

LIMIT THEOREMS FOR BIRKHOFF SUMS AND LOCAL TIMES OF THE PERIODIC LORENTZ GAS WITH INFINITE HORIZON

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ABSTRACT. This work is a contribution to the study of the ergodic and stochastic properties of \mathbb{Z}^d -periodic dynamical systems preserving an infinite measure. We establish functional limit theorems for natural Birkhoff sums related to local times of the \mathbb{Z}^d -periodic Lorentz gas with infinite horizon, for both the collision map and the flow. In particular, our results apply to the difference between the numbers of collisions in two different cells. Because of the \mathbb{Z}^d -periodicity of the model we are interested in, these Birkhoff sums can be rewritten as additive functionals of a Birkhoff sum of the Sinai billiard. Our proofs rely on a general argument valid in a general framework. For completeness and in view of future studies, we state a general result of convergence of additive functionals of Birkhoff sums of chaotic probability preserving dynamical systems under general assumptions.

INTRODUCTION

Let $d \in \{1, 2\}$. The \mathbb{Z}^d -periodic Lorentz gas coming from [14] models the displacement of a point particle moving at unit speed in $\mathbb{R}^d \times \mathbb{T}^{2-d}$ (i.e. in the plane \mathbb{R}^2 if $d = 2$ and on the tube $\mathbb{R} \times \mathbb{T}$ if $d = 1$) between a (non-empty) \mathbb{Z}^d -periodic configuration of obstacles, with elastic collisions off these obstacles. The obstacles are assumed to be open, convex, with boundary \mathcal{C}^3 , non null curvature and with pairwise disjoint closures. The number of obstacles is assumed to be locally finite. We study the \mathbb{Z}^d -periodic Lorentz gas flow $(Y_t)_{t \in \mathbb{R}}$ (continuous time model) via the \mathbb{Z}^d -periodic Lorentz gas map which describes the dynamics at collision times (discrete time model). Both for the continuous time and for the discrete time model, a state is a couple (q, \vec{v}) made of a position and a unit velocity vector.

For the Lorentz gas flow $(Y_t)_{t \in \mathbb{R}}$, the positions are taken in the domain (outside obstacles), and to avoid any confusion, we identify, when the position is on a obstacle, pre-collisional and post-collisional vectors. The flow Y_t maps a state (q, \vec{v}) to the state (q_t, \vec{v}_t) at time t of a point particle that was in state (q, \vec{v}) at time 0. This map preserves the Lebesgue measure \mathbf{m} .

For the Lorentz gas map T , the states are the couples of a position on the boundary of an obstacle and of a unit post-collisional vector. We write M for the set of such states and we consider the collision map $T : M \rightarrow M$ which maps a state at a collision time to the state at the next reflection time. This map T preserves a measure μ , invariant by translation, equivalent to the Lebesgue measure and normalized so that $\mu(M \cap ([0; 1]^2 \times \mathbb{S}^1)) = 1$. This measure μ is infinite.

The horizon is said to be finite if there exist no line touching no obstacle, then the horizon is actually bounded, meaning that the distance of a trajectory between two collisions is uniformly bounded. It is said to be infinite otherwise.

Some results are now known to be true both in finite and infinite horizon. For $d \in \{1, 2\}$, the \mathbb{Z}^d -periodic Lorentz gas is known to be recurrent (both in finite horizon [7, 33] and in infinite horizon [34]) and ergodic ([31, 17]). A crucial point in the study of this model is that it is

a \mathbb{Z}^d -extension of the Sinai billiard map $(\bar{M}, \bar{T}, \bar{\mu})$, which enjoys a chaotic behaviour. This probability measure preserving system corresponding to the Lorentz gas modulo \mathbb{Z}^d for the position is mixing [32], enjoys exponential decorrelation for Hölder observables (both in finite horizon [35] and in infinite horizon [5] and enjoys a standard Central Limit Theorem [5] for Hölder observables (both in finite horizon [3, 4] and in infinite horizon [5]).

When the horizon is finite, the position of a particle after n collisions satisfies a standard Central Limit Theorem [3, 4] (due to a general argument [36], this result holds true with respect to any probability measure absolutely continuous with respect to the Lebesgue measure). Among the results proved for the Lorentz gas in finite horizon, let us mention mixing rate in infinite measure [33, 19, 20, 8] including expansions of any order (both for the map [20] and for the flow [8]), limit theorems for Birkhoff sums (both for integrable observables [9], and for smooth integrable observables with null integral [26, 27]), quantitative recurrence estimates (estimate for the tail probability of the first return time in the initial cell [9], limit theorem for the return time in a neighbourhood of the initial state or of its initial position [22]), limit theorem for the self-intersections number [18, 29], study of differential equations perturbed by the Lorentz gas [28], etc.

In the present article, we focus on the case when the horizon is fully-dimensional infinite, i.e. when there exist d non parallel infinite lines touching no obstacle. In this case, the time between two consecutive collisions is not bounded anymore, and even worth it is not square integrable with respect to the invariant probability measure $\bar{\mu}$ of the Sinai billiard map. It is still possible to apply operator techniques as in the finite horizon case, but with a loss of important nice properties, and the study requires much more delicate study. Roughly speaking, whereas the Fourier-perturbed operators family $t \mapsto P_t$ is \mathcal{C}^∞ from \mathbb{R} to $\mathcal{L}(\mathcal{B})$ for some nice Banach space \mathcal{B} when the horizon is finite, when the horizon is infinite $t \mapsto P_t$ is only smooth (and less than \mathcal{C}^2) from \mathbb{R} to $\mathcal{L}(\mathcal{B} \rightarrow L^1)$, this complicates seriously the work with iterates or expansion this operator. Nevertheless some results have been established in this infinite horizon context, overcoming these difficulties by creative ideas combined with additional technicality. A first specific result is the non-standard Central Limit Theorem satisfied by the position at the n -th collision time [34]. This result was established together with a non-standard local limit theorem for the cell label [34], leading to a mixing rate in $(n \log n)^{-\frac{d}{2}}$ for the Lorentz gas map. Whereas a mixing expansion has been established in the finite horizon case, this does not seem reachable in the infinite horizon case because of the weak smoothness properties of $t \mapsto P_t$. Nevertheless, further mixing estimates, including an error term and also different mixing rates for some specific null integral smooth observables have been established in [24]. These shows the variety of different possible mixing rates in the infinite horizon case. Among the recent results in infinite horizon, let us mention an estimate on the tail probability of the first return time of the map T to the initial cell [24], a Local Large Deviation (LLD) estimate [15], a mixing rate in $(t \log t)^{-\frac{d}{2}}$ for the flow [25] for natural observables (such as indicator functions of balls). The proof of this last result required a coupled version of the above mentioned LLD, combined with several new tricks such as a large deviation estimate on the time of the n -th collision, a joint mixing local limit theorem, a new tightness-type criteria, etc.

The goal of the present article is to investigate the behaviour of ergodic sums of the Lorentz gas map T , that is of partial sums of the form $\sum_{k=0}^{n-1} f \circ T^k$ for integrable observables, including the case of null integral observables. We restrict our study to the case where the observables only depend on the cell label. This restriction allows us to treat natural interesting cases such

as the number of visits to the 0-cell, or the difference between the number of visits to two different cells. Unfortunately the general study of [26] does not apply to this context because of the lack of smoothness of $t \rightarrow P_t \in \mathcal{L}(\mathcal{B})$. Nevertheless, we found a way to implement the moment method used in [26] despite these difficulties and to establish a limit theorem for some Birkhoff sums of null integral observables of the \mathbb{Z}^d -periodic Lorentz gas (Theorem 1.1). The observables we consider are functions of the local time in the Lorentz-gas cells. In particular it applies to the difference between the number of collisions (among the n first collisions) in the cell labeled by $a \in \mathbb{Z}^d$ and the number of collisions in another cell labeled by $b \in \mathbb{Z}^d$ with $b \neq a$. Setting $\mathbf{a}_k := \max(1, \sqrt{k \log k})$, when the observable is integrable, the Birkhoff sum will be normalized by $\mathfrak{A}_n := \sum_{k=1}^n \mathbf{a}_k^{-d}$, whereas it will be normalized by $\sqrt{\mathfrak{A}_n}$ when the observable has null integral (and satisfies some integrability assumption). We reinforce Theorem 1.1 in a joint functional limit theorem (Theorem 1.3) valid for the Birkhoff sum of a couple (g, f) with g integrable and f having a null integral, and obtain as a consequence an analogous result for the Lorentz gas flow (Theorem 1.4). Observe that

$$\mathfrak{A}_n \sim 2\sqrt{\frac{n}{\log n}} \quad \text{if } d = 1, \quad \text{and} \quad \mathfrak{A}_n \sim \log \log n \quad \text{if } d = 2.$$

The present article is organized as follows. In Section 1 we present our main results for the periodic Lorentz gas flow in infinite horizon. These results will appear as an application of general results stated in a general framework in Section 2 completed with Appendix A. In Section 3, we present a general strategy to prove our general assumptions of Section 2 via Fourier type operator perturbation techniques and we use this approach to prove our main results stated in Section 1.

1. MAIN RESULTS FOR THE PERIODIC LORENTZ GAS IN INFINITE HORIZON

1.1. Limit theorem for Birkhoff sums for the map. Let us start by introducing some additional notations. The obstacles are given by $\mathcal{O}_i + \ell$ with $\ell \in \mathbb{Z}^d$ and with $i = 1, \dots, I$ for some $I \in \mathbb{N}^*$ (up to identifying \mathbb{Z}^1 with $\mathbb{Z} \times \{0\}$ when $d = 1$). We write \mathcal{C}_ℓ (and call it ℓ -cell) for the set of states $(q, \vec{v}) \in M$ based on $\bigcup_{i=1}^I \mathcal{O}_i + \ell$. We identify \bar{M} with \mathcal{C}_0 and the Sinai billiard map \bar{T} to the map $\bar{T} : \bar{M} \rightarrow \bar{M}$ corresponding to the quotient map of T modulo \mathbb{Z}^d for positions. Let $\bar{\Psi} : \bar{M} \rightarrow \mathbb{Z}^d$ be the cell change function, i.e., for all $\bar{x} \in \bar{M} = \mathcal{C}_0$, $T(\bar{x}) \in \mathcal{C}_{\bar{\Psi}(\bar{x})}$. This can be rewritten as follows

$$\forall \bar{x} = (q, \vec{v}) \in \bar{M} = \mathcal{C}_0, \quad \bar{T}(\bar{x}) = (q', \vec{v}') \Rightarrow T(\bar{x}) = (q' + \bar{\Psi}(\bar{x}), \vec{v}').$$

More generally, by \mathbb{Z}^d -periodicity,

$$\forall \bar{x} = (q, \vec{v}) \in \bar{M} = \mathcal{C}_0, \quad \forall \ell \in \mathbb{Z}^d, \quad \bar{T}(\bar{x}) = (q', \vec{v}') \Rightarrow T((q + \ell, \vec{v})) = (q' + \ell + \bar{\Psi}(\bar{x}), \vec{v}').$$

It then follows by a direct induction that

$$\forall \bar{x} = (q, \vec{v}) \in \bar{M} = \mathcal{C}_0, \quad \forall \ell \in \mathbb{Z}^d, \quad \bar{T}^n(\bar{x}) = (q_n, \vec{v}_n) \Rightarrow T^n((q + \ell, \vec{v})) = (q_n + \ell + \bar{S}_n(\bar{x}), \vec{v}_n),$$

where $\bar{S}_n := \sum_{k=0}^{n-1} \bar{\psi} \circ \bar{T}^k$. This means that the dynamics of the Lorentz gas is totally determined by the joint dynamics of the Sinai billiard and of the Birkhoff sum \bar{S}_n . In other words, (M, T, μ) can be represented as the \mathbb{Z}^d -extension of $(\bar{M}, \bar{T}, \bar{\mu})$ by $\bar{\Psi}$. Recall that Szász and Varjú proved in [34] that $(\bar{S}_n/\mathbf{a}_n)_n$ converges in distribution to a Gaussian distribution. Let us write Φ for the density function of this Gaussian random variable.

Theorem 1.1. *Let f be a μ -integrable function constant on the cells \mathcal{C}_ℓ . Then $(\sum_{k=0}^{n-1} f \circ T^k / \mathfrak{A}_n)_n$ converges in distribution (with respect to any probability measure absolutely continuous with respect to the Lebesgue measure on M) to $\Phi(0) \int_M f d\mu |Z|$, where*

- Z is a standard gaussian distribution if $d = 1$,
- Z is a random variable with standard exponential distribution if $d = 2$.

If furthermore $\int_M f d\mu = 0$ and $\int_M (1 + d(0, \cdot))^{\frac{2+\varepsilon-d}{2}} |f| d\mu < \infty$ for some $\varepsilon \in (0, 1/2)$, then $(\sum_{k=0}^{n-1} f \circ T^k / \sqrt{\mathfrak{A}_n})_n$ converges in distribution (with respect to any probability measure absolutely continuous with respect to the Lebesgue measure on M) to $\sqrt{\sigma_f^2 \Phi(0) |Z|} \mathcal{N}$ where \mathcal{N} is a random variable, independent of Z , with standard gaussian and where $\sigma_f^2 = \sum_{k \in \mathbb{Z}} \int_M f \circ T^k d\mu$.

The first part of Theorem 1.1 is a direct consequence of [34] via moment estimates used e.g. in [9] for the periodic Lorentz gas with finite horizon. Even in the case of the finite horizon, the second part of Theorem 1.1 (study of Birkhoff sums of null integral) is more delicate to establish since it requires a delicate care of cancellations in order to identify the main order terms in compositions of perturbed operators using their expansions (see [26]). When the horizon is infinite, taking care of these cancellations become even more challenging since the perturbed operators do not admit expansion as a family of operators, but only expansions as a family of linear maps from \mathcal{B} to L^1 , forbidding direct compositions of these expansions. This additional difficulty comes from the fact that the cell change function $\bar{\Psi}$ is **not square integrable** with respect to $\bar{\mu}$ (whereas it is bounded and so finite-valued when the horizon of the Lorentz process is finite).

Remark 1.2. *It follows from the first part of Theorem 1.1 with the notations therein combined with the Hopf ergodic theorem¹ that for any μ -integrable $h : M \rightarrow \mathbb{R}$, $(\sum_{k=0}^{n-1} h \circ T^k / \mathfrak{A}_n)_n$ converges in distribution to $\Phi(0) \int_M h d\mu |Z|$. This result can be seen as a weak Law of Large Number for the infinite measure preserving dynamical system (M, T, μ) .*

1.2. Functional limit theorem for the map and for the flow.

Theorem 1.3. *Let $f, g : M \rightarrow \mathbb{R}$ be two integrable functions, with f constant on each cell \mathcal{C}_ℓ . Assume furthermore that $\int_M (1 + d(0, \cdot))^{\frac{2+\varepsilon-d}{2}} |f| d\mu < \infty$ for some $\varepsilon \in (0, \frac{1}{2})$ and $\int_M f d\mu = 0$, then the following family of couple of processes*

$$(1) \quad \left(\left(\sum_{k=0}^{\lfloor nt \rfloor - 1} g \circ T^k / \mathfrak{A}_n \right)_t, \left(\sum_{k=0}^{\lfloor nt \rfloor - 1} f \circ T^k / \sqrt{\mathfrak{A}_n} \right)_t \right)_{n \geq 1}$$

converges in distribution (with respect to any probability measure absolutely continuous with respect to the Lebesgue measure on M) to $((\int_M g d\mu \mathcal{L}_t)_t, (B_{\sigma_f^2 \mathcal{L}_t})_t)$ (in $(\mathcal{D}([0; T]))^2$ for all $T > 0$ if $d = 1$ and in $(\mathcal{D}([T_0; T]))^2$ for all $0 < T_0 < T$ if $d = 2$), where σ_f^2 is the quantity introduced in Theorem 1.1, where B is a standard brownian motion independent of the process \mathcal{L}_t where

- if $d = 1$, \mathcal{L}_t is the local time at 0 in the time interval $[0; t]$ of the Brownian motion W limit in distribution of $(\bar{S}_{\lfloor nt \rfloor} / \sqrt{n})_t$ as $n \rightarrow +\infty$,

¹The Hopf ergodic theorem states that, since (M, μ, T) is recurrent ergodic, the sequence of ergodic ratios $(\sum_{k=0}^{n-1} h \circ T^k / \sum_{k=0}^{n-1} \mathbf{1}_{\mathcal{C}_0} \circ T^k)_{n \geq 1}$ converges almost surely to $\int_M h d\mu / \mu(\mathcal{C}_0) = \int_M h d\mu$.

- if $d = 2$, for all $t > 0$ $\mathcal{L}_t = \mathcal{L}_1$ is a random variable with exponential distribution with mean $\Phi(0)$.

Our proofs of Theorems 1.1 and 1.3 are given in Section 3. They rely on the general results (in a general framework) stated in the Section 2.

As an immediate consequence of Theorem 1.3 combined with the classical random time change result (see e.g. [1, Chapter 14]), we obtain the following result valid for the periodic Lorentz gas flow. Let us write \mathcal{M} for the set of states of the Lorentz gas flow, and \mathbf{m} for the Lebesgue measure on \mathcal{M} , and $\mathcal{N}_t(\ell)$ for the number of collisions of the flow in the cell \mathcal{C}_ℓ up to time t .

Theorem 1.4. *For any real valued sequence $(\beta_\ell)_{\ell \in \mathbb{Z}^d}$ such that $\sum_{\ell \in \mathbb{Z}^d} (1 + |\ell|)^{\frac{2+\varepsilon-d}{2}} |\beta_\ell| < \infty$ for some $\varepsilon > 0$ and $\sum_{\ell \in \mathbb{Z}^d} \beta_\ell = 0$, then the family of processes*

$$(2) \quad \left(\left(\mathfrak{A}_n^{-1} \mathcal{N}_{nt}(0), \mathfrak{A}_n^{-\frac{1}{2}} \sum_{\ell \in \mathbb{Z}^d} \beta_\ell \mathcal{N}_{nt}(\ell) \right) \right)_t \Big|_{n \geq 1}$$

converges in distribution (with respect to any probability measure absolutely continuous with respect to the Lebesgue measure on \mathcal{M}) to $((\mathcal{L}'_t)_t, (B_{\sigma^2 \mathcal{L}'_t})_t)$ (in $(\mathcal{D}([0; T]))^2$ for all $T > 0$ if $d = 1$ and in $(\mathcal{D}([T_0; T]))^2$ for all $0 < T_0 < T$ if $d = 2$), where σ^2 is quantity σ_f^2 introduced in Theorem 1.1 for the function f such that $f|_{\mathcal{C}_\ell} \equiv \beta_\ell$ for all $\ell \in \mathbb{Z}^d$, or equivalently, with

$$\begin{aligned} \sigma^2 &:= \sum_{k \in \mathbb{Z}} \sum_{a, b \in \mathbb{Z}^d} \beta_a \beta_b \bar{\mu}(\bar{S}_k = b - a) \\ &= \sum_{k \in \mathbb{Z}} \sum_{a, b \in \mathbb{Z}^d} \beta_a \beta_b (\bar{\mu}(\bar{S}_k = b - a) - \bar{\mu}(\bar{S}_k = b) - \bar{\mu}(\bar{S}_k = -a) + \bar{\mu}(\bar{S}_k = 0)) . \end{aligned}$$

where B is a standard brownian motion independent of the process $\tilde{\mathcal{L}}_t$ where

- if $d = 1$, \mathcal{L}'_t is the local time at 0 in the time interval $[0; t]$ of the Brownian motion W' limit in distribution, as $n \rightarrow +\infty$, of $((q_{nt}/\sqrt{n})_t)_n$ where q_{nt} is the first coordinate of the position of $Y_{nt}(\cdot)$, i.e. the first coordinate of the position at time nt of the particle,
- if $d = 2$, for all $t > 0$ $\mathcal{L}'_t = \mathcal{L}'_1$ is a random variable with exponential distribution with mean $\Phi(0)$.

Proof of Theorem 1.4. Recall that the Sinai billiard flow (at unit speed) endowed with the Lebesgue measure \mathbf{m} can naturally be represented by the suspension flow over $(\bar{M}, \bar{T}, 2 \sum_{i=1}^I |\partial O_i| \bar{\mu})$ with roof function τ , the time before the next collision. Indeed, this representation consists in identifying each $y \in \mathcal{M}$ with the unique couple (x, s) such that $x \in M$, $s \in [0; \tau(x))$ and $y = Y_s(x)$ (x corresponds to the state at the previous collision time and s to the time spent since this previous collision time). Since the Sinai billiard system is ergodic, it follows from the Birkhoff ergodic theorem that the number of collisions \mathbf{n}_{nt} for the billiard flow (or equivalently for the \mathbb{Z}^d -periodic Lorentz gas flow) in the time interval $[0; nt]$ is almost surely equivalent to

$nt/\mathbb{E}_{\bar{\mu}}[\tau]$ as $n \rightarrow +\infty$. We conclude by applying Theorem 1.3 to $g = \mathbf{1}_{c_0}$ and to $f = \sum_{\ell \in \mathbb{Z}^d} \beta_\ell \mathbf{1}_{c_\ell}$, and using the random time \mathbf{n}_{nt} , that

$$\left(\left(\mathfrak{A}_{\mathbf{n}_n}^{-1} \mathcal{N}_{nt}(0), \mathfrak{A}_{\mathbf{n}_n}^{-\frac{1}{2}} \sum_{\ell \in \mathbb{Z}^d} \beta_\ell \mathcal{N}_{nt}(\ell) \right) \right)_t \Big|_n$$

has the same limit in distribution $(X^{(1)}, X^{(2)})$ as (1) for the above choice of (g, f) . Now, using the fact that, almost surely, as $t \rightarrow +\infty$, $\mathfrak{A}_{\mathbf{n}_n} \sim \mathfrak{A}_n / (\mathbb{E}_{\bar{\mu}}[\tau])^{\frac{2-d}{2}}$, we conclude that the joint process (2) converges in distribution to the joint process $((\mathbb{E}_{\bar{\mu}}[\tau])^{\frac{2-d}{2}} X^{(1)}, (\mathbb{E}_{\bar{\mu}}[\tau])^{\frac{2-d}{4}} X^{(2)})$. But, by the same random time change argument, the Brownian motion W' corresponds to the Brownian motion limit of $(\bar{S}_{\mathbf{n}_{nt}}/\sqrt{n})_t$, i.e. to the Brownian motion limit of $(\bar{S}_{nt/\mathbb{E}_{\bar{\mu}}[\tau]}/\sqrt{n})_n$, i.e. to $W/\sqrt{\mathbb{E}_{\bar{\mu}}[\tau]}$. Thus the local time $(\mathcal{L}'_t)_t$ of W' at 0, is equal to $(\sqrt{\mathbb{E}_{\bar{\mu}}[\tau]} \mathcal{L}_t)_t$, recalling that \mathcal{L} is the local time of W at 0. This ends the proof of the corollary. \square

Remark 1.5. Let $G : \mathcal{M} \rightarrow \mathbb{R}$ be an integrable function with respect to the Lebesgue measure \mathbf{m} on \mathcal{M} (velocity vectors $\vec{v} \in \mathbb{S}^1$ being identified with an angle in \mathbb{R}/\mathbb{Z}). The proof of Theorem 1.3 can be adapted, by taking defined on M by $g(x) := \int_0^{\tau(x)} G(Y_s(x)) ds$ and by noticing that

$$\int_M g d\mu = \frac{\int_{\mathcal{M}} G d\mathbf{m}}{2 \sum_{i=1}^I |\partial O_i|}$$

to prove that the family of processes

$$\left(\left(\mathfrak{A}_{\mathbf{n}_n}^{-1} \int_0^{nt} G \circ Y_s ds, \mathfrak{A}_{\mathbf{n}_n}^{-\frac{1}{2}} \sum_{\ell \in \mathbb{Z}^d} \beta_\ell \mathcal{N}_{nt}(\ell) \right) \right)_t \Big|_n$$

converges in distribution (with respect to any probability measure absolutely continuous with respect to the Lebesgue measure on $Q \times \mathbb{S}^1$) to $\left(\left(\frac{1}{2 \sum_{i=1}^I |\partial O_i|} \int_M G d\mathbf{m} \mathcal{L}'_t \right)_t, (B_{\sigma^2 \mathcal{L}'_t})_t \right)$ in the same sense as in Theorem 1.4, with σ^2 the quantity appearing in Theorem 1.4.

2. GENERAL RESULTS

Observe that the Birkhoff sums considered in Theorem 1.1 can be rewritten as additive functionals $(\sum_{k=0}^{n-1} \beta_{\bar{S}_k})_n$ of the Birkhoff sums $(\bar{S}_n)_n$ with respect to the Sinai billiard system $(\bar{M}, \bar{T}, \bar{\mu})$. We keep this formulation in the present section and state limit theorems for additive functionals of Birkhoff sums of a probability preserving dynamical systems under general assumptions expressed in terms of operators. We will see in Section 3 how these assumptions can be proved using Fourier-perturbations of the transfer operator and how this result can be used to prove Theorem 1.1.

2.1. General assumptions.

Hypothesis 2.1. Let $d \in \{1, 2\}$ and $\alpha \in [d; 2]$. Let (Δ, F, ν) be a probability preserving dynamical system with transfer operator P . Let $\Psi : \Delta \rightarrow \mathbb{Z}^d$. For any $a \in \mathbb{Z}^d$ and any non-negative integer n , we set $S_n := \sum_{k=0}^{n-1} \Psi \circ F^k$ and we set $Q_{n,a}$ for the operator given by

$$Q_{n,a} := P^n(\mathbf{1}_{\{S_n=a\}} \cdot).$$

There exist a $(1/\alpha)$ -regularly varying sequence $(\mathbf{a}_n)_{n \geq 0}$ such that $\mathfrak{A}_n := \sum_{k=0}^n \mathbf{a}_k^{-d} \rightarrow +\infty$ as $n \rightarrow +\infty$ and a Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ preserved by the operators $Q_{n,a}$ such that

$$(3) \quad \mathbf{1}_{\Delta} \in \mathcal{B} \hookrightarrow L^1(\nu),$$

where the notation \hookrightarrow means a continuous inclusion, we assume furthermore that

$$(4) \quad \|Q_{n,0}\|_{\mathcal{B}} = \mathcal{O}(\mathbf{a}_n^{-d})$$

and that there exists $\Phi(0) > 0$ such that ²

$$(5) \quad Q_{n,0} = \Phi(0)\mathbf{a}_n^{-d}\mathbb{E}_{\nu}[\cdot] + o(\mathbf{a}_n^{-d}) \quad \text{in } \mathcal{L}(\mathcal{B} \rightarrow L^1(\nu)).$$

In [26], to study Birkhoff sums of the periodic Lorentz gas with finite horizon, we used the following condition

$$Q_{n,a} = \Phi(a/\mathbf{a}_n)\mathbf{a}_n^{-d}\mathbb{E}_{\nu}[\cdot] + o(\mathbf{a}_n^{-d}) \quad \text{in } \mathcal{L}(\mathcal{B}).$$

A crucial difference between this condition and the assumptions of the present article is that (5) is much weaker since it holds in $\mathcal{L}(\mathcal{B} \rightarrow L^1(\nu))$ instead of $\mathcal{L}(\mathcal{B})$. In practice, this weaker condition comes from the fact that the family of perturbed operators $t \mapsto P_t \in \mathcal{L}(\mathcal{B})$ behind (see Section 3) is not continuous, but that $t \mapsto P_t \in \mathcal{L}(\mathcal{B} \rightarrow \mathcal{L}^1(\nu))$ is continuous.

To study additive functionals $\sum_{k=0}^{n-1} \beta_{S_k}$ with $\sum_{a \in \mathbb{Z}^d} \beta_a = 0$, we will reinforce the previous assumption as follows.

Hypothesis 2.2. *Assume Hypothesis 2.1 and that*

$$(6) \quad \sup_{a \in \mathbb{Z}^d} \|Q_{k,a}\|_{\mathcal{B}} = \mathcal{O}(\mathbf{a}_k^{-d}),$$

and that, for all $\eta \in [0; 1]$,

$$(7) \quad \|Q'_{k,a,b} := Q_{k,b} - Q_{k,a}\|_{\mathcal{B}} = \mathcal{O}\left(|b-a|^{\eta} \mathbf{a}_k^{-d-\eta}\right),$$

and

$$(8) \quad \|Q''_{k,a,b} := Q_{k,b-a} - Q_{k,b} - Q_{k,-a} + Q_{k,0}\|_{\mathcal{B}} = \mathcal{O}\left((|a||b|)^{\eta} \mathbf{a}_k^{-d-2\eta}\right),$$

uniformly in $a, b \in \mathbb{Z}^d$.

2.2. Limit theorem for additive functionals of Birkhoff sums.

Theorem 2.3. *Assume Hypothesis 2.1. Then $(\sum_{k=0}^{n-1} \mathbf{1}_{\{S_k=0\}}/\mathfrak{A}_n)_{n \geq 1}$ converges in distribution (and in the sense of moments), with respect to ν , to $\Phi(0)\mathcal{Y}$, where \mathcal{Y} is a Mittag-Leffler distribution of index $\frac{\alpha-d}{\alpha}$, i.e.*

$$\mathbb{E}[\mathcal{Y}^N] := N! \frac{\Gamma(1 + \frac{\alpha-d}{\alpha})^N}{\Gamma(1 + N \frac{\alpha-d}{\alpha})}.$$

If furthermore $\sum_{\ell \in \mathbb{Z}^d} |1 + |\ell|^{\eta}| \beta_{\ell} < \infty$ with $\eta := \frac{\alpha+\varepsilon-d}{2}$ for some $\varepsilon \in (0, 1/2)$ and $\sum_{\ell \in \mathbb{Z}^d} \beta_{\ell} = 0$, and if Hypothesis 2.2 holds true, then $(\sum_{k=0}^{n-1} \beta_{S_k}/\sqrt{\mathfrak{A}_n})_{n \geq 1}$ converges in distribution (and in

² $\Phi(0)$ will appear to be the value at 0 of the density function Φ of the limit in distribution of $(S_n/\mathbf{a}_n)_n$ (see Section 3).

the sense of moments), with respect to ν , to $\sqrt{\sigma_\beta^2 \Phi(0)} \mathcal{YN}$ where \mathcal{N} is a random variable, independent of \mathcal{Z} , with standard gaussian and where

$$(9) \quad \sigma_\beta^2 := \sum_{k \in \mathbb{Z}} \sum_{a, b \in \mathbb{Z}^d} \beta_a \beta_b \nu(S_{|k|} = b - a)$$

$$(10) \quad = \sum_{k \in \mathbb{Z}} \sum_{a, b \in \mathbb{Z}^d} \beta_a \beta_b (\nu(S_{|k|} = b - a) - \nu(S_{|k|} = b) - \nu(S_{|k|} = -a) + \nu(S_{|k|} = 0)) .$$

Remark 2.4. The summability assumption of β_ℓ appearing in Theorem 2.3 is to our knowledge the optimal one even in the case of additive observables of random walks with i.i.d. increments.

Remark 2.5. It follows from our assumptions that, if β is not identically null, only the second sum (10) defining σ_β^2 is absolutely convergent in k, a, b . Indeed

$$\nu(S_k = b - a) - \nu(S_k = b) - \nu(S_k = -a) + \nu(S_k = 0) = \mathbb{E}_\nu[Q''_{k,a,b}(\mathbf{1})]$$

is summable in $(k, a, b) \in \mathbb{N} \times \mathbb{Z}^d \times \mathbb{Z}^d$, whereas $\nu(S_k = 0) = \mathbb{E}_\nu[Q_{k,0}(\mathbf{1})] \sim \Phi(0) \mathbf{a}_k^{-d}$ is not summable. The summability of (10) combined with the fact that

$$\forall k \geq 0, \quad \sum_{a, b \in \mathbb{Z}^d} \beta_a \beta_b \nu(S_k = b - a) = \sum_{a, b \in \mathbb{Z}^d} \beta_a \beta_b \mathbb{E}_\nu[Q''_{k,a,b}(\mathbf{1})]$$

implies the absolute convergence in k of the sum appearing in the right hand side of (9).

Proof of Theorem 2.3. Let $(\beta_\ell)_{\ell \in \mathbb{Z}^d}$ be as in the assumption of Theorem 2.3. We start by writing

$$(11) \quad \mathbb{E}_\nu \left[\left(\sum_{k=0}^{n-1} \beta_{S_k} \right)^N \right] = \sum_{k_1, \dots, k_N=0}^{n-1} \mathbb{E}_\nu \left[\prod_{j=1}^N \beta_{S_{k_j}} \right] \\ = \sum_{0 \leq k_1 \leq \dots \leq k_N \leq n-1} c_{(k_1, \dots, k_N)} \sum_{a_1, \dots, a_N \in \mathbb{Z}^d} \mathbb{E}_\nu \left[\prod_{j=1}^N \left(\beta_{a_j} \mathbf{1}_{\{S_{k_j} = a_j\}} \right) \right],$$

where we denote by $c_{(k_1, \dots, k_N)}$ the number of N -uples $(k'_1, \dots, k'_N) \in \{0, \dots, n-1\}^N$ such that there exists a permutation $\sigma \in \mathfrak{S}_N$ such that $k'_i = k_{\sigma(i)}$ for all $i = 1, \dots, N$. We observe that

$$(12) \quad \mathbb{E}_\nu \left[\prod_{j=1}^N \left(\beta_{a_j} \mathbf{1}_{\{S_{k_j} = a_j\}} \right) \right] = \mathbb{E}_\nu \left[\prod_{j=1}^N \left(\beta_{a_j} \mathbf{1}_{\{S_{k_j - k_{j-1}} = a_j - a_{j-1}\}} \circ F^{k_{j-1}} \right) \right] \\ = \mathbb{E}_\nu \left[P^{k_N} \left(\prod_{j=1}^N \left(\beta_{a_j} \mathbf{1}_{\{S_{k_j - k_{j-1}} = a_j - a_{j-1}\}} \circ F^{k_{j-1}} \right) \right) \right],$$

setting $a_0 := 0$ and P for the transfer operator of F with respect to ν , using the fact that $\mathbb{E}_\nu[\cdot] = \mathbb{E}_\nu[P^{k_N}(\cdot)]$. Since $P^k(f \circ g \circ F^k) = g P^k(f)$, we observe that, for any $j = 1, \dots, N$, for any $k_1 \leq \dots \leq k_j$ and any $b_1, \dots, b_j \in \mathbb{Z}^d$,

$$(13) \quad P^{k_j} \left(\prod_{i=1}^j \mathbf{1}_{\{S_{k_i - k_{i-1}} = b_i\}} \circ F^{k_{i-1}} \right) = P^{k_j - k_{j-1}} \left(\mathbf{1}_{\{S_{k_j - k_{j-1}} = b_j\}} P^{k_{j-1}} \left(\prod_{i=1}^{j-1} \left(\mathbf{1}_{\{S_{k_i - k_{i-1}} = b_i\}} \circ F^{k_{i-1}} \right) \right) \right) .$$

It follows from (12) and (13) that

$$(14) \quad \mathbb{E}_\nu \left[\prod_{j=1}^N \left(\beta_{a_j} \mathbf{1}_{\{S_{k_j}=a_j\}} \right) \right] = \mathbb{E}_\nu \left[\beta_{a_N} Q_{k_N-k_{N-1}, a_N-a_{N-1}} (\cdots (\beta_{a_2} Q_{k_2-k_1, a_2-a_1} (\beta_{a_1} Q_{k_1, a_1}(\mathbf{1}))) \cdots) \right].$$

For the first part of Theorem 2.3, we apply (11) with $\beta_\ell = \mathbf{1}_{\{\ell=0\}}$ with (14). In that case, applying repeatedly (5), combined with (4) and (3), the right hand side of (14) becomes

$$\mathbb{E}_\nu \left[Q_{k_N-k_{N-1}, 0} (\cdots (Q_{k_1, 0}(\mathbf{1}))) \right] = (\Phi(0))^N \prod_{j=1}^N \mathbf{a}_{k_j-k_{j-1}}^{-d} + \sum_{j=1}^N o(\mathbf{a}_{k_j-k_{j-1}}^{-d}) \prod_{i \neq j} \mathcal{O}(\mathbf{a}_{k_i-k_{i-1}}^{-d})$$

and so that

$$\begin{aligned} \mathbb{E}_\nu \left[\left(\sum_{k=0}^{n-1} \mathbf{1}_{\{S_k=0\}} \right)^N \right] &= o(\mathfrak{A}_n^N) + N! \left(\Phi(0) \int_M f d\mu \right)^N \sum_{1 < k_1 < \cdots < k_N \leq n-1} \prod_{j=1}^N \mathbf{a}_{k_j-k_{j-1}}^{-d} \\ &= o(\mathfrak{A}_n^N) + N! \mathfrak{A}_n^N \left[\left(\Phi(0) \int_M f d\mu \right)^N \frac{\Gamma(1 + \frac{2-d}{2})^N}{\Gamma(1 + N \frac{2-d}{2})} + o(1) \right], \end{aligned}$$

using [26, Lemma 2.7] for the last estimate. This ends the proof of the first part of Theorem 2.3.

Now let us prove the second part. We assume from now on that $\sum_{a \in \mathbb{Z}^d} \beta_a = 0$ and that $\sum_{a \in \mathbb{Z}^d} (1 + |a|^\eta) |\beta_a| < \infty$ with $\eta := \frac{\alpha-d+\varepsilon}{2}$ for some $\varepsilon \in (0, \frac{1}{2})$. Recall that it follows from (11) combined with (14) that

$$(15) \quad \mathbb{E}_\nu \left[\left(\sum_{k=0}^{n-1} \beta_{S_k} \right)^N \right] = \sum_{0 \leq k_1 \leq \cdots \leq k_N \leq n-1} c(k_1, \dots, k_N) \sum_{a_1, \dots, a_N \in \mathbb{Z}^d} \mathbb{E}_\nu \left[\beta_{a_N} Q_{k_N-k_{N-1}, a_N-a_{N-1}} (\cdots (\beta_{a_2} Q_{k_2-k_1, a_2-a_1} (\beta_{a_1} Q_{k_1, a_1}(\mathbf{1}))) \cdots) \right].$$

In Formula (15), we decompose each $Q_{k,a}$ in $Q_{k,a}^{(0)} + Q_{k,a}^{(1)}$, with $Q_{k,a}^{(0)} := Q_{k,0}$ and $Q_{k,a}^{(1)} := Q'_{k,a} := Q'_{k,0,a} = Q_{k,a} - Q_{k,0}$. Thus

$$(16) \quad \mathbb{E}_\nu \left[\left(\sum_{k=0}^{n-1} \beta_{S_k} \right)^N \right] = \sum_{\varepsilon_1, \dots, \varepsilon_N \in \{0,1\}} H_{\varepsilon_1, \dots, \varepsilon_N}^{(n,N)},$$

with

$$H_{\varepsilon_1, \dots, \varepsilon_N}^{(n,N)} = \sum_{0 \leq k_1 \leq \cdots \leq k_N \leq n-1} c(k_1, \dots, k_N) H_{\mathbf{k}, \boldsymbol{\varepsilon}}^{n,N}$$

setting $\mathbf{k} = (k_1, \dots, k_N)$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_N)$ and

$$\begin{aligned} H_{\mathbf{k}, \boldsymbol{\varepsilon}}^{n,N} &:= \sum_{a_1, \dots, a_N \in \mathbb{Z}^d} \mathbb{E}_\nu \left[\beta_{a_N} Q_{k_N-k_{N-1}, a_N-a_{N-1}}^{(\varepsilon_N)} (\cdots (\beta_{a_2} Q_{k_2-k_1, a_2-a_1}^{(\varepsilon_2)} (\beta_{a_1} Q_{k_1, a_1}(\mathbf{1}))) \cdots) \right] \\ &= \sum_{a_N \in \mathbb{Z}^d} \mathbb{E}_\nu \left[\beta_{a_N} \left(\sum_{a_{N-1} \in \mathbb{Z}^d} \beta_{a_{N-1}} Q_{k_N-k_{N-1}, a_N-a_{N-1}}^{(\varepsilon_N)} \left(\cdots \left(\sum_{a_1 \in \mathbb{Z}^d} \beta_{a_1} Q_{k_2-k_1, a_2-a_1}^{(\varepsilon_2)} \left(Q_{k_1, a_1}^{(\varepsilon_1)}(\mathbf{1}) \right) \right) \cdots \right) \right) \right]. \end{aligned}$$

We observe that $H_{\varepsilon_1, \dots, \varepsilon_N}^{(n, N)} = 0$ as soon as $\varepsilon_N = 0$ or if there exists $j_0 = 1, \dots, N-1$ such that $\varepsilon_{j_0} = \varepsilon_{j_0+1} = 0$ (since $\sum_{a_{j_0} \in \mathbb{Z}^d} \beta_{a_{j_0}} = 0$). Thus, we restrict our study to the case of the ε'_j 's for which the j_0 's such that $\varepsilon_{j_0} = 0$ are isolated and do not include N . Let such an $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$. Observe that there are at most $N/2$ indices j 's such that $\varepsilon_j = 0$. We set, by convention, $\varepsilon_{N+1} = \varepsilon_0 = 0$. We will prove that

$$(17) \quad H_{\varepsilon_1, \dots, \varepsilon_N}^{n, N} = o\left(\mathfrak{A}_n^{\frac{N}{2}}\right) \quad \text{unless if } \#\{j : \varepsilon_j = 0\} = \frac{N}{2},$$

i.e. $H_{\varepsilon_1, \dots, \varepsilon_N}^{n, N} = o\left(\mathfrak{A}_n^{\frac{N}{2}}\right)$ unless if N is even and if $(\varepsilon_1, \dots, \varepsilon_N) = (0, 1, \dots, 0, 1)$. For reader's convenience, we first give a short proof of this estimate in a particular case. A proof of this estimate in the general case $\sum_{a \in \mathbb{Z}^d} |a|^{\frac{\alpha-d+\varepsilon}{2}} |\beta_a| < \infty$ with $\eta = \frac{\alpha-d+\varepsilon}{2}$, with $\varepsilon \in (0, \frac{1}{2})$ is given in Appendix A.

- A short proof of (17) in a particular case. Assume in this item only that $\alpha < d+1$ and $\sum_{a \in \mathbb{Z}^d} (1+|a|)^{2\bar{\eta}} |\beta_a| < \infty$ with $\alpha - d < \bar{\eta} < 1$. It follows from Hypothesis 2.2 that

$$\begin{aligned} Q_{k_j - k_{j-1}, a_j - a_{j-1}}^{(\varepsilon_j)} &= \mathcal{O}\left(|a_j - a_{j-1}|^{\varepsilon_j \bar{\eta}} \mathfrak{a}_{k_j - k_{j-1}}^{-d - \varepsilon_j \bar{\eta}}\right) \\ &= \mathcal{O}\left(\left((1+|a_j|)(1+|a_{j-1}|)\right)^{\varepsilon_j \bar{\eta}} \mathfrak{a}_{k_j - k_{j-1}}^{-d - \varepsilon_j \bar{\eta}}\right), \end{aligned}$$

and so that

$$\begin{aligned} |H_{\varepsilon_1, \dots, \varepsilon_N}^{n, N}| &= \mathcal{O}\left(\left(\sum_{a \in \mathbb{Z}^d} (1+|a|)^{2\bar{\eta}} |\beta_a|\right)^N \prod_{j=1}^N \sum_{k_j=0}^{n-1} \mathfrak{a}_k^{-d - \varepsilon_j \bar{\eta}}\right) \\ &= \mathcal{O}\left(\mathfrak{A}_n^{\#\{j: \varepsilon_j=0\}}\right), \end{aligned}$$

where we used the fact that $\sum_{a \in \mathbb{Z}^d} (1+|a|)^{2\bar{\eta}} |\beta_a| < \infty$ combined with $\sum_{k \geq 0} \mathfrak{a}_k^{-d - \bar{\eta}} < +\infty$, since $d + \bar{\eta} > \alpha$ and since $(\mathfrak{a}_n)_n$ is $\frac{1}{\alpha}$ -regularly varying. This concludes the proof of (17).

In particular, Formula (17) ensures that

$$(18) \quad \mathbb{E}_\nu \left[\left(\sum_{k=0}^{n-1} \beta_{S_k} \right)^N \right] = o\left(\mathfrak{A}_n^{\frac{N}{2}}\right) \quad \text{if } N \text{ is odd,}$$

and that

$$(19) \quad \mathbb{E}_\nu \left[\left(\sum_{k=0}^{n-1} \beta_{S_k} \right)^N \right] = H_{0,1,\dots,0,1}^{(n,N)} + o\left(\mathfrak{A}_n^{\frac{N}{2}}\right) \quad \text{if } N \text{ is even.}$$

Assume from now on that N is even, then

$$\begin{aligned} H_{0,1,\dots,0,1}^{(n,N)} &= \sum_{0 \leq k_1 \leq \dots \leq k_N \leq n-1} c_{k_1, \dots, k_N} \sum_{a_1, \dots, a_N \in \mathbb{Z}^d} \left(\prod_{j=1}^N \beta_{a_j} \right) \\ &\mathbb{E}_\nu [Q'_{k_N - k_{N-1}, a_N - a_{N-1}} (Q_{k_{N-1} - k_{N-2}, 0} (\dots (Q'_{k_2 - k_1, a_2 - a_1} (Q_{k_1, 0}(\mathbf{1}))) \dots))] \end{aligned}$$

Hence

$$(20) \quad H_{0,1,\dots,0,1}^{(n,N)} = \sum_{0 \leq k_1 \leq \dots \leq k_N \leq n-1} c_{k_1, \dots, k_N} \mathbb{E}_\nu [\bar{Q}_{k_N - k_{N-1}} (Q_{k_{N-1} - k_{N-2}, 0} (\dots (\bar{Q}_{k_2 - k_1} (Q_{k_1, 0}(\mathbf{1}))) \dots))],$$

with

$$(21) \quad \bar{Q}_k := \sum_{a,b \in \mathbb{Z}^d} \beta_a \beta_b Q'_{k,b-a} = \sum_{a,b \in \mathbb{Z}^d} \beta_a \beta_b Q_{k,b-a} = \sum_{a,b \in \mathbb{Z}^d} \beta_a \beta_b Q''_{k,a,b},$$

where we used the fact that $\sum_{a \in \mathbb{Z}^d} \beta_a = 0$ and the notation $Q''_{k,a,b} = Q_{k,b-a} - Q_{k,b} - Q_{k,-a} + Q_{k,0}$ introduced in (8). Combining (8) with $\eta = \frac{\alpha + \varepsilon - d}{2}$ with (21), since $\sum_{a \in \mathbb{Z}^d} (1 + |a|)^\eta |\beta_a| < \infty$, we infer that

$$(22) \quad \bar{Q}_k = \mathcal{O}(\mathbf{a}_k^{-\alpha - \varepsilon}) \quad \text{in } \mathcal{L}(\mathcal{B}),$$

since $d + 2\eta = \alpha + \varepsilon$, which ensures the summability of $\|\bar{Q}_k\|_{\mathcal{B}}$. The study of (20) leads us to the question of estimating $\mathbb{E}_\nu[\bar{Q}_{k'}(Q_{k,0}(h))]$. Unfortunately we cannot compose directly (8) and (5) since this last estimate is in $\mathcal{L}(\mathcal{B} \rightarrow L^1(\nu))$ and not in $\mathcal{L}(\mathcal{B})$. But, proceeding in two steps, we will prove that

$$(23) \quad \mathbb{E}_\nu[\bar{Q}_{k'}(Q_{k,0}(h))] - \Phi(0) \mathbf{a}_k^{-d} \mathbb{E}_\nu[\bar{Q}_{k'}(\mathbf{1})] \mathbb{E}_\nu[h] = \mathcal{O}(\mathbf{a}_k^{-\alpha - \varepsilon'} \|h\|_{\mathcal{B}}) o(\mathbf{a}_k^{-d}),$$

where $\varepsilon' \in (0, \frac{1}{2})$ is small enough so that $\frac{(\alpha - d + 2\varepsilon')(\alpha + \varepsilon')}{\alpha + 2\varepsilon'} \leq \alpha - d + \varepsilon = 2\eta$.

First, dominating separately both terms, it follows from (8) with $\eta' = \frac{\alpha + 2\varepsilon' - d}{2} \in (0, 1)$ and from (3) and (4) that

$$(24) \quad \mathbb{E}_\nu[Q''_{k',a,b}(Q_{k,0}(h))] - \Phi(0) \mathbf{a}_k^{-d} \mathbb{E}_\nu[Q''_{k',a,b}(\mathbf{1})] \mathbb{E}_\nu[h] = \mathcal{O}(|a|^{\eta'} |b|^{\eta'} \mathbf{a}_k^{-\alpha - 2\varepsilon'} \mathbf{a}_k^{-d} \|h\|_{\mathcal{B}}),$$

since $-d - 2\eta' = -\alpha - 2\varepsilon'$. Second, it follows from the definition of $Q''_{k',a,b}$ that

$$\mathbb{E}_\nu[Q''_{k',a,b}(h_0)] = \mathbb{E}_\nu[(\mathbf{1}_{\{S_{k'}=b-a\}} - \mathbf{1}_{\{S_{k'}=b\}} - \mathbf{1}_{\{S_{k'}=-a\}} + \mathbf{1}_{\{S_{k'}=0\}}).h_0] = \mathcal{O}(\|h_0\|_{L^1(\nu)}).$$

This combined with (5) ensures that

$$(25) \quad \begin{aligned} & \mathbb{E}_\nu[Q''_{k',a,b}(Q_{k,0}(h))] - \Phi(0) \mathbf{a}_k^{-d} \mathbb{E}_\nu[Q''_{k',a,b}(\mathbf{1})] \mathbb{E}_\nu[h] \\ & = \mathbb{E}_\nu[Q''_{k',a,b}(Q_{k,0}(h) - \Phi(0) \mathbf{a}_k^{-d} \mathbb{E}_\nu[h])] = \mathcal{O}(\|h\|_{\mathcal{B}}) o(\mathbf{a}_k^{-d}). \end{aligned}$$

Thus, combining (24) and (25), we obtain that

$$\begin{aligned} & \mathbb{E}_\nu[Q''_{k',a,b}(Q_{k,0}(h))] - \Phi(0) \mathbf{a}_k^{-d} \mathbb{E}_\nu[Q''_{k',a,b}(\mathbf{1})] \mathbb{E}_\nu[h] \\ & = \left(\mathcal{O}(|a|^{\eta'} |b|^{\eta'} \mathbf{a}_k^{-\alpha - 2\varepsilon'} \mathbf{a}_k^{-d} \|h\|_{\mathcal{B}}) \right)^{(\alpha + \varepsilon')/(\alpha + 2\varepsilon')} \left(\mathcal{O}(\|h\|_{\mathcal{B}}) o(\mathbf{a}_k^{-d}) \right)^{\varepsilon'/(\alpha + 2\varepsilon')} \\ & = \mathcal{O}(|a|^{\frac{\eta'(\alpha + \varepsilon')}{\alpha + 2\varepsilon'}} |b|^{\frac{\eta'(\alpha + \varepsilon')}{\alpha + 2\varepsilon'}} \mathbf{a}_k^{-\alpha - \varepsilon'} \|h\|_{\mathcal{B}}) o(\mathbf{a}_k^{-d}). \end{aligned}$$

After summation over $a, b \in \mathbb{Z}^d$, we obtain (23), since $\frac{\eta'(\alpha + \varepsilon')}{\alpha + 2\varepsilon'} \leq \eta$ and since $\sum_{a \in \mathbb{Z}^d} (1 + |a|^\eta) |\beta_a| < \infty$.

Using (23) inductively in (20) (combined with the fact that $\|Q_{k,0}\|_{\mathcal{B}} = \mathcal{O}(\mathbf{a}_k^{-d})$ and that $\|\bar{Q}_k\|_{\mathcal{B}}$ is summable which follows from (22)), we conclude that, when N is even

$$\begin{aligned} H_{0,1,\dots,0,1}^{(n,N)} & = \sum_{0 \leq k_1 \leq \dots \leq k_N} c_{k_1, \dots, k_N} \prod_{j=1}^{N/2} \left(\Phi(0) \left(\mathbf{a}_{k_{2j-1} - k_{2j-2}}^{-d} \mathbb{E}_\nu[\bar{Q}_{k_{2j} - k_{2j-1}}(\mathbf{1})] \right) \right) \\ & + \mathcal{O} \left(\left(\sum_{k=1}^n \mathbf{a}_k^{-d} \right)^{N/2-1} \sum_{k=1}^n o(\mathbf{a}_k^{-d}) \right). \end{aligned}$$

Recall that

$$\sigma_\beta^2 = \sum_{k \in \mathbb{Z}} \mathbb{E}_\nu [\bar{Q}_{|k|}(\mathbf{1})] = \mathbb{E}_\nu [\bar{Q}_0(\mathbf{1})] + 2 \sum_{k \geq 1} \mathbb{E}_\nu [\bar{Q}_k(\mathbf{1})].$$

Therefore, proceeding exactly as in [26, p. 1918-1919], we obtain that

$$\begin{aligned} H_{0,1,\dots,0,1}^{(n,N)} &= \frac{\Gamma(1 + \frac{\alpha-d}{\alpha})^{N/2}}{\Gamma(1 + \frac{N}{2} \frac{\alpha-d}{\alpha})} \frac{N!}{2^{N/2}} (\Phi(0) \sigma_\beta^2 \mathfrak{A}_n)^{N/2} + o(\mathfrak{A}_n^{N/2}) \\ &= \mathfrak{A}_n^{N/2} \mathbb{E} \left[\left(\sqrt{\sigma_\beta^2 \Phi(0) |Z| \mathcal{N}} \right)^N \right] + o(\mathfrak{A}_n^{N/2}). \end{aligned}$$

This, combined with (18) and (19), ends the proof of the convergence of every moments. We conclude the convergence in distribution by the Carleman criterion [30]. \square

2.3. Joint Limit theorem for additive functional of Birkhoff sums.

Theorem 2.6. *Assume Assumptions 2.1 and 2.2. Let $\eta := \frac{\alpha+\varepsilon-d}{2}$ for some $\varepsilon \in (0, 1/2)$. Let $(\beta_a^{(0)})_{a \in \mathbb{Z}^d}$ and $(\beta_a^{(1)})_{a \in \mathbb{Z}^d}$ be two families of real numbers such that $\sum_{a \in \mathbb{Z}^d} (1 + |a|)^\eta |\beta_a^{(j)}| < \infty$ and $\sum_{a \in \mathbb{Z}^d} \beta_a^{(1)} = 0$. Then the following family of couples of processes*

$$(26) \quad \left(\left(\sum_{k=0}^{\lfloor nt \rfloor - 1} \beta_{S_k}^{(0)} / \mathfrak{A}_n \right)_t, \left(\sum_{k=0}^{\lfloor nt \rfloor - 1} \beta_{S_k}^{(1)} / \sqrt{\mathfrak{A}_n} \right)_t \right)_{n \geq 1}$$

converges in distribution, with respect to ν , to $((\sum_{a \in \mathbb{Z}^d} \beta_a^{(0)} \mathcal{L}_t)_t, (\sigma_{\beta^{(1)}} B_{\mathcal{L}_t})_t)$, (in $(\mathcal{D}([0; T]))^2$ for all $T > 0$ if $d = 1$ and in $(\mathcal{D}([T_0; T]))^2$ for all $0 < T_0 < T$ if $d = 2$), where $\sigma_{\beta^{(1)}}^2$ is defined in Formula (9) of Theorem 2.3 taking $\beta = \beta^{(1)}$, where B is a Brownian motion and where \mathcal{L}_t is the following process

- if $\alpha > d$, \mathcal{L}_t is the local time at 0 in the time interval $[0; t]$ of an centered α -stable process W , independent of B , such that W_1 has density probability Φ with $\Phi(0)$ satisfying (5),
- if $\alpha = d$, $\mathcal{L}_t = \mathbf{1}_{\{t > 0\}} \mathcal{L}_1$, where \mathcal{L}_1 is a random variable with exponential distribution with mean $\Phi(0)$.

Proof of Theorem 2.6. We start by proving the convergence of the finite distributions and we will then prove the tightness. For the convergence of the finite distributions, we use again the convergence of moments. It is enough to study the asymptotic behaviour as n goes to infinity of every moments of the following form

$$E_n := \mathbb{E}_\nu \left[\prod_{j=1}^M \left(\sum_{k_j = \lfloor nt_{j-1} \rfloor}^{\lfloor nt_j \rfloor - 1} \beta_{S_k}^{(0)} \right)^{N_j^{(0)}} \left(\sum_{k'_j = \lfloor nt_{j-1} \rfloor}^{\lfloor nt_j \rfloor - 1} \beta_{S_k}^{(1)} \right)^{N_j^{(1)}} \right],$$

for any $M \in \mathbb{N}^*$, any $N_j^{(0)}, N_j^{(1)} \in \mathbb{N}$, any $t_0 = 0 < t_1 < \dots < t_M$. We set $\Gamma_k := \sum_{j=1}^M N_j^{(k)}$ for $k \in \{0, 1\}$ and will prove that

$$\lim_{n \rightarrow +\infty} \mathfrak{A}_n^{-\Gamma_0 - \frac{\Gamma_1}{2}} E_n = (\Phi(0))^{\Gamma_0 + \frac{\Gamma_1}{2}} \left(\sum_{a \in \mathbb{Z}^d} \beta_a^{(0)} \right)^{\Gamma_0} \sigma_{\beta^{(1)}}^{\Gamma_1} \mathbb{E} \left[\prod_{j=1}^M (\mathcal{L}_{t_j} - \mathcal{L}_{t_{j-1}})^{N_j^{(0)} + \frac{N_j^{(1)}}{2}} \right] \prod_{j=1}^M \mathbb{E}[\mathcal{N}^{N_j^{(1)}}].$$

We set $m_i := \sum_{r=1}^i (N_r^{(0)} + N_r^{(1)})$ and $N := m_M$. Proceeding as in the proof of Theorem 1.1, we observe that E_n can be rewritten as follows

$$(27) \quad \sum_{\gamma_1, \dots, \gamma_N} \sum_{(k_1, \dots, k_N) \in \mathcal{K}_n} \prod_{i=1}^N c_{(k_{m_{i-1}+1}, \dots, k_{m_i})}^{(\gamma_{m_{i-1}+1}, \dots, \gamma_{m_i})} \sum_{a_1, \dots, a_N \in \mathbb{Z}^d} \left(\prod_{j=1}^N \beta_{a_j}^{(\gamma_j)} \right) \nu (\forall i = 1, \dots, N, S_{k_i} = a_i),$$

where \mathcal{K}_n is the set of increasing N -uples (k_1, \dots, k_N) such that $\lfloor nt_{j-1} \rfloor \leq k_m \leq \lfloor nt_j \rfloor - 1$ for all m such that $m_{j-1} < m \leq m_j$, where the first sum is taken over $\gamma_j \in \{0, 1\}$ such that, for all $\gamma \in \{0, 1\}$ and all $i = 1, \dots, M$, $\#\{j = m_{i-1} + 1, \dots, m_i : \gamma_j = \gamma\} = N_i^{(\gamma)}$ and where $c_{(k_{m_{i-1}+1}, \dots, k_{m_i})}^{(\gamma'_{m_{i-1}+1}, \dots, \gamma'_{m_i})}$ is the number of $(\gamma'_{m_{i-1}+1}, \dots, \gamma'_{m_i}, k'_{m_{i-1}+1}, \dots, k'_{m_i}) \in \{0, 1\}^{N_i^{(1)} + N_i^{(2)}} \times \{\lfloor nt_{i-1} \rfloor, \dots, \lfloor nt_i \rfloor - 1\}^{N_i^{(0)} + N_i^{(1)}}$ such that there exists a permutation σ of $\{m_{i-1} + 1, \dots, m_i\}$ such that $\gamma'_{\sigma(r)} = \gamma_r$ and $k'_{\sigma(r)} = k_r$ for all $r \in \{m_{i-1} + 1, \dots, m_i\}$. Furthermore we use (14) to express $\nu(\forall i = 1, \dots, j-1, N, k_i = a_i)$ using a composition of operators $Q_{k_j - k_{j-1}, a_j - a_{j-1}}$ and, as in Appendix A, for each j , we decompose each $Q_{k_j - k_{j-1}, a_j - a_{j-1}}$ in a sum of $\tilde{Q}^{(\varepsilon)}$. This leads to

$$(28) \quad E_n := \sum_{\gamma = (\gamma_1, \dots, \gamma_N)} \sum_{\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)} \tilde{H}_{\varepsilon}^{(n)}(\gamma),$$

with

$$\begin{aligned} \tilde{H}_{\varepsilon_1, \dots, \varepsilon_N}^{(n)}(\gamma_1, \dots, \gamma_N) &= \sum_{(k_1, \dots, k_N) \in \mathcal{K}_n} \prod_{i=1}^M c_{(k_{m_{i-1}+1}, \dots, k_{m_i})}^{(h_{m_{i-1}+1}, \dots, h_{m_i})} \sum_{a_1, \dots, a_N \in \mathbb{Z}^d} \\ \mathbb{E}_{\nu} \left[\beta_{a_N}^{(\gamma_N)} \left(\beta_{a_{N-1}}^{(\gamma_{N-1})} \tilde{Q}_{k_N - k_{N-1}, a_N - a_{N-1}}^{(\varepsilon_N)} \left(\dots \left(\beta_{a_1}^{(\gamma_1)} \tilde{Q}_{k_2 - k_1, a_1, a_2}^{(\varepsilon_2)} \left(\tilde{Q}_{k_1, 0, a_1}^{(\varepsilon_1)}(\mathbf{1}) \right) \right) \dots \right) \right) \right], \end{aligned}$$

with the use of the operators $\tilde{Q}_{(k, a, b)}^{(\varepsilon)}$ defined in Appendix A and where the sum over $\varepsilon_1, \dots, \varepsilon_N$ is taken over $\varepsilon_2, \dots, \varepsilon_N \in \{0, 1\}^{2^d}$ and $\varepsilon_1 \in \{(0, 0), (0, 1)\}$. We write $\varepsilon_j = (\varepsilon_{j,1}, \varepsilon_{j,2})$. Since $\sum_{a \in \mathbb{Z}^d} \beta_a^{(1)} = 0$, $H_{\varepsilon_1, \dots, \varepsilon_N} = 0$ if there exists $j = 1, \dots, N$ such that $\gamma_j = 1$ and $\varepsilon_{j,2} + \varepsilon_{j+1,1} = 0$ (with convention $\varepsilon_{N+1,1} = 0$). Therefore we assume from now on that (γ, ε) is such that $\varepsilon_{j,2} + \varepsilon_{j+1,1} \geq \gamma_j$, for all j (with the convention $\varepsilon_{N+1,1} = 0$, and with $\varepsilon_{1,1} = 0$) and we call **admissible** such a pair (γ, ε) . Then

$$\begin{aligned} \tilde{H}_{\varepsilon}^{(n)}(\gamma) &= \mathcal{O} \left(\sum_{(k_1, \dots, k_N)} \sum_{a_1, \dots, a_N \in \mathbb{Z}^d} \prod_{j=1}^N \left(|\beta_{a_j}^{(\gamma_j)}| \left\| \tilde{Q}_{k_j - k_{j-1}, a_{j-1}, a_j}^{(\varepsilon_j)} \right\|_{\mathcal{B}} \right) \right) \\ &= \mathcal{O} \left(\sum_{(k_1, \dots, k_N)} \sum_{a_1, \dots, a_N \in \mathbb{Z}^d} \prod_{j=1}^N \left(|\beta_{a_j}^{(\gamma_j)}| (1 + |a_j|) \eta_{\mathbf{a}_{k_j - k_{j-1}}^{-d - \eta_{j,1} \varepsilon_{j,1} - \eta_{j,2} \varepsilon_{j,2}}} \right) \right), \end{aligned}$$

with $(\eta_{j,i})_{j,i}$ a sequence in $\{0, \eta\}$ such that $\eta_{j,2} \varepsilon_{j,2} + \eta_{j+1,1} \varepsilon_{j+1,1} = \eta \gamma_j$ and such that

- $(\eta_{j,1}, \eta_{j,2}) = \eta \varepsilon_j$ if $\varepsilon_j \neq (1, 1)$,
- $\eta_{j,1} = \eta(1 - \varepsilon_{j-1,2})$ if $\varepsilon_j = (1, 1)$,
- $\eta_{j,2} = \eta$ if $\varepsilon_j = (1, 1)$ and $\varepsilon_{j+1} \neq (1, 0)$,
- $\eta_{j,2} = 0$ if $\varepsilon_j = (1, 1)$ and $\varepsilon_{j+1} = (1, 0)$.

As seen in Appendix A, we use the fact that there exists $u_0 \in (0, 1]$ such that $\sum_{k=0}^{n-1} \mathbf{a}_k^{-d-\eta} = \mathcal{O}\left(\mathfrak{A}_n^{\frac{1-u_0}{2}}\right)$ and that $\sum_{k \geq 0} \mathbf{a}_k^{-d-2\eta} < \infty$. Using the summability assumption on $\beta_a^{(\gamma)}$, we infer that

$$\tilde{H}_\varepsilon^{(n)}(\gamma) = \mathcal{O}\left(\sum_{(k_1, \dots, k_N)} \prod_{j: \varepsilon_{j,1} + \varepsilon_{j,2} = 0} \mathbf{a}_{k_j - k_{j-1}}^{-d - \eta_{j,1}\varepsilon_{j,1} - \eta_{j,2}\varepsilon_{j,2}}\right) = \mathcal{O}\left(\mathcal{A}_n^{\varepsilon_0^\eta + \frac{1-u_0}{2}\varepsilon_1^\eta}\right),$$

where

$$\mathcal{E}_k^\eta := \#\{j = 1, \dots, N : \eta_{j,1}\varepsilon_{j,1} + \eta_{j,2}\varepsilon_{j,2} = k\eta\}.$$

We also set $\mathcal{E}_k := \#\{j = 1, \dots, N : \varepsilon_{j,1} + \varepsilon_{j,2} = k\}$ and observe that $\mathcal{E}_1 \leq \mathcal{E}_1^\eta$. Recall $\Gamma_k = \sum_{j=0}^N N_j^{(k)} = \#\{j : \gamma_j = k\}$. Then

$$N = \mathcal{E}_0^\eta + \mathcal{E}_1^\eta + \mathcal{E}_2^\eta = \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2 = \Gamma_0 + \Gamma_1.$$

On the other side

$$\begin{aligned} \eta\Gamma_1 &= \eta \sum_{j=1}^N \gamma_j = \sum_{j=1}^N (\eta_{j,2}\varepsilon_{j,2} + \eta_{j+1,1}\varepsilon_{j+1,1}) \\ &= \sum_{j=1}^N (\eta_{j,1}\varepsilon_{j,1} + \eta_{j+1,1}\varepsilon_{j,2}) - \eta_{1,1}\varepsilon_{1,1} = \eta(\mathcal{E}_1^\eta + 2\mathcal{E}_2^\eta), \end{aligned}$$

and so

$$\begin{aligned} \mathcal{E}_0^\eta + \frac{1-u_0}{2}\mathcal{E}_1^\eta &= N - \mathcal{E}_2^\eta - \mathcal{E}_1^\eta + \frac{1-u_0}{2}\mathcal{E}_1^\eta \\ &= (\Gamma_0 + \Gamma_1) - \left(\frac{\Gamma_1 - \mathcal{E}_1^\eta}{2}\right) - \frac{1+u_0}{2}\mathcal{E}_1^\eta \\ &= \Gamma_0 + \frac{\Gamma_1}{2} - \frac{u_0}{2}\mathcal{E}_1^\eta. \end{aligned}$$

Hence we have proved that

$$H_\varepsilon^{(n)}(\gamma) = o\left(\mathcal{A}_n^{\Gamma_0 + \frac{\Gamma_1}{2}}\right) \quad \text{if } \mathcal{E}_1^\eta > 0,$$

Now we assume that $\mathcal{E}_1^\eta = 0$, this implies that $\mathcal{E}_1 = 0$ and so that the j 's such that $\varepsilon_j = (1, 1)$ are isolated and $\mathcal{E}_2 = \mathcal{E}_2^\eta = \frac{\Gamma_1}{2}$.

Observe that this implies that $\Gamma_1 = \sum_{j=1}^M N_j^{(1)}$ is even and we have proved that

$$(29) \quad E_n = o\left(\mathfrak{A}_n^{\Gamma_0 + \frac{\Gamma_1}{2}}\right) \quad \text{if } n \text{ is odd.}$$

The above conditions on γ and ε imply that ε is a sequence of $(0, 0)$ and $(1, 1)$ (which are isolated) and that $\varepsilon_{j,2} + \varepsilon_{j+1,1} \geq \gamma_j$, thus $\varepsilon_{j,2} + \varepsilon_{j+1,1} = \gamma_j$ and

- if $\gamma_j = 0$, then both $\varepsilon_j = (\varepsilon_{j,1}, \varepsilon_{j,2})$ and $\varepsilon_{j+1} = (\varepsilon_{j+1,1}, \varepsilon_{j+1,2})$ are $(0, 0)$.
- if $\gamma_j = 1$, then either $\varepsilon_j = (\varepsilon_{j,1}, \varepsilon_{j,2})$ or $\varepsilon_j = (\varepsilon_{j,1}, \varepsilon_{j,2})$ is $(1, 1)$, and the other one is $(0, 0)$.

This means that the j 's such that $\gamma_j = 1$ appear in pairwise disjoint pairs $(j-1, j)$ such that $(\gamma_{j-1}, \gamma_j) = (1, 1)$, and that $\varepsilon_j = (1, 1)$ if and only if $(j-1, j)$ is such a couple. Fixing such a pair (γ, ε) , let us write \mathcal{J} for the set of such j such that $\varepsilon_j = (1, 1)$. Then, using repeatedly (5) and (24), we obtain

$$\begin{aligned} \mathfrak{A}_n^{-\Gamma_0 - \frac{\Gamma_1}{2}} \tilde{H}_\varepsilon^{(n)}(\gamma) &= \sum_{(k_1, \dots, k_N) \in \mathcal{K}_n} \prod_{i=1}^M c_{(k_{m_{i-1}+1}, \dots, k_{m_i})}^{(\gamma_{m_{i-1}+1}, \dots, \gamma_{m_i})} \prod_{j': \gamma_{j'}=0} \left(\frac{\mathfrak{a}_{k_{j'}-k_{j'-1}}^{-d}}{\mathfrak{A}_n} \Phi(0) \sum_{a \in \mathbb{Z}^d} \beta_a^{(0)} \right) \\ &\prod_{j \in \mathcal{J}} \frac{\Phi(0) \mathfrak{a}_{k_{j-1}-k_{j-2}}^{-d}}{2 \mathfrak{A}_n} \sum_{a, b \in \mathbb{Z}^d} \beta_a^{(1)} \beta_b^{(1)} \mathbb{E}_\nu \left[Q''_{k_j - k_{j-1}, a, b}(\mathbf{1}) \right] + o(1) \end{aligned}$$

and so

$$\begin{aligned} \mathfrak{A}_n^{-\Gamma_0 - \frac{\Gamma_1}{2}} \tilde{H}_\varepsilon^{(n)}(\gamma) &= o(1) + \Phi(0)^{\Gamma_0 + \frac{\Gamma_1}{2}} \left(\sum_{a \in \mathbb{Z}^d} \beta_a^{(0)} \right)^{\Gamma_0} \sum_{(k_1, \dots, k_N) \in \mathcal{K}_n: k_j < k_{j+1}} \prod_{i=1}^M (N_i^{(0)})! (N_i^{(1)})! \\ &\left(\prod_{j: \gamma_j=0} \frac{\mathfrak{a}_{k_j - k_{j-1}}^{-d}}{\mathfrak{A}_n} \right) \prod_{j \in \mathcal{J}} \frac{\mathfrak{a}_{k_{j-1} - k_{j-2}}^{-d}}{2 \mathfrak{A}_n} \sum_{a, b \in \mathbb{Z}^d} \beta_a^{(1)} \beta_b^{(1)} \mathbb{E}_\nu \left[Q''_{k_j - k_{j-1}, a, b}(\mathbf{1}) \right]. \end{aligned}$$

It can be worthwhile to notice that we can restrict the above sum on the $(k_1, \dots, k_N) \in \mathcal{K}_n$ such that $k_j - k_{j-1} < \log n$ if $j \in \mathcal{J}$, and $k_j - k_{j-1} > \log n$ for the other values of j 's. This implies that

$$\mathfrak{A}_n^{-\Gamma_0 - \frac{\Gamma_1}{2}} \tilde{H}_\varepsilon^{(n)}(\gamma) = o(1)$$

as soon as there exist $j \in \mathcal{J}$ and $i \in \{1, \dots, M\}$ such that $k_{j-1} < \lfloor nt_i \rfloor \leq k_j$ (indeed this combined with $k_j - k_{j-1} < \log n$ implies that $0 < k_j - nt_i < \log n$ and $0 < nt_i - k_{j-1} < \log n$). In particular

$$E_n = o \left(\mathfrak{A}_n^{\Gamma_0 + \frac{\Gamma_1}{2}} \right) \quad \text{if } \exists j \in \{1, \dots, M\}, N_j^{(1)} \in 2\mathbb{Z} + 1.$$

We assume from now on that the $N_j^{(1)}$'s are even and that \mathcal{J} is such that, for every $j \in \mathcal{J}$, there exists $i = 1, \dots, M$ such that k_{j-1}, k_j are in a same set $\{\lfloor nt_{i-1} \rfloor, \dots, \lfloor nt_i \rfloor - 1\}$. Then

$$\begin{aligned} \mathfrak{A}_n^{-\Gamma_0 - \frac{\Gamma_1}{2}} \tilde{H}_\varepsilon^{(n)}(\gamma) &= o(1) + \prod_{j=1}^M \left(N_j^{(0)}! N_j^{(1)}! (\Phi(0))^{N_j''} \left(\sum_{a \in \mathbb{Z}^d} \beta_a^{(0)} \right)^{N_j^{(0)}} \right. \\ &\left. \left(\frac{1}{2} \sum_{k \geq 0} (1 + \mathbf{1}_{\{k \neq 0\}}) \sum_{a, b \in \mathbb{Z}^d} \beta_a^{(1)} \beta_b^{(1)} \mathbb{E}_\nu \left[\bar{Q}''_{k_j - k_{j-1}, a, b}(\mathbf{1}) \right] \right)^{\frac{N_j^{(1)}}{2}} \sum_{(k'_1, \dots, k'_{N''}) \in \mathcal{K}'_n} \prod_{j=1}^{N''} \frac{\mathfrak{a}_{k'_j - k'_{j-1}}^{-d}}{\mathfrak{A}_n} \right), \end{aligned}$$

with $N_j'' := N_j^{(0)} + \frac{N_j^{(1)}}{2}$ and $N'' := \sum_{j=1}^M N_j''$, and where \mathcal{K}'_n is the set of strictly increasing sequences $k'_1 \leq \dots \leq k'_{N''}$ with exactly N_j'' elements between $\lfloor nt_{j-1} \rfloor$ and $\lfloor nt_j \rfloor - 1$ and with the convention $k'_0 = 0$. Thus

$$\frac{\tilde{H}_\varepsilon^{(n)}(\gamma)}{\mathfrak{A}_n^{\sum_{j=1}^M N_j''}} = o(1) + E_n \prod_{j=1}^M \left(N_j^{(0)}! N_j^{(1)}! \left(\sum_{a \in \mathbb{Z}^d} \beta_a^{(0)} \right)^{N_j^{(0)}} 2^{-\frac{N_j^{(1)}}{2}} \sigma_{\beta^{(1)}}^{N_j^{(1)}} \frac{1}{N_j''!} \right)$$

with

$$\begin{aligned}
E'_n &:= (\Phi(0))^{N''} \left(\prod_{j=1}^M N_j''! \right) \sum_{(k'_1, \dots, k'_{N''}) \in \mathcal{K}_n} \prod_{j=1}^{N''} \frac{\mathfrak{a}_n^{-d} \mathfrak{a}_n^{k'_j - k'_{j-1}}}{\mathfrak{A}_n} \\
(30) \quad &= o(1) + \mathfrak{A}_n^{-N''} \mathbb{E}_\nu \left[\prod_{j=1}^M \left(\sum_{k_j = \lfloor nt_{j-1} \rfloor}^{\lfloor nt_j \rfloor - 1} \mathbf{1}_{\{S_{k_j} = 0\}} \right)^{N_j''} \right].
\end{aligned}$$

We observe that there exist $\frac{N_j''!}{N_j^{(0)}!(N_j^{(1)}/2)!}$ sequences $(\gamma_{m_j+1}, \dots, \gamma_{m_j+1}) \in \{0, 1\}^{N_j^{(0)} + N_j^{(1)}}$ in which the 1's appear in $N_j^{(1)}/2$ pairwise distinct pairs (γ_{j-1}, γ_j) . Therefore

$$\begin{aligned}
\frac{E_n}{\mathfrak{A}_n^{\sum_{j=1}^M N_j''}} &= o(1) + E'_n \prod_{j=1}^M \left(\frac{N_j^{(1)}!}{(N_j^{(1)}/2)! 2^{\frac{N_j^{(1)}}{2}}} \left(\sum_{a \in \mathbb{Z}^d} \beta_a^{(0)} \right)^{N_j^{(0)}} \sigma_{\beta^{(1)}}^{N_j^{(1)}} \right) \\
&= o(1) + E'_n \prod_{j=1}^M \left(\left(\sum_{a \in \mathbb{Z}^d} \beta_a^{(0)} \right)^{N_j^{(0)}} \mathbb{E} \left[(\sigma_{\beta^{(1)}} \mathcal{N})^{N_j^{(1)}} \right] \right).
\end{aligned}$$

It remains to study the asymptotics of E'_n .

- If $d = 1 < \alpha$, we consider a \mathbb{Z} -valued non-arithmetic random walk $(\tilde{S}_n)_n$ (with i.i.d. increments) such that $(\tilde{S}_{\lfloor nt \rfloor} / \mathfrak{a}_n)_n$ converges in distribution to the α -stable process W . The previous computations hold also true (more easily) for \tilde{S}_n instead of S_n and lead to

$$E'_n \sim \mathfrak{A}_n^{-N''} \mathbb{E} \left[\prod_{j=1}^M \left(\sum_{k_j = \lfloor nt_{j-1} \rfloor}^{\lfloor nt_j \rfloor - 1} \mathbf{1}_{\{\tilde{S}_{k_j} = 0\}} \right)^{N_j''} \right] \quad \text{as } n \rightarrow +\infty.$$

But the process $\left(\sum_{k=0}^{\lfloor nt \rfloor - 1} \mathbf{1}_{\{S_k = 0\}} \right)_t$ of local time at 0 of \tilde{S}_n converges in distribution to the process $(\mathcal{L}_t)_t$ of local time at 0 of W . This combined with the dominations of the moments of any order ensures that $(E'_n)_n$ converges in distribution to $\mathbb{E} \left[\prod_{j=1}^M (\mathcal{L}_{t_j} - \mathcal{L}_{t_{j-1}})^{N_j''} \right]$. We conclude that

$$\begin{aligned}
\frac{E_n}{\mathfrak{A}_n^{\sum_{j=1}^M N_j''}} &= o(1) + \mathbb{E} \left[\prod_{j=1}^M (\mathcal{L}_{t_j} - \mathcal{L}_{t_{j-1}})^{N_j''} \right] \prod_{j=1}^M \left(\left(\sum_{a \in \mathbb{Z}^d} \beta_a^{(0)} \right)^{N_j^{(0)}} \mathbb{E} \left[(\sigma_{\beta^{(1)}} \mathcal{N})^{N_j^{(1)}} \right] \right) \\
&= o(1) + \mathbb{E} \left[\prod_{j=1}^M \left(\sum_{a \in \mathbb{Z}^d} \beta_a^{(0)} (\mathcal{L}_{t_j} - \mathcal{L}_{t_{j-1}}) \right)^{N_j^{(0)}} \prod_{j=1}^M (\sigma_{\beta^{(1)}} (\mathcal{N}_{\mathcal{L}_{t_j}} - \mathcal{N}_{\mathcal{L}_{t_{j-1}}}))^{N_j^{(1)}} \right].
\end{aligned}$$

- If $d = \alpha$, then

$$\begin{aligned} \mathfrak{A}_n^{-N''} \mathbb{E}_\nu \left[\left(\sum_{k=\lfloor NT_0 \rfloor}^{\lfloor Nt_M \rfloor - 1} \mathbf{1}_{\{S_k=0\}} \right)^{N''} \right] &= \mathcal{O} \left(\mathfrak{A}_n^{-N''} \left(\sum_{k=\lfloor NT_0 \rfloor}^{\lfloor Nt_M \rfloor - 1} \mathfrak{a}_k^{-d} \right)^{N''} \right) \\ &= \mathcal{O} \left(\left((\mathfrak{A}_{\lfloor nt_M \rfloor} - \mathfrak{A}_{\lfloor nT_0 \rfloor}) / \mathfrak{A}_n^{N''} \right)^{N''} \right), \end{aligned}$$

converges to 0 as $n \rightarrow +\infty$, since $(\mathfrak{A}_n)_{n \geq 0}$ is slowly varying. Thus

$$E'_n = o(1) \quad \text{if } M \geq 2.$$

Furthermore, it follows from the proof of Theorem 1.1 that if $M = 1$,

$$\begin{aligned} E'_n &= o(1) + \mathfrak{A}_n^{-N''} \mathbb{E}_\nu \left[\left(\sum_{k_j=\lfloor nt_{j-1} \rfloor}^{\lfloor nt_j \rfloor - 1} \mathbf{1}_{\{S_{k_j}=0\}} \right)^{N''_1} \right] \\ &= (\Phi(0))^{N''_1} N''_1! = (\Phi(0))^{N''_1} \mathbb{E}[\mathcal{E}^{N''_1}], \end{aligned}$$

where \mathcal{E} is a random variable with standard exponential distribution due to theorem 2.3. We infer that

$$\begin{aligned} \frac{E_n}{\mathfrak{A}_n^{\sum_{j=1}^M N''_j}} &= o(1) + \mathbb{E} \left[(\Phi(0)\mathcal{E})^{N''_1} \prod_{j=2}^M (\Phi(0)\mathcal{E} - \Phi(0)\mathcal{E})^{N''_j} \prod_{j=1}^M \left(\left(\sum_{a \in \mathbb{Z}^d} \beta_a^{(0)} \right)^{N''_j} \mathbb{E} \left[(\sigma_{\beta^{(1)}} \mathcal{N})^{N''_j} \right] \right) \right] \\ &= o(1) + \mathbb{E} \left[\prod_{j=1}^M \left(\sum_{a \in \mathbb{Z}^d} \beta_a^{(0)} (\Phi(0)\mathcal{E}(\mathbf{1}_{\{t_j > 0\}} - \mathbf{1}_{\{t_{j-1} > 0\}})) \right)^{N''_j} \prod_{j=1}^M (\sigma_{\beta^{(1)}}(\mathcal{N}_{\Phi(0)\mathcal{E}\mathbf{1}_{\{t_j > 0\}}} - \mathcal{N}_{\Phi(0)\mathcal{E}\mathbf{1}_{\{t_{j-1} > 0\}}}))^{N''_j} \right]. \end{aligned}$$

This combined with (29) and the Carleman's criteria [30] ends the proof of the convergence of the finite dimensional distributions.

Let us write $((X_t^{(1,n)}, X_t^{(2,n)})_t)_{n \geq 1}$ for the joint process (26) and let us prove its tightness. When $d = 1 < \alpha$, we set $T_0 = 0$, otherwise we fix some $T_0 \in (0; T)$. We use the tightness criterion of [1, Theorem 13.5, (13.4)]. We have proved the convergence of the finite dimensional distributions. It remains to prove that there exist $\alpha_1 > 1$ and $C > 0$ such that, for every r, s, t such that $T_0 \leq r \leq s \leq t \leq T$, for all $j \in \{1, 2\}$,

$$(31) \quad \exists p_j \in \mathbb{N}^*, \quad \mathbb{E}_\nu \left[|X_t^{(j,n)} - X_s^{(j,n)}|^{p_j} |X_s^{(j,n)} - X_r^{(j,n)}|^{p_j} \right] \leq C |t - r|^{\alpha_1}.$$

Observe first that, if $0 \leq r \leq s \leq t \leq T$ and $t - r < 1/n$, then $X_t^{(n)} - X_s^{(n)} = 0$ or $X_s^{(n)} - X_r^{(n)} = 0$, thus the left hand side of (31) is null and so (31) holds true. Assume from now on that $T_0 \leq r \leq s \leq t \leq T$ and that $t - r \geq 1/n$. We will use the following inequality

$$\mathbb{E}_\nu \left[|X_t^{(j,n)} - X_s^{(j,n)}|^{p_j} |X_s^{(j,n)} - X_r^{(j,n)}|^{p_j} \right] \leq \left\| X_t^{(j,n)} - X_s^{(j,n)} \right\|_{L^{2p_j}(\nu)}^{p_j} \left\| X_s^{(j,n)} - X_r^{(j,n)} \right\|_{L^{2p_j}(\nu)}^{p_j}.$$

Thus (31) will follow from the fact that, for any $T_0 \leq r < t \leq T$, $|t - r| > 1/n$, and $j \in \{1, 2\}$,

$$(32) \quad \exists p_j \in \mathbb{N}^*, \quad \sup_{a, b: r \leq a < b \leq t} \left\| X_b^{(j,n)} - X_a^{(j,n)} \right\|_{L^{2p_j}(\nu)}^{2p_j} \leq C |t - r|^{\alpha_1}.$$

It follows from our previous moment computation that

$$\mathbb{E}_{\mu_{c_0}} \left[|X_b^{(j,n)} - X_a^{(j,n)}|^{2p} \right] = \mathcal{O} \left(\left(\mathfrak{A}_n^{-1} \sum_{k=\lfloor nr \rfloor}^{\lfloor nt \rfloor - 1} \mathfrak{a}_k^{-d} \right)^{\frac{2p}{j}} \right).$$

Thus it is enough to prove that

$$(33) \quad \exists \alpha_0 > 0, \quad \sup_{r, t: T_0 \leq r < t < T, |t-r| \geq 1/n} \mathfrak{A}_n^{-1} \sum_{k=\lfloor nr \rfloor}^{\lfloor nt \rfloor - 1} \mathfrak{a}_k^{-d} = \mathcal{O}((t-r)^{\alpha_0}).$$

Indeed, we will conclude by taking $p_j = j \lceil (2\alpha_0)^{-1} \rceil$ so that (32) and so (31) hold true with $\alpha_1 := \frac{2p_j \alpha_0}{j} > 1$.

Since $(\mathfrak{a}_n)_{n \geq 0}$ is $(1/\alpha)$ -regularly varying, it follows from Karamata's theorem [12, 2] that there exist three bounded convergent sequences $(c(n))_{n \geq 0}$ (positive, with positive limit), $(b(n))_{n \geq 0}$ (converging to 0) and $(\theta_n)_{n \geq 0}$ (positive, converging to $\frac{\alpha-1}{\alpha}$ if $d < \alpha$ and to 0 if $d = \alpha$, see [2, Proposition 1.5.9.b]) such that

$$\forall n \geq 0, \quad na_n^{-d} = \mathfrak{A}_n \theta_n \quad \text{and} \quad \mathfrak{A}_n = c(n) n^{\frac{\alpha-d}{\alpha}} e^{\int_1^n \frac{b(t)}{t} dt}.$$

Now let us choose α_0 . If $d = 1 < \alpha$, we set $\alpha_0 := \frac{\alpha-1}{2\alpha}$. If $d = \alpha$, we take $\alpha_0 := 1$. Up to change, if necessary, the first terms of $(b(n))_{n \geq 0}$ and $(c(n))_{n \geq 0}$, we assume without loss of generality that the sequence $(b(n))_{n \geq 0}$ is bounded by α_0 .

If $d = 1 < \alpha$, if $r \leq 2/n$ (observe that, when $d = \alpha$, this case does not happen for large values of n since $r \geq T_0 > 0$) and $|t-r| \geq 1/n$, then

$$(34) \quad \mathfrak{A}_n^{-1} \sum_{k=0}^{\lfloor nt \rfloor - 1} \mathfrak{a}_k^{-d} = \mathcal{O} \left(\frac{\mathfrak{A}_{\lfloor nt \rfloor}}{\mathfrak{A}_n} \right) = \mathcal{O} \left(t^{1-\frac{1}{\alpha}-\alpha_0} \right) = \mathcal{O}((t-r)^{\alpha_0}),$$

implying (33) and so (32) and (31) in this case.

We assume from now on that $T_0 \leq r < t \leq T$ and $2/n < r < t \leq T$ (so that $\lfloor nr \rfloor - 1 > 0$) and $t-r \geq 1/n$ (so that $\lfloor nt \rfloor - \lfloor nr \rfloor \leq 2(nt-nr)$). Then, using the uniform dominations on $(c(n), \theta_n, b(n))_n$ combined with a series-integral comparison, we obtain that

$$\mathfrak{A}_n^{-1} \sum_{k=\lfloor nr \rfloor}^{\lfloor nt \rfloor - 1} \mathfrak{a}_k^{-d} = \sum_{k=\lfloor nr \rfloor}^{\lfloor nt \rfloor - 1} \frac{\alpha-1}{\alpha k} \frac{c(k)}{c(n)} \frac{k^{\frac{\alpha-d}{\alpha}}}{n^{\frac{\alpha-d}{\alpha}}} \theta_k e^{\int_n^k \frac{b(u)}{u} du} = \mathcal{O} \left(\sum_{k=\lfloor nr \rfloor}^{\lfloor nt \rfloor - 1} \frac{1}{n} (k/n)^{-\frac{d}{\alpha}-\alpha_0} \right).$$

- If $d = 1 < \alpha$, it follows that

$$(35) \quad \begin{aligned} \mathfrak{A}_n^{-1} \sum_{k=\lfloor nr \rfloor}^{\lfloor nt \rfloor - 1} \mathfrak{a}_k^{-d} &= \mathcal{O} \left(\left(\frac{\lfloor nt \rfloor - 1}{n} \right)^{\alpha_0} - \left(\frac{\lfloor nr \rfloor - 1}{n} \right)^{\alpha_0} \right) \\ &= \mathcal{O} \left(\left(\frac{\lfloor nt \rfloor - \lfloor nr \rfloor}{n} \right)^{\alpha_0} \right) = \mathcal{O}((t-r)^{\alpha_0}), \end{aligned}$$

ending the proof of (33), from which we infer (32) and (31). This ends the proof of the tightness when $d < \alpha$.

- When $d = \alpha$, we obtain

$$(36) \quad \begin{aligned} \mathfrak{A}_n^{-1} \sum_{k=\lfloor nr \rfloor}^{\lfloor nt \rfloor - 1} \mathfrak{a}_k^{-d} &= O \left(\left(\frac{\lfloor nr \rfloor}{n} \right)^{-\alpha_0} - \left(\frac{\lfloor nt \rfloor}{n} \right)^{-\alpha_0} \right) \\ &= O \left(\frac{\lfloor nt \rfloor - \lfloor nr \rfloor}{n} \right) = \mathcal{O}(t - r), \end{aligned}$$

from which we infer (33), (32), (31), and so the tightness in the case where $d = \alpha$.

This ends the proof of Theorem 2.6. \square

3. PROOF OF THEOREMS 1.1 AND 1.3 VIA FOURIER PERTURBATIONS

A strategy to prove Assumptions 2.1 and 2.2 consists in noticing that

$$(37) \quad Q_{k,a} = \frac{1}{(2\pi)^d} \int_{[-\pi; \pi]^d} e^{-i\langle t, a \rangle} P_t^k(\cdot) dt, \quad \text{with } P_t(h) := P(e^{i\langle t, \Psi \rangle}),$$

and to establish nice properties for P_t as the one listed in the next result. Recall that $P_t^n(h) = P^n(h e^{itS_n})$.

Proposition 3.1. *Assume \mathcal{B} is a Banach space satisfying (3) and that there exist two constants $b \in (0, \pi)$ and $\alpha_0 > 0$ such that :*

$$(38) \quad \forall t \in [-b, b]^d, P_t^k = \lambda_t^k \Pi_t + \mathcal{O}(e^{-\alpha_0 k}) \quad \text{and} \quad \sup_{b < |u|_\infty < \pi} \|P_u^k\|_{\mathcal{B}} = \mathcal{O}(e^{-\alpha_0 k}),$$

with $(\lambda_{t/\mathfrak{a}_k}^k)_{k \geq 0}$ converging to the characteristic function φ of an α -stable distribution, with $\Pi_t = \mathbb{E}_\nu[\cdot] + o(1)$ in $\mathcal{L}(\mathcal{B} \rightarrow L^1(\nu))$ as $t \rightarrow 0$, and with

$$(39) \quad \sup_{t \in [-b; b]^d} \|\Pi_t\|_{\mathcal{B}} < \infty \quad \text{and} \quad \int_{\mathbb{R}} (1 + |t|^2) \left(\sup_{k \geq 1} |\lambda_{t/\mathfrak{a}_k}^k| \mathbf{1}_{\{|t| < b\mathfrak{a}_k\}} \right) dt < \infty.$$

Then Hypotheses 2.2 (and so 2.1) hold true with $\Phi(0)$ the value at 0 of the density function Φ of the α -stable distribution with characteristic function φ .

Proof of Proposition 3.1. It follows from our assumptions that

$$(40) \quad Q_{k,0} = \frac{1}{(2\pi)^d} \int_{[-\pi; \pi]^d} P_t^k dt = \frac{\mathfrak{a}_k^{-d}}{(2\pi)^d} \int_{[-b\mathfrak{a}_k; b\mathfrak{a}_k]^d} \lambda_{t/\mathfrak{a}_k}^k \Pi_{t/\mathfrak{a}_k} dt + \mathcal{O}(e^{-\alpha_0 k}) = \Phi(0) \mathfrak{a}_k^{-d} + o(\mathfrak{a}_k^{-d}),$$

in $\mathcal{L}(\mathcal{B} \rightarrow L^1(\nu))$, via the dominated convergence theorem since $\lim_{n \rightarrow +\infty} \lambda_{t/\mathfrak{a}_n}^n \Pi_{t/\mathfrak{a}_n} = \varphi(t) \mathbb{E}_\nu[\cdot]$ in $\mathcal{L}(\mathcal{B} \rightarrow L^1(\nu))$ and since $\Phi(0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}} \varphi(t) dt$. Thus (5) holds true.

Furthermore, for all $\eta \in [0; 2]$ and $a \in \mathbb{Z}^d$, in $\mathcal{L}(\mathcal{B})$,

$$(41) \quad \int_{[-\pi; \pi]^d} |t|^\eta \|P_t^k\|_{\mathcal{B}} dt = \mathfrak{a}_k^{-d-\eta} \int_{[-b\mathfrak{a}_k; b\mathfrak{a}_k]^d} |t|^\eta |\lambda_{t/\mathfrak{a}_k}^k| \|\Pi_{t/\mathfrak{a}_k}\|_{\mathcal{B}} dt + \mathcal{O}(e^{-\alpha_0 k}) = \mathcal{O}(\mathfrak{a}_k^{-d-\eta}).$$

Using the expression (37) of $Q_{k,a}$ combined with (41) with $m = 0$, we obtain (4) and (6). Fix $\eta \in [0; 1]$. Then

$$\begin{aligned} Q'_{k,a,b} &= Q_{k,b} - Q_{k,a} = \frac{1}{(2\pi)^d} \int_{[-\pi;\pi]^d} (e^{i\langle t,b \rangle} - e^{i\langle t,a \rangle}) P_t^k(\cdot) dt \\ &= \frac{1}{(2\pi)^d} \int_{[-\pi;\pi]^d} \mathcal{O}(\langle t, b-a \rangle)^\eta P_t^k(\cdot) dt \\ &= \mathcal{O}(|b-a|^\eta \mathbf{a}_k^{-d-\eta}) \quad \text{in } \mathcal{L}(\mathcal{B}), \end{aligned}$$

where we used again (41) combined with the bound $|e^{ix} - e^{iy}| \leq \min(2, |x-y|) \leq 2^{1-\eta}|x-y|^\eta$, and so we have proved (7). For (8), in the same way, we obtain

$$\begin{aligned} Q''_{k,a,b} &= Q_{k,b-a} - Q_{k,b} - Q_{k,-a} + Q_{k,0} \\ &= \frac{1}{(2\pi)^d} \int_{[-\pi;\pi]^d} (e^{i\langle t,b-a \rangle} - e^{i\langle t,b \rangle} - e^{-i\langle t,a \rangle} + 1) P_t^k(\cdot) dt \\ &= \frac{1}{(2\pi)^d} \int_{[-\pi;\pi]^d} (e^{i\langle t,b \rangle} - 1)(e^{-i\langle t,a \rangle} - 1) P_t^k(\cdot) dt \\ &= \frac{1}{(2\pi)^d} \int_{[-\pi;\pi]^d} \mathcal{O}(\langle t,b \rangle \langle t,a \rangle)^\eta P_t^k(\cdot) dt \\ (42) \quad &= \mathcal{O}(|a||b|^\eta \mathbf{a}_k^{-d-2\eta}) \quad \text{in } \mathcal{L}(\mathcal{B}), \end{aligned}$$

and so (8). □

Proof of Theorems 1.1 and 1.3. Let us write β_ℓ for the constant to which f is equal on the ℓ -cell \mathcal{C}_ℓ . The integrability assumption means that $\sum_{\ell \in \mathbb{Z}} |\beta_\ell| < \infty$. Due to [36], since μ is equivalent to the Lebesgue measure on M , it is enough to prove the results with respect to the measure $\mu_{\mathcal{C}_0}$ (the restriction of μ to \mathcal{C}_0). Thus, we consider this reference measure and establish the convergence of every moment with respect to this probability measure. We observe that, with the identification of \bar{M} to \mathcal{C}_0 , $\mu_{\mathcal{C}_0}$ is identified with $\bar{\mu}$ and $f \circ T^k$ is identified with $\beta_{\bar{S}_k}$.

As in [34, 24], we use the two Young towers [35, 5]. We write $(\hat{\Delta}, \hat{F}, \hat{\nu})$ for the hyperbolic tower which is an extension of $(\bar{M}, \bar{T}, \bar{\mu})$, and write (Δ, F, ν) for the expanding tower obtained by quotienting $(\hat{\Delta}, \hat{F}, \hat{\nu})$ along stable curves. We write $\bar{\pi} : \hat{\Delta} \rightarrow \bar{M}$ and $\pi : \hat{\Delta} \rightarrow \Delta$ for the two measurable maps such that $\bar{\pi}_* \hat{\nu} = \bar{\mu}$, $\pi_* \hat{\nu} = \nu$, $\bar{T} \circ \bar{\pi} = \bar{\pi} \circ \hat{F}$ and $F \circ \pi = \pi \circ \hat{F}$. Since $\bar{\Psi}$ is constant on stable curves, there exists a function $\Psi : \Delta \rightarrow \mathbb{Z}^d$ such that $\Psi \circ \pi = \bar{\Psi} \circ \bar{\pi}$. Setting $S_n := \sum_{k=0}^{n-1} \Psi \circ F^k$, it follows that $S_n \circ \pi = \bar{S}_n \circ \bar{\pi}$. For the first part of Theorem 1.1, as noticed in Remark 1.2, it is enough to prove the first convergence result for $\beta_\ell = \mathbf{1}_{\{\ell=0\}}$. We will conclude by Theorem 2.3. To prove the assumptions of Theorem 2.3, we show that the criterion given in Proposition 3.1 is satisfied here with our choice of \mathbf{a}_n , with $\alpha = 2$ and with Φ the characteristic function of the Gaussian limit distribution of $(S_n/\mathbf{a}_n)_n$. The assumptions of Proposition 3.1 have been proved in [34] with the use of the Banach spaces introduced in [35] combined with the use of the Nagaev-Guivarch perturbation method [16, 10, 11] via the Keller and Liverani theorem [13] (see also [21] for a general reference on this method). The fact that $(\lambda_{t/\mathbf{a}_k}^k)_{k \geq 1}$ converges pointwise to the characteristic function of a Gaussian random variable follows from the existence of a positive symmetric matrix A such that $1 - \lambda_t \sim \langle At, t \rangle |\log |t||$ as $t \rightarrow 0$ (this was proved in [34]). For the second part of (39), one can e.g. use the fact that

$|\lambda_{t/\alpha_k}^k| \mathbf{1}_{\{|t| < b\alpha_k\}} \leq e^{-c_0 \min(|t|^{2-\varepsilon}, |t|^{2+\varepsilon})}$. Thus Proposition 3.1 holds true and Theorems 2.3 and 2.6 apply. Finally, we identify the formulas of the asymptotic variances σ_f^2 and σ_β^2 by noticing that

$$\sum_{a,b \in \mathbb{Z}^2} \beta_a \beta_b \nu(S_{|k|} = b - a) = \int_M f \cdot f \circ T^{|k|} d\mu = \int_M f \cdot f \circ T^k d\mu.$$

For Theorem 1.3, we deduce the result for general g using ergodicity of (M, T, μ) together with the Hopf ergodic ratio theorem. \square

APPENDIX A. PROOF OF (17) IN THE GENERAL CASE

We assume here that $\sum_{a \in \mathbb{Z}^d} (1 + |a|^\eta) |\beta_a| < \infty$ with $\eta := \frac{\alpha-d+\varepsilon}{2}$ for some $\varepsilon \in (0; 1/2)$.

In Formula (15), we decompose $Q_{k,a}$ using the operators $Q''_{k,a,b}$ and $Q'_{k,c} := Q'_{k,0,c} = Q_{k,c} - Q_{k,0}$ as follows

$$Q_{k,a} = Q''_{k,a,b} + Q_{k,b} + Q_{k,-a} - Q_{k,0} = Q''_{k,a,b} + Q'_{k,b} + Q'_{k,-a} + Q_{k,0}.$$

In (15), we replace each $Q_{k_j - k_{j-1}, a_j - a_{j-1}}$ by this decomposition, we develop and obtain

$$(43) \quad \mathbb{E}_\nu \left[\left(\sum_{k=0}^{n-1} \beta_{S_k} \right)^N \right] = \sum_{\varepsilon_1, \dots, \varepsilon_N} D_{\varepsilon_1, \dots, \varepsilon_N}^{(n, N)},$$

summing a priori over $(\varepsilon_1, \dots, \varepsilon_N) \in (\{0, 1\}^2)^N$ such that $\varepsilon_1 \in \{(0, 0), (0, 1)\}$ the following quantity

$$D_{\varepsilon_1, \dots, \varepsilon_N}^{(n, N)} = \sum_{0 \leq k_1 \leq \dots \leq k_N \leq n-1} c_{(k_1, \dots, k_N)} \sum_{a_1, \dots, a_N \in \mathbb{Z}^d} \left(\prod_{j=1}^N \beta_{a_j} \right) \mathbb{E}_\nu \left[\tilde{Q}_{k_N - k_{N-1}, a_{N-1}, a_N}^{(\varepsilon_N)} \dots \tilde{Q}_{k_2 - k_1, a_1, a_2}^{(\varepsilon_2)} \tilde{Q}_{k_1, a_0, a_1}^{(\varepsilon_1)}(\mathbf{1}) \right],$$

with $a_0 = 0$ and where $\tilde{Q}_{k,a,b}^{(0,0)} = Q_{k_j - k_{j-1}, 0}$, $\tilde{Q}_{k,a,b}^{(1,0)} = Q'_{k,-a}$, $\tilde{Q}_{k,a,b}^{(0,1)} = Q'_{k,b}$, $\tilde{Q}_{k,a,b}^{(1,1)} = Q''_{k,a,b}$. We first restrict the sum over $\varepsilon_1, \dots, \varepsilon_N$. We observe that, since $\sum_{a_j \in \mathbb{Z}^d} \beta_{a_j} = 0$, $D_{\varepsilon_1, \dots, \varepsilon_N}^{n, N} = 0$ if there exists $j = 1, \dots, N$ such that $\varepsilon_{j,2} + \varepsilon_{j+1,1} = 0$ (with convention $\varepsilon_{N+1,1} = 0$). Thus we restrict the sum in (43) to the sum over the $\varepsilon_1, \dots, \varepsilon_N$ such that for all $j = 1, \dots, N$, such that $\varepsilon_{j,2} + \varepsilon_{j+1,1} \geq 1$. We call **admissible** any such sequence $\varepsilon := (\varepsilon_1, \dots, \varepsilon_N)$. Let $\varepsilon := (\varepsilon_1, \dots, \varepsilon_N)$ be an admissible sequence.

- We observe that $\#\{j : \varepsilon_j = (0, 0)\} \leq N/2$.
- The contribution to (43) of an admissible sequence $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$ is

$$D_{\varepsilon_1, \dots, \varepsilon_N}^{(n, N)} = \mathcal{O} \left(\sum_{a_1, \dots, a_N \in \mathbb{Z}^d} \prod_{j=1}^N \left(|\beta_{a_j}| \sum_{k_j=0}^n \|\tilde{Q}_{k_j, a_{j-1}, a_j}^{(\varepsilon_j)}\|_{\mathcal{B}} \right) \right).$$

- We observe that there exists $u_0 > 0$ such that

$$(44) \quad \sum_{k=0}^n \mathbf{a}_k^{-d} = \mathfrak{A}_n, \quad \sum_{k=0}^n \mathbf{a}_k^{-d-2\eta} = \mathcal{O}(1), \quad \sum_{k=0}^n \mathbf{a}_k^{-d-\eta} = \mathcal{O} \left(\mathfrak{A}_n^{\frac{1-u_0}{2}} \right).$$

Indeed $d+2\eta = \alpha + \varepsilon > \alpha$. For the last estimate, we use the fact that $(\mathbf{a}_k)_k$ is $\frac{1}{\alpha}$ -regularly varying, and infer that $(\sum_{k=0}^n \mathbf{a}_k^{-d-\eta})^2$ is either bounded or $2 - \frac{2d+2\eta}{\alpha}$ -regularly varying

whereas $(\mathfrak{Q}_n)_n$ is $(1 - \frac{d}{\alpha})$ -regularly varying and diverges to infinity (and $2 - \frac{2d+2\eta}{\alpha} = 1 - \frac{d+\varepsilon}{\alpha} < 1 - \frac{d}{\alpha}$).

- If, for all $j = 1, \dots, N$, $\varepsilon_{j,2} + \varepsilon_{j+1,1} = 1$, then, it follows from Hypothesis 2.2 that

$$D_{\varepsilon_1, \dots, \varepsilon_N}^{(n, N)} = \mathcal{O} \left(d_{\varepsilon_1, \dots, \varepsilon_N}^{(n, N)} \right),$$

with

$$(45) \quad d_{\varepsilon_1, \dots, \varepsilon_N}^{(n, N)} := \sum_{a_1, \dots, a_N \in \mathbb{Z}^d} \prod_{j=1}^N \left(|\beta_{a_j}| |a_j|^{\eta(\varepsilon_{j,2} + \varepsilon_{j+1,1})} \sum_{k_j=0}^n \mathfrak{a}_{k_j}^{-d - \eta\varepsilon_{j,1} - \eta\varepsilon_{j,2}} \right),$$

and so, using (44), that

$$D_{\varepsilon_1, \dots, \varepsilon_N}^{(n, N)} = \mathcal{O} \left(d_{\varepsilon_1, \dots, \varepsilon_N}^{(n, N)} \right) = \mathcal{O} \left(\mathfrak{Q}_n^{N_0 + N_1 \frac{1-u_0}{2}} \right),$$

where $N_k := \#\{j : \varepsilon_{j,1} + \varepsilon_{j,2} = k\}$, since $\sum_{a \in \mathbb{Z}^d} (1 + |a|)^\eta |\beta_a| < \infty$. Observe that $N_0 + N_1 + N_2 = N$ and $N = \sum_{k=1}^2 \sum_{j=1}^N \varepsilon_{j,k} = N_1 + 2N_2$ and so $N_2 = N_0$ and $N_0 = \frac{N - N_1}{2}$. Therefore, in this case,

$$D_{\varepsilon_1, \dots, \varepsilon_N}^{(n, N)} = \mathcal{O} \left(d_{\varepsilon_1, \dots, \varepsilon_N}^{(n, N)} \right) = \mathcal{O} \left(\mathfrak{Q}_n^{\frac{N - u_0 N_1}{2}} \right) = o \left(\mathfrak{Q}_n^{N/2} \right)$$

unless if $N_1 = 0$, i.e. unless if N is even and if $\varepsilon_1, \dots, \varepsilon_N$ is the alternate sequence $(0, 0), (1, 1), \dots, (0, 0), (1, 1)$.

- Assume now that there exists some $j_0 \in \{1, \dots, N\}$ such that $\varepsilon_{j_0,2} + \varepsilon_{j_0+1,1} = 2$. Recall that it follows from Hypothesis 2.2 that, for all $\eta'_{j,1}, \eta'_{j,2} \in \{0, \eta\}$,

$$(46) \quad \left\| \tilde{Q}_{k, a_j, a_{j-1}}^{(\varepsilon_j)} \right\|_{\mathcal{B}} = \mathcal{O} \left(|a_{j-1}| \mathfrak{a}_k^{\eta'_{j,1} \varepsilon_{j,1}} |a_j| \mathfrak{a}_k^{\eta'_{j,2} \varepsilon_{j,2}} \mathfrak{a}_k^{-d} \right).$$

Indeed this follows from (6), (7) and (8) combined with the two following facts

$$\forall \eta \in [0; 1], \quad \left\| Q''_{k, a, b} \right\|_{\mathcal{B}} = \left\| Q'_{k, -a, b-a} - Q'_{k, 0, b} \right\|_{\mathcal{B}} = \mathcal{O}(|b|^\eta) \mathfrak{a}_k^{-d-\eta}$$

and

$$\forall \eta \in [0; 1], \quad \left\| Q''_{k, a, b} \right\|_{\mathcal{B}} = \left\| Q'_{k, b, b-a} - Q'_{k, 0, -a} \right\|_{\mathcal{B}} = \mathcal{O}(|a|^\eta) \mathfrak{a}_k^{-d-\eta}.$$

We choose a family $(\eta'_{j,i})_{j=1, \dots, N; i=1, 2}$ of $\{0, 1\}$ such that, for all $j = 1, \dots, N$, $\eta'_{j,2} \varepsilon_{j,2} + \eta'_{j+1,1} \varepsilon_{j+1,1} = \eta = \frac{\alpha + \varepsilon - d}{2}$ and $\eta'_{1,1} = 0$, with convention $\eta'_{N+1,1} = 0$. Therefore

$$D_{\varepsilon_1, \dots, \varepsilon_N}^{(n, N)} = \mathcal{O} \left(\sum_{a_1, \dots, a_N \in \mathbb{Z}^d} \prod_{j=1}^N \left(|\beta_{a_j}| |a_j|^\eta \sum_{k_j=0}^n \mathfrak{a}_{k_j}^{-d - \eta'_{j,1} \varepsilon_{j,1} - \eta'_{j,2} \varepsilon_{j,2}} \right) \right) = \mathcal{O} \left(d_{\varepsilon'_1, \dots, \varepsilon'_N}^{(n, N)} \right),$$

where we set $\varepsilon'_j := (\eta'_{j,1} \varepsilon_{j,1}, \eta'_{j,2} \varepsilon_{j,2}) / \eta$. We also consider the sequence $\varepsilon''_1, \dots, \varepsilon''_N$ obtained from ε' by permuting the values of $\varepsilon'_{j_0,2}$ and $\varepsilon'_{j_0+1,1}$. Both sequences ε' and ε'' are admissible and satisfy $\varepsilon'_{j,2} + \varepsilon'_{j+1,1} = \varepsilon''_{j,2} + \varepsilon''_{j+1,1} = 1$ for all $j = 1, \dots, N$. Thus it follows from the previous item that

$$D_{\varepsilon_1, \dots, \varepsilon_N}^{(n, N)} = \mathcal{O} \left(\min \left(d_{\varepsilon'_1, \dots, \varepsilon'_N}^{(n, N)}, d_{\varepsilon''_1, \dots, \varepsilon''_N}^{(n, N)} \right) \right) = o \left(\mathfrak{Q}_n^{N/2} \right),$$

since ε' and ε'' cannot both coincide with the alternate sequence $(0, 0), (1, 1), \dots, (0, 0), (1, 1)$.

Estimate (17) follows from the two last items.

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