol{\varepsilon}=(\varepsilon_1,\dots,\varepsilon_N)} \tilde{H}^{(n)}_{\boldsymbol{\varepsilon}}(\boldsymbol{\gamma})\,, \tag{52}$$

with

$$\tilde{H}^{(n)}_{\varepsilon_1,\dots,\varepsilon_N}(\gamma_1,\dots,\gamma_N) := \sum_{(k_1,\dots,k_N)\in\mathcal{K}_n} \left(\prod_{i=1}^M c^{(\gamma_{m_{i-1}+1},\dots,\gamma_{m_i})}_{(k_{m_{i-1}+1},\dots,k_{m_i})}\right) \sum_{a_1,\dots,a_N\in\mathbb{Z}^d} \mathbb{E}_\nu\left[\beta_{a_N}^{(\gamma_N)}\left(\beta_{a_{N-1}}^{(\gamma_{N-1})}\widetilde{Q}^{(\varepsilon_N)}_{k_N-k_{N-1},a_{N-1},a_N}\left(\cdots\left(\beta_{a_1}^{(\gamma_1)}\widetilde{Q}^{(\varepsilon_2)}_{k_2-k_1,a_1,a_2}\left(\widetilde{Q}^{(\varepsilon_1)}_{k_1,0,a_1}(\mathbf{1})\right)\right)\cdots\right)\right)\right]\,, \tag{53}$$

with the use of the operators $\widetilde{Q}^{(\varepsilon)}_{(k,a,b)}$ defined in Section 4.2 and where the sum over $\varepsilon_1,\dots,\varepsilon_N$ is taken over $\varepsilon_2,\dots,\varepsilon_N \in \{0,1\}^2$ and

$$\varepsilon_{1,1} = 0\,.$$

We write $\varepsilon_j = (\varepsilon_{j,1},\varepsilon_{j,2})$. Since $\sum_{a\in\mathbb{Z}^d}\beta_a^{(1)} = 0$, $H_{\varepsilon_1,\dots,\varepsilon_N} = 0$ if there exists $j = 1,\dots,N$ such that $\gamma_j = 1$ and $\varepsilon_{j,2}+\varepsilon_{j+1,1} = 0$ (with convention $\varepsilon_{N+1,1} = 0$). Therefore we assume from now on that $(\boldsymbol{\gamma},\boldsymbol{\varepsilon})$ is such that

$$\forall j = 1,\dots,N, \quad \varepsilon_{j,2}+\varepsilon_{j+1,1} \ge \gamma_j\,,$$

with the convention $\varepsilon_{N+1,1} = 0$ and we call **admissible** such a pair $(\boldsymbol{\gamma},\boldsymbol{\varepsilon})$. We want to determine the $\boldsymbol{\varepsilon}$ such that

$$\tilde{H}^{(n)}_{\varepsilon_1,\dots,\varepsilon_N}(\gamma_1,\dots,\gamma_N) \ne o\left(\mathfrak{A}_n^{\Gamma_0+\frac{\Gamma_1}{2}}\right)\,.$$

Then

$$\begin{aligned}\tilde{H}^{(n)}_{\boldsymbol{\varepsilon}}(\boldsymbol{\gamma}) &= \mathcal{O}\left(\sum_{(k_1,\dots,k_N)} \sum_{a_1,\dots,a_N\in\mathbb{Z}^d} \prod_{j=1}^N \left(|\beta_{a_j}^{(\gamma_j)}|\left\|\widetilde{Q}^{(\varepsilon_j)}_{k_j-k_{j-1},a_{j-1},a_j}\right\|_{\mathcal{B}}\right)\right)\\ &= \mathcal{O}\left(\sum_{(k_1,\dots,k_N)} \sum_{a_1,\dots,a_N\in\mathbb{Z}^d} \prod_{j=1}^N \left(|\beta_{a_j}^{(\gamma_j)}|\,(1+|a_j|)^\eta \mathfrak{a}_{k_j-k_{j-1}}^{-d-\eta_{j,1}\varepsilon_{j,1}-\eta_{j,2}\varepsilon_{j,2}}\right)\right)\,,\end{aligned}$$

with $(\eta_{j,i})_{j,i}$ a sequence in $\{0,\eta\}$ such that

$$\eta\gamma_j \le \eta_{j,2}\varepsilon_{j,2} + \eta_{j+1,1}\varepsilon_{j+1,1} \le \eta\,. \tag{54}$$

We set

$$\varepsilon'_{j,i} := \eta_{j,i}\varepsilon_{j,i}\,.$$

As seen in Section 4.2, we use the fact that there exists $u_0 \in (0,1]$ such that $\sum_{k=0}^{n-1} \mathfrak{a}_k^{-d-\eta} = \mathcal{O}\left(\mathfrak{A}_n^{\frac{1-u_0}{2}}\right)$ and that $\sum_{k\ge 0} \mathfrak{a}_k^{-d-2\eta} < \infty$. Using the summability assumption on $\beta_a^{(\gamma)}$, we infer that

$$\tilde{H}_{\boldsymbol{\varepsilon}}^{(n)}(\boldsymbol{\gamma}) = \mathcal{O}\left(\sum_{(k_1,\dots,k_N)} \prod_{j:\varepsilon_{j,1}+\varepsilon_{j,2}=0} \mathfrak{a}_{k_j-k_{j-1}}^{-d-\varepsilon'_{j,1}-\varepsilon'_{j,2}}\right) = \mathcal{O}\left(\mathcal{A}_n^{\mathcal{E}_0^{\boldsymbol{\eta}}+\frac{1-u_0}{2}\mathcal{E}_1^{\boldsymbol{\eta}}}\right)\,, \tag{55}$$

where

$$\mathcal{E}_k^{\boldsymbol{\eta}} := \#\left\{j=1,\dots,N : \varepsilon'_{j,1}+\varepsilon'_{j,2} = k\eta\right\}\,.$$

We also set $\mathcal{E}_k := \#\{j=1,\dots,N : \varepsilon_{j,1}+\varepsilon_{j,2} = k\}$. Recall that

$$\Gamma_k = \sum_{j=1}^N N_j^{(k)} = \#\{j=1,\dots,N : \gamma_j = k\}\,.$$

We observe that

$$N = \mathcal{E}_0^{\boldsymbol{\eta}} + \mathcal{E}_1^{\boldsymbol{\eta}} + \mathcal{E}_2^{\boldsymbol{\eta}} = \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2 = \Gamma_0 + \Gamma_1\,. \tag{56}$$

It follows from (54) that

$$\begin{aligned}\eta\Gamma_1 = \eta\sum_{j=1}^N \gamma_j &\le \sum_{j=1}^N (\varepsilon'_{j,2} + \varepsilon'_{j+1,1}) \\ &= \sum_{j=1}^N (\varepsilon'_{j,1} + \varepsilon'_{j,2})\,,\end{aligned}$$

since $\varepsilon_{1,1} = 0$ and so $\varepsilon'_{1,1} = 0$. Thus

$$\eta\Gamma_1 + \sum_{j=1}^N \left(\varepsilon'_{j,2} + \varepsilon'_{j+1,1} - \eta\gamma_j\right) = \eta\left(\mathcal{E}_1^{\boldsymbol{\eta}} + 2\mathcal{E}_2^{\boldsymbol{\eta}}\right)\,. \tag{57}$$

We now use this estimate combined with (56) to control the exponent of $\mathfrak{A}_n$ in (55) as follows

$$\begin{aligned}\mathcal{E}_0^{\eta} + \frac{1-u_0}{2}\mathcal{E}_1^{\boldsymbol{\eta}} &= N - \mathcal{E}_2^{\boldsymbol{\eta}} - \mathcal{E}_1^{\boldsymbol{\eta}} + \frac{1-u_0}{2}\mathcal{E}_1^{\boldsymbol{\eta}} \\ &\le (\Gamma_0+\Gamma_1) - \left(\frac{\Gamma_1 - \mathcal{E}_1^{\boldsymbol{\eta}}}{2}\right) - \frac{w_{\boldsymbol{\varepsilon}'}/\eta + (1+u_0)\mathcal{E}_1^{\boldsymbol{\eta}}}{2} \\ &= \Gamma_0 + \frac{\Gamma_1}{2} - \frac{w_{\boldsymbol{\varepsilon}'}/\eta + u_0\mathcal{E}_1^{\boldsymbol{\eta}}}{2}\,,\end{aligned}$$

with

$$\begin{aligned}w_{\boldsymbol{\varepsilon}'} &:= \sum_{j=1}^N \left(\varepsilon'_{j,2} + \varepsilon'_{j+1,1} - \eta\gamma_j\right) \\ &= \eta\,\#\left\{j=1,\dots,N \,:\, \gamma_j = 0,\ \varepsilon'_{j,2} + \varepsilon'_{j+1,1} > 0\right\}\,.\end{aligned}$$

Hence we have proved that

$$H_{\boldsymbol{\varepsilon}}^{(n)}(\boldsymbol{\gamma}) = \mathcal{O}\left(\mathcal{A}_n^{\Gamma_0+\frac{\Gamma_1}{2}}\right).$$

and that

$$H_{\boldsymbol{\varepsilon}}^{(n)}(\boldsymbol{\gamma}) = o\left(\mathcal{A}_n^{\Gamma_0+\frac{\Gamma_1}{2}}\right) \quad \text{if } \mathcal{E}_1^{\boldsymbol{\eta}} > 0 \quad \text{or} \quad w_{\boldsymbol{\varepsilon}'} > 0\,.$$

We assume now that

$$H_{\boldsymbol{\varepsilon}}^{(n)}(\boldsymbol{\gamma}) \neq o\left(\mathcal{A}_n^{\Gamma_0+\frac{\Gamma_1}{2}}\right).$$

This implies that $\mathcal{E}_1^{\boldsymbol{\eta}} = w_{\boldsymbol{\varepsilon}'} = 0$ for any $(\varepsilon'_{j,i})$ as above.

(i) We consider $\varepsilon'_{j,i}$ such that
  - $\varepsilon'_{1,1} = 0$,
  - If $(\varepsilon_{j,2}, \varepsilon_{j+1,1}) \neq (1,1)$, then $\varepsilon'_{j,2} = \eta\varepsilon_{j,2}$ and $\varepsilon'_{j+1,1} = \eta\varepsilon_{j+1,1}$,
  - If $(\varepsilon_{j,2}, \varepsilon_{j+1,1}) = (1,1)$, then $\varepsilon'_{j,2} = \eta$ and $\varepsilon'_{j+1,1} = 0$.

  Let $j \in \{1, ..., N\}$ such that $\gamma_j = 0$. The fact that $w_{\boldsymbol{\varepsilon}'} = 0$ ensures that $\varepsilon'_{j,2} = \varepsilon'_{j+1,1} = 0$. By definition of $\varepsilon'_{j,i}$, this implies that $\varepsilon_{j,2} = \varepsilon_{j+1,1} = 0$. Since $\mathcal{E}_1^{\boldsymbol{\eta}} = 0$, this implies also that, for each $j = 1, ..., N$, $\varepsilon'_{j+1,2} = 0$ and so that $\varepsilon_{j+1,2} = 0$.
  The fact that $w_{\boldsymbol{\varepsilon}'} = 0$ combined with (57) ensures that $\Gamma_1 = \mathcal{E}_1^{\boldsymbol{\eta}} + 2\mathcal{E}_2^{\boldsymbol{\eta}}$. But, since $\mathcal{E}_1^{\boldsymbol{\eta}} = 0$, we conclude that $\mathcal{E}_2^{\boldsymbol{\eta}} = \frac{\Gamma_1}{2}$.
  Let $j'$ such that $(\varepsilon'_{j',1}, \varepsilon'_{j',2}) = (\eta, \eta)$, then $\gamma_{j'} = 1$ (due to the previous analysis of the $j$'s such that $\gamma_j = 0$). Furthermore, since $\varepsilon'_{j',i} \leq \eta\varepsilon_{j,i}$, it follows that $\varepsilon_{j'} = (1,1)$, that $j' \in \{2, ..., N-1\}$ (since $\varepsilon_{1,1} = 0$) and $\varepsilon_{j'-1,2} = 0$ (otherwise we would have $\varepsilon'_{j'-1,2} = \eta$ and $\varepsilon'_{j',1} = 0$).

(ii) Now, exchanging the role played by $\varepsilon_{j,2}$ and $\varepsilon_{j+1,1}$, we consider $\varepsilon'_{j,i}$ such that
  - $\varepsilon'_{1,1} = 0$,
  - If $(\varepsilon_{j,2}, \varepsilon_{j+1,1}) \neq (1,1)$, then $\varepsilon'_{j,2} = \eta\varepsilon_{j,2}$ and $\varepsilon'_{j+1,1} = \eta\varepsilon_{j+1,1}$,
  - If $(\varepsilon_{j,2}, \varepsilon_{j+1,1}) = (1,1)$, then $\varepsilon'_{j,2} = 0$ and $\varepsilon'_{j+1,1} = \eta$.

  Arguing as in the previous item, we conclude that
  - if $\gamma_j = 0$, then $\varepsilon_{j,2} = \varepsilon_{j+1,1} = 0$, and, since $\mathcal{E}_1^{\boldsymbol{\eta}} = 0$, then $\varepsilon'_{j,1} = 0$ and so $\varepsilon_{j,1} = 0$,
  - if $(\varepsilon'_{j,1}, \varepsilon'_{j,2}) = (\eta, \eta)$, then $\gamma_j = 1$ (due to the previous item) and $\varepsilon_j = (1,1)$, and so $j \in \{2, ...N-1\}$ (since $\varepsilon_{1,1} = 0$) and $\varepsilon_{j+1,1} = 0$.

Gathering, all these facts, we conclude that :

- $\gamma_j = 0$ implies that $\varepsilon_j = (0,0)$ and $\varepsilon_{j+1} = (0,0)$.
- $\Gamma_1$ is even and the set of $j$'s such that $\gamma_j = 1$ is a disjoint union of two sets $\mathcal{J}$ and $\mathcal{J}'$ of same cardinal $\Gamma_1/2$ such that
  - The set $\mathcal{J}$ is the set of $j \in \{1, ..., N\}$ such that $\gamma_j = 1$ and $\varepsilon_j = (1,1)$. For any $j \in \mathcal{J}$, we also have $\gamma_{j-1} = 1$, $\varepsilon_{j-1,2} = 0$, $\varepsilon_{j+1,1} = 0$.
  - The set $\mathcal{J}'$ is the set of $j$'s such that $\gamma_j = 1$, $(j+1) \in \mathcal{J}$ and $\varepsilon_{j,2} = 0$. Furthermore, since these points are either $j = 1$, or after a $j-1$ such that $\gamma_{j-1} = 0$, or after $j-1 \in \mathcal{J}$, we conclude that they satisfy also $\varepsilon_j = (0,0)$.

Hence we have proved that

$$H_{\boldsymbol{\varepsilon}}^{(n)}(\boldsymbol{\gamma}) \neq o\left(\mathcal{A}_n^{\Gamma_0+\frac{\Gamma_1}{2}}\right)$$

implies that $\Gamma_1$ is even, that $\boldsymbol{\varepsilon}$ is a sequence of $(0,0)$ and $(1,1)$ and that

- if $\gamma_j=0$, then $\varepsilon_j=\varepsilon_{j+1}=(0,0)$,
- if $\gamma_j=1$, then either $[\varepsilon_j=(0,0)$ and $\gamma_{j+1}=1$ and $\varepsilon_{j+1}=(1,1)]$ or $[\varepsilon_j=(1,1)$ and $\gamma_{j-1}=1$ and $\varepsilon_{j-1}=(0,0)]$.

This means that the $j$'s such that $\gamma_j=1$ appear in pairwise disjoint pairs $(j-1,j)$ such that $(\gamma_{j-1},\gamma_j)=(1,1)$, and that $\varepsilon_j=(1,1)$ if and only if $(j-1,j)$ is such a couple. We fix such a pair $(\boldsymbol{\gamma},\boldsymbol{\varepsilon})$, and still write $\mathcal{J}$ for the set of $j$ such that $\varepsilon_j=(1,1)$.

Then, using repeatedly (16), (17) and (45), we obtain

$$\mathfrak{A}_n^{-\Gamma_0-\frac{\Gamma_1}{2}}\tilde H_{\boldsymbol{\varepsilon}}^{(n)}(\gamma)=\sum_{(k_1,\dots,k_N)\in\mathcal{K}_n}\left(\prod_{i=1}^M c^{(\gamma_{m_{i-1}+1},\dots,\gamma_{m_i})}_{(k_{m_{i-1}+1},\dots,k_{m_i})}\right)\prod_{j':\gamma_{j'}=0}\left(\frac{\mathfrak{a}^{-d}_{k_{j'}-k_{j'-1}}}{\mathfrak{A}_n}\Phi(0)\sum_{a\in\mathbb{Z}^d}\beta_a^{(0)}\right)$$
$$\prod_{j\in\mathcal{J}}\frac{\Phi(0)\mathfrak{a}^{-d}_{k_{j-1}-k_{j-2}}}{\mathfrak{A}_n}\sum_{a,b\in\mathbb{Z}^d}\beta_a^{(1)}\beta_b^{(1)}\mathbb{E}_\nu\left[Q''_{k_j-k_{j-1},a,b}(\mathbf{1})\right]+o(1)$$

and so

$$\mathfrak{A}_n^{-\Gamma_0-\frac{\Gamma_1}{2}}\tilde H_{\boldsymbol{\varepsilon}}^{(n)}(\boldsymbol{\gamma})=o\left(1\right)+\Phi(0)^{\Gamma_0+\frac{\Gamma_1}{2}}\left(\sum_{a\in\mathbb{Z}^d}\beta_a^{(0)}\right)^{\Gamma_0}\sum_{(k_1,\dots,k_N)\in\mathcal{K}_n:k_j<k_{j+1}}\prod_{i=1}^M(N_i^{(0)})!(N_i^{(1)})!$$
$$\left(\prod_{j:\gamma_j=0}\frac{\mathfrak{a}^{-d}_{k_j-k_{j-1}}}{\mathfrak{A}_n}\right)\prod_{j\in\mathcal{J}}\frac{\mathfrak{a}^{-d}_{k_{j-1}-k_{j-2}}}{\mathfrak{A}_n}\sum_{a,b\in\mathbb{Z}^d}\beta_a^{(1)}\beta_b^{(1)}\mathbb{E}_\nu\left[Q''_{k_j-k_{j-1},a,b}(\mathbf{1})\right].$$

It can be worthwhile to notice that we can restrict the above sum on the set $\mathcal{K}'_n$ made of the $(k_1,...,k_N)\in\mathcal{K}_n$ such that $k_j-k_{j-1}<\log n$ if $j\in\mathcal{J}$, and $k_j-k_{j-1}>\log n$ for the other values of $j$'s. Let us observe that, for any $(k_1,...,k_N)\in\mathcal{K}'_n$, since $k_j>k_{j-1}$ for all $j\notin\mathcal{J}$, we have

$$c^{(\gamma_{m_{i-1}+1},\dots,\gamma_{m_i})}_{(k_{m_{i-1}+1},\dots,k_{m_i})}=\left(\prod_{i=1}^N N_i^{(0)}!\,N_i^{(1)}!\right)\prod_{j\in\mathcal{J}}\frac{1+\mathbf{1}_{\{k_j-k_{j-1}\neq0\}}}{2}.\tag{58}$$

The fact that we can neglect the sum over $\mathcal{K}_n\setminus\mathcal{K}'_n$ implies that

$$\mathfrak{A}_n^{-\Gamma_0-\frac{\Gamma_1}{2}}\tilde H_{\boldsymbol{\varepsilon}}^{(n)}(\boldsymbol{\gamma})=o(1)$$

as soon as there exist $j\in\mathcal{J}$ and $i\in\{1,...,M\}$ such that $k_{j-1}<\lfloor nt_i\rfloor\le k_j$ (indeed this combined with $k_j-k_{j-1}<\log n$ implies that $0<k_j-nt_i<\log n$ and $0<nt_i-k_{j-1}<\log n$. In particular

$$E_n=o\left(\mathfrak{A}_n^{\Gamma_0+\frac{\Gamma_1}{2}}\right)\quad\text{if }\exists j\in\{1,...,M\},\ N_j^{(1)}\in2\mathbb{Z}+1\,.$$

We assume from now on that the $N_j^{(1)}$'s are even and that $\mathcal{J}$ is such that, for every $j\in\mathcal{J}$, there exists $i=1,...,M$ such that $k_{j-1},k_j$ are in a same set $\{\lfloor nt_{i-1}\rfloor,...,\lfloor nt_i\rfloor-1\}$. Then, using the fact that the sum over $\mathcal{K}_n\setminus\mathcal{K}'_n$ is neglectable and (58), we obtain that

$$\mathfrak{A}_n^{-\Gamma_0-\frac{\Gamma_1}{2}}\tilde H_{\boldsymbol\varepsilon}^{(n)}(\boldsymbol\gamma)=o(1)+\prod_{j=1}^M\left(N_j^{(0)}!\,N_j^{(1)}!(\Phi(0))^{N''_j}\left(\sum_{a\in\mathbb{Z}^d}\beta_a^{(0)}\right)^{N_j^{(0)}}\right.$$

$$\left.\left(\frac12\sum_{k\ge0}(1+\mathbf{1}_{\{k\ne0\}})\sum_{a,b\in\mathbb{Z}^d}\beta_a^{(1)}\beta_b^{(1)}\mathbb{E}_\nu\left[\bar Q''_{k_j-k_{j-1},a,b}(\mathbf{1})\right]\right)^{\frac{N_j^{(1)}}{2}}\sum_{(k'_1,\dots,k'_{N''})\in\mathcal{K}''_n}\prod_{j=1}^{N''}\frac{\mathfrak{a}^{-d}_{k'_j-k'_{j-1}}}{\mathfrak{A}_n}\right),$$

with $N''_j:=N_j^{(0)}+\frac{N_j^{(1)}}{2}$ and $N'':=\sum_{j=1}^M N''_j$, and where $\mathcal{K}''_n$ is the set of strictly increasing sequences $k'_1<\dots<k'_{N''}$ with exactly $N''_j$ elements between $\lfloor nt_{j-1}\rfloor$ and $\lfloor nt_j\rfloor-1$ and with the convention $k'_0=0$. Thus

$$\frac{\tilde H_{\boldsymbol\varepsilon}^{(n)}(\boldsymbol\gamma)}{\mathfrak{A}_n^{\sum_{j=1}^M N''_j}}=o(1)+E'_n\prod_{j=1}^M\left(N_j^{(0)}!N_j^{(1)}!\left(\sum_{a\in\mathbb{Z}^d}\beta_a^{(0)}\right)^{N_j^{(0)}}2^{-\frac{N_j^{(1)}}{2}}\sigma_{\beta^{(1)}}^{N_j^{(1)}}\frac{1}{N''_j!}\right)$$

with

$$E'_n:=(\Phi(0))^{N''}\left(\prod_{j=1}^M N''_j!\right)\sum_{(k'_1,\dots,k'_{N''})\in\mathcal{K}'_n}\prod_{j=1}^{N''}\frac{\mathfrak{a}^{-d}_{k'_j-k'_{j-1}}}{\mathfrak{A}_n}$$

$$=o(1)+\mathfrak{A}_n^{-N''}\mathbb{E}_\nu\left[\prod_{j=1}^M\left(\sum_{k_j=\lfloor nt_{j-1}\rfloor}^{\lfloor nt_j\rfloor-1}\mathbf{1}_{\{S_{k_j}=0\}}\right)^{N''_j}\right].\tag{59}$$

We observe that there exist $\frac{N''_j!}{N_j^{(0)}!(N_j^{(1)}/2)!}$ sequences $(\gamma_{m_j+1},\dots,\gamma_{m_{j+1}})\in\{0,1\}^{N_j^{(0)}+N_j^{(1)}}$ in which the 1's appear in $N_j^{(1)}/2$ pairwise distinct pairs $(\gamma_{j-1},\gamma_j)$. Therefore

$$\frac{E_n}{\mathfrak{A}_n^{\sum_{j=1}^M N''_j}}=o(1)+E'_n\prod_{j=1}^M\left(\frac{N_j^{(1)}!}{(N_j^{(1)}/2)!2^{\frac{N_j^{(1)}}{2}}}\left(\sum_{a\in\mathbb{Z}^d}\beta_a^{(0)}\right)^{N_j^{(0)}}\sigma_{\beta^{(1)}}^{N_j^{(1)}}\right)$$

$$=o(1)+E'_n\prod_{j=1}^M\left(\left(\sum_{a\in\mathbb{Z}^d}\beta_a^{(0)}\right)^{N_j^{(0)}}\mathbb{E}\left[(\sigma_{\beta^{(1)}}\mathcal{N})^{N_j^{(1)}}\right]\right).$$

It remains to study the asymptotics of $E'_n$.

- If $d=1<\alpha$, we consider a $\mathbb{Z}$-valued non-arithmetic random walk $(\widetilde S_n)_n$ (with i.i.d. increments) such that $(\widetilde S_{\lfloor nt\rfloor}/\mathfrak{a}_n)_n$ converges in distribution to the $\alpha$-stable process $W$. The previous computations hold also true (more easily) for $\widetilde S_n$ instead of $S_n$ and lead to

$$E'_n\sim\mathfrak{A}_n^{-N''}\mathbb{E}\left[\prod_{j=1}^M\left(\sum_{k_j=\lfloor nt_{j-1}\rfloor}^{\lfloor nt_j\rfloor-1}\mathbf{1}_{\{\widetilde S_{k_j}=0\}}\right)^{N''_j}\right]\quad\text{as } n\to+\infty\,.$$

But the process $\left(\sum_{k=0}^{\lfloor nt\rfloor -1}\mathbf{1}_{\{S_k=0\}}\right)_t$ of local time at 0 of $\widetilde{S}_n$ converges in distribution to the process $(\mathcal{L}_t)_t$ of local time at 0 of $W$. This combined with the dominations of the moments of any order ensures that $(E'_n)_n$ converges in distribution to $\mathbb{E}\left[\prod_{j=1}^M\left(\mathcal{L}_{t_j}-\mathcal{L}_{t_{j-1}}\right)^{N''_j}\right]$. We conclude that

$$\frac{E_n}{\mathfrak{A}_n^{\sum_{j=1}^M N''_j}}=o\left(1\right)+\mathbb{E}\left[\prod_{j=1}^M\left(\mathcal{L}_{t_j}-\mathcal{L}_{t_{j-1}}\right)^{N''_j}\right]\prod_{j=1}^M\left(\left(\sum_{a\in\mathbb{Z}^d}\beta_a^{(0)}\right)^{N_j^{(0)}}\mathbb{E}\left[\left(\sigma_{\beta^{(1)}}\mathcal{N}\right)^{N_j^{(1)}}\right]\right)$$

$$=o\left(1\right)+\mathbb{E}\left[\prod_{j=1}^M\left(\sum_{a\in\mathbb{Z}^d}\beta_a^{(0)}\left(\mathcal{L}_{t_j}-\mathcal{L}_{t_{j-1}}\right)\right)^{N_j^{(0)}}\prod_{j=1}^M\left(\sigma_{\beta^{(1)}}(B_{\mathcal{L}_{t_j}}-B_{\mathcal{L}_{t_{j-1}}})\right)^{N_j^{(1)}}\right].$$

- If $d=\alpha$, then

$$\mathfrak{A}_n^{-N''}\mathbb{E}_\nu\left[\left(\sum_{k=\lfloor NT_0\rfloor}^{\lfloor Nt_M\rfloor-1}\mathbf{1}_{\{S_k=0\}}\right)^{N''}\right]=\mathcal{O}\left(\mathfrak{A}_n^{-N''}\left(\sum_{k=\lfloor NT_0\rfloor}^{\lfloor nt_M\rfloor-1}\mathfrak{a}_k^{-d}\right)^{N''}\right)$$
$$=\mathcal{O}\left(\left((\mathfrak{A}_{\lfloor nt_M\rfloor}-\mathfrak{A}_{\lfloor nT_0\rfloor})/\mathfrak{A}_n\right)^{N''}\right),$$

converges to 0 as $n\to+\infty$, since $(\mathfrak{A}_n)_{n\ge 0}$ is slowly varying. Thus

$$E'_n=o(1)\quad\text{if } M\ge 2\,.$$

Furthermore, it follows from the proof of Theorem 1.1 that if $M=1$,

$$E'_n=o(1)+\mathfrak{A}_n^{-N''}\mathbb{E}_\nu\left[\left(\sum_{k_j=\lfloor nt_{j-1}\rfloor}^{\lfloor nt_j\rfloor-1}\mathbf{1}_{\{S_{k_j}=0\}}\right)^{N''_1}\right]$$
$$=(\Phi(0))^{N''_1}N''_1!=(\Phi(0))^{N''_1}\mathbb{E}[\mathcal{E}^{N''_1}],$$

where $\mathcal{E}$ is a random variable with standard exponential distribution due to theorem 2.3. We infer that

$$\frac{E_n}{\mathfrak{A}_n^{\sum_{j=1}^M N''_j}}=o\left(1\right)+\mathbb{E}\left[(\Phi(0)\mathcal{E})^{N''_1}\prod_{j=2}^M\left(\Phi(0)\mathcal{E}-\Phi(0)\mathcal{E}\right)^{N''_j}\right]\times$$
$$\times\prod_{j=1}^M\left(\left(\sum_{a\in\mathbb{Z}^d}\beta_a^{(0)}\right)^{N_j^{(0)}}\mathbb{E}\left[\left(\sigma_{\beta^{(1)}}\mathcal{N}\right)^{N_j^{(1)}}\right]\right).$$

Therefore

$$\frac{E_n}{\mathfrak{A}_n^{\sum_{j=1}^M N''_j}}=o\left(1\right)+\mathbb{E}\left[\prod_{j=1}^M\left(\sum_{a\in\mathbb{Z}^d}\beta_a^{(0)}\left(\Phi(0)\mathcal{E}(\mathbf{1}_{\{t_j>0\}}-\mathbf{1}_{\{t_{j-1}>0\}})\right)\right)^{N_j^{(0)}}\times\right.$$
$$\left.\times\prod_{j=1}^M(\sigma_{\beta^{(1)}}(B_{\Phi(0)\mathcal{E}\mathbf{1}_{\{t_j>0\}}}-B_{\Phi(0)\mathcal{E}\mathbf{1}_{\{t_j>0\}}}))^{N_j^{(1)}}\right].$$

This combined with the Carleman's criteria [47] ends the proof of the convergence of the finite dimensional distributions.

Let us write $((X_t^{(1,n)},X_t^{(2,n)})_t)_{n\ge 1}$ for the sequence of joint processes (23) and let us prove its tightness. When $d=1<\alpha$, we set $T_0=0$, otherwise we fix some $T_0\in(0;T)$. We use the tightness criterion of [2, Theorem 13.5, (13.4)]. We have proved the convergence of the finite dimensional distributions. It remains to prove that there exist $\alpha_1>1$ and $C>0$ such that, for every $r,s,t$ such that $T_0\le r\le s\le t\le T$, for all $j\in\{1,2\}$,

$$\exists p_j\in\mathbb{N}^*,\quad \mathbb{E}_\nu\left[|X_t^{(j,n)}-X_s^{(j,n)}|^{p_j}|X_s^{(j,n)}-X_r^{(j,n)}|^{p_j}\right]\le C|t-r|^{\alpha_1}\,. \tag{60}$$

Observe first that, if $0\le r\le s\le t\le T$ and $t-r<1/n$, then $X_t^{(n)}-X_s^{(n)}=0$ or $X_s^{(n)}-X_r^{(n)}=0$, thus the left hand side of (60) is null and so (60) holds true. Assume from now on that $T_0\le r\le s\le t\le T$ and that $t-r\ge 1/n$. We will use the following inequality

$$\mathbb{E}_\nu\left[|X_t^{(j,n)}-X_s^{(j,n)}|^{p_j}|X_s^{(n)}-X_r^{(n)}|^{p_j}\right]\le\left\|X_t^{(j,n)}-X_s^{(j,n)}\right\|_{L^{2p_j}(\nu)}^{p_j}\left\|X_s^{(j,n)}-X_r^{(j,n)}\right\|_{L^{2p_j}(\nu)}^{p_j}\,.$$

Thus (60) will follow from the fact that, for any $T_0\le r<t\le T$, $|t-r|>1/n$, and $j\in\{1,2\}$,

$$\exists p_j\in\mathbb{N}^*,\quad \sup_{a,b:r\le a<b\le t}\left\|X_b^{(j,n)}-X_a^{(j,n)}\right\|_{L^{2p_j}(\nu)}^{2p_j}\le C|t-r|^{\alpha_1}\,. \tag{61}$$

It follows from our previous moment computation that

$$\sup_{a,b:r\le a<b\le t}\mathbb{E}_\nu\left[|X_b^{(j,n)}-X_a^{(j,n)}|^{2p}\right]=\mathcal{O}\left(\left(\mathfrak{A}_n^{-1}\sum_{k=\lfloor nr\rfloor}^{\lfloor nt\rfloor-1}\mathfrak{a}_k^{-d}\right)^{\frac{2p}{j}}\right)\,.$$

Thus it is enough to prove that

$$\exists\alpha_0>0,\quad \sup_{r,t:T_0\le r<t<T,\,|t-r|\ge 1/n}\mathfrak{A}_n^{-1}\sum_{k=\lfloor nr\rfloor}^{\lfloor nt\rfloor-1}\mathfrak{a}_k^{-d}=\mathcal{O}\left((t-r)^{\alpha_0}\right)\,. \tag{62}$$

Indeed, we will conclude by taking $p_j=j(\lfloor(2\alpha_0)^{-1}\rfloor+1)$ so that (61) and so (60) hold true with $\alpha_1:=\frac{2p_j\alpha_0}{j}>1$.

Since $(\mathfrak{a}_n)_{n\ge 0}$ is $(1/\alpha)$-regularly varying and since $\lim_{n\to+\infty}\mathfrak{A}_n=+\infty$, it follows from Karamata's theorem [21, 3] that there exist three bounded convergent sequences $(c(n))_{n\ge 0}$ (positive, with positive limit), $(b(n))_{n\ge 0}$ (converging to 0) and $(\theta_n)_{n\ge 0}$ (positive, converging to $\frac{\alpha-1}{\alpha}$ if $d<\alpha$ and to 0 if $d=\alpha$, see [3, Proposition 1.5.9.b]) such that

$$\forall n\ge 0,\quad n\mathfrak{a}_n^{-d}=\mathfrak{A}_n\theta_n\quad\text{and}\quad\mathfrak{A}_n=c(n)n^{\frac{\alpha-d}{\alpha}}e^{\int_1^n\frac{b(t)}{t}\,dt}\,.$$

Now let us choose $\alpha_0$. If $d=1<\alpha$, we set $\alpha_0:=\frac{\alpha-1}{2\alpha}$. If $d=\alpha$, we take $\alpha_0:=1$. Up to change, if necessary, the first terms of $(b(n))_{n\ge 0}$ and $(c(n))_{n\ge 0}$, we assume without loss of generality that the sequence $(b(n))_{n\ge 0}$ is bounded by $\alpha_0$.

If $d=1<\alpha$, if $r\le 2/n$ (observe that, when $d=\alpha$, this case does not happen for large values of $n$ since $r\ge T_0>0$) and $|t-r|\ge 1/n$, then

$$\mathfrak{A}_n^{-1}\sum_{k=0}^{\lfloor nt\rfloor-1}\mathfrak{a}_k^{-d}=\mathcal{O}\left(\frac{\mathfrak{A}_{\lfloor nt\rfloor}}{\mathfrak{A}_n}\right)=\mathcal{O}\left(t^{1-\frac{1}{\alpha}-\alpha_0}\right)=\mathcal{O}\left((t-r)^{\alpha_0}\right)\,, \tag{63}$$

implying (62) and so (61) and (60) in this case.

We assume from now on that $T_0 \le r < t \le T$ and $2/n < r < t \le T$ (so that $\lfloor nr \rfloor - 1 > 0$) and $t - r \ge 1/n$ (so that $\lfloor nt \rfloor - \lfloor nr \rfloor \le 2(nt - nr)$). Then, using the uniform dominations on $(c(n), \theta_n, b(n))_n$ combined with a series-integral comparison, we obtain that

$$\mathfrak{A}_n^{-1} \sum_{k=\lfloor nr \rfloor}^{\lfloor nt \rfloor - 1} \mathfrak{a}_k^{-d} = \sum_{k=\lfloor nr \rfloor}^{\lfloor nt \rfloor - 1} \frac{1}{k} \frac{c(k)}{c(n)} \frac{k^{\frac{\alpha-d}{\alpha}}}{n^{\frac{\alpha-d}{\alpha}}} \theta_k e^{\int_n^k \frac{b(u)}{u}\, du}$$

$$= \mathcal{O}\left( \sum_{k=\lfloor nr \rfloor}^{\lfloor nt \rfloor - 1} \frac{1}{n} (k/n)^{-\frac{d}{\alpha} - \alpha_0} \right) . \tag{64}$$

- If $d = 1 < \alpha$, then $1 - \frac{d}{\alpha} - \alpha_0 = \alpha_0$ and so

$$\mathfrak{A}_n^{-1} \sum_{k=\lfloor nr \rfloor}^{\lfloor nt \rfloor - 1} \mathfrak{a}_k^{-d} = \mathcal{O}\left( \left( \frac{\lfloor nt \rfloor - 1}{n} \right)^{\alpha_0} - \left( \frac{\lfloor nr \rfloor - 1}{n} \right)^{\alpha_0} \right)$$

$$= \mathcal{O}\left( \left( \frac{\lfloor nt \rfloor - \lfloor nr \rfloor}{n} \right)^{\alpha_0} \right) = \mathcal{O}\left( (t-r)^{\alpha_0} \right) , \tag{65}$$

  ending the proof of (62), from which we infer (61) and (60). This ends the proof of the tightness when $d < \alpha$.
- When $d = \alpha$, we obtain

$$\mathfrak{A}_n^{-1} \sum_{k=\lfloor nr \rfloor}^{\lfloor nt \rfloor - 1} \mathfrak{a}_k^{-d} = O\left( \left( \frac{\lfloor nr \rfloor}{n} \right)^{-\alpha_0} - \left( \frac{\lfloor nt \rfloor}{n} \right)^{-\alpha_0} \right)$$

$$= O\left( \frac{\lfloor nt \rfloor - \lfloor nr \rfloor}{n} \right) = \mathcal{O}\left( t - r \right) , \tag{66}$$

  from which we infer (62), (61), (60), and so the tightness in the case where $d = \alpha$.

This ends the proof of Theorem 2.6.

**Acknowledgement :** The author conduced this work within the framework of the Henri Lebesgue Center ANR-11-LABX-0020-01 and is supported by the ANR projects GALS (ANR-23-CE40-0001) and RAWABRANCH (ANR-23-CE40-0008).

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