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UNIFORM BOUNDEDNESS FOR THE OPTIMAL CONTROLS OF A DISCONTINUOUS, NON-CONVEX BOLZA PROBLEM*,**

PIERNICOLA BETTIOL[1] AND CARLO MARICONDA[2,***]

Abstract. We consider a Bolza type optimal control problem of the form

$$\min J_t(y,u) := \int_t^T \Lambda(s, y(s), u(s))\, \mathrm{d}s + g(y(T))$$

Subject to:

$$\begin{cases} y \in \mathrm{W}^{1,1}([t,T];\mathbb{R}^n) \\ y' = b(y)u \text{ a.e. } s \in [t,T],\ y(t) = x \\ u(s) \in \mathcal{U} \text{ a.e. } s \in [t,T],\ y(s) \in \mathcal{S}\ \forall s \in [t,T], \end{cases}$$

where $\Lambda(s,y,u)$ is locally Lipschitz in s, just Borel in (y,u), b has at most a linear growth and both the Lagrangian Λ and the end-point cost function g may take the value $+\infty$. If $b \equiv 1$, $g \equiv 0$, $(P_{t,x})$ is the classical problem of the Calculus of Variations. We suppose the validity of a slow growth condition in u, introduced by Clarke in 1993, including Lagrangians of the type $\Lambda(s,y,u) = \sqrt{1+|u|^2}$ and $\Lambda(s,y,u) = |u| - \sqrt{|u|}$ and the superlinear case. We show that, if Λ is real valued, any family of optimal pairs (y_*, u_*) for $(\mathrm{P}_{t,x})$ whose energy $J_t(y_*, u_*)$ is equi-bounded as (t,x) vary in a compact set, has L^∞ – equibounded controls. Moreover, if Λ is extended valued, the same conclusion holds under an additional lower semicontinuity assumption on $(s,u) \mapsto \Lambda(s,y,u)$ and requiring a condition on the structure of the effective domain. No convexity, nor local Lipschitzianity is assumed on the variables (y,u). As an application we obtain the local Lipschitz continuity of the value function under slow growth assumptions.

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1. Introduction

A major issue arising in the basic problem of the Calculus of Variations is the Lipschitz regularity of the minimizers. Providing positive answers on this issue is often a first step towards higher regularity properties, and it allows numerical methods to catch the value of the infimum.

We consider here optimal control problems, such as $(P_{t,x})$ below, imposing very weak assumptions on the Lagrangian $\Lambda(s, y, u)$, where $s \in [t_0, T]$ (the time variable), $y \in \mathbb{R}^n$ (the state variable) and $u \in \mathbb{R}^m$ (the control variable), motivated by the fact that, starting from the Calculus of Variations case (*i.e.* when $b \equiv 1$, $u \in \mathbb{R}^n$) there are discontinuous and non-convex problems that admit existence of minimizers, even if the classical Tonelli's existence conditions are not satisfied.

In the Calculus of Variations setting several results appeared on the subject following Tonelli himself [20]: see, for instance Clarke–Vinter [16], Ambrosio–Ascenzi–Buttazzo [2], Cellina [9]. In the autonomous case, just superlinearity or even slower growths suffice to obtain Lipschitzianity of the minimizers, whether they exist among the absolutely continuous functions (Dal Maso–Frankowska [17], Mariconda–Treu [18]).

In the nonautonomous case growth conditions in general do not guarantee the Lipschitzianity of the minimizers. A celebrated example by Ball–Mizel [3] shows that there are polynomial Lagrangians that satisfy Tonelli's existence assumptions (convexity in the velocity variable and superlinearity) for which even the Lavrentiev phenomenon occurs (*i.e.*, the infimum of the functional among Lipschitz functions is strictly greater than the infimum taken over the absolutely continuous ones). So, extra hypotheses are needed in the nonautonomous setting to make sure that minimizers are Lipschitz continuous.

A well established approach consists in imposing superlinearity together with some regularity conditions on the state or velocity variables in order to ensure the validity of both the Euler condition and Weierstrass inequality, see [14] for a minimal set of assumptions.

Alternatively, one can impose a local Lipschitz condition on the time variable alone of the Lagrangian, that we call here Condition (S) (see Sect. 2.2). Condition (S) was known in the smooth setting for providing the validity of the Du Bois-Reymond equation (see [12]) at any minimizer x_*: namely,

$$p(s) := \Lambda(s, x_*(s), x_*'(s)) - x_*'(s) \cdot \nabla_v \Lambda(s, x_*(s), x_*'(s))$$

is absolutely continuous and

$$p'(s) = (D_s \Lambda)(s, x_*(s), x_*'(s)),$$

where $D_s \Lambda$ denotes the partial derivative of Λ with respect to the first variable. In the nonsmooth setting it became a key assumption for several recent results concerning important aspects such as existence and regularity of minimizers:

– Existence: Clarke introduced in his seminal paper [13] the essential idea of using an indirect method which relies on a weak growth condition of type (H), that we consider here for simplicity just in the autonomous case $\Lambda = \Lambda(y, v)$: it subsumes the convexity of $v \mapsto \Lambda(y, \cdot) = \Lambda(y, v)$ for each y; denoting by $\partial_v \Lambda(y, v)$ the convex subdifferential of $\Lambda(y, \cdot)$ at v and by $J(y) = \int_{t_0}^{T} \Lambda(y(s), y'(s))\, ds$ it requires that there is $c > 0$ such that, for every admissible trajectory on a suitable finite sublevel of J, one has

$$\operatorname{essinf}_{s \in [t_0, T]} |x'(s)| < c,$$

and

$$\lim_{\nu \to +\infty} \sup_{y \in \mathbb{R}^n,\, v \in \mathcal{U},\ |v| > \nu} \{\Lambda(y, v) - v \cdot \partial_v \Lambda(y, v)\} \ < \inf_{y \in \mathbb{R}^n,\, v \in \mathcal{U},\ |v| \le c} \{\Lambda(y, v) - v \cdot \partial_v \Lambda(y, v)\}.$$

We notice that the term $\Lambda(y,v) - v \cdot \partial_v \Lambda(y,v)$ is the level of the intersection of the supporting hyperplane to the graph of $u \mapsto \Lambda(y,u)$ at $u = v$ with the ordinate axis. Condition (H) is fulfilled, for instance, by Lagrangians of the form $\Lambda(s,y,u) = \sqrt{1+|u|^2}$, and superlinear ones. In [13] it is shown that Condition (S) with Condition (H) allow to replace the superlinearity assumption in Tonelli's existence theorem (leaving unchanged lower semicontinuity of the Lagrangian and convexity in the velocity variable), with the advantage that minimizers turn out to be Lipschitz.

– Regularity: Condition (S) alone yields the validity of a Du Bois-Reymond (DBR) type condition expressed in terms of convex subdifferentials, without any convexity assumption (see [4, 6]). The fact that (S) is satisfied whenever the Lagrangian is autonomous implies in particular the validity of the (DBR) condition for any Borel autonomous Lagrangian. Once Condition (S) is fulfilled, the weak growth condition (H) (alone if Λ is real valued) yields the Lipschitzianity of the minimizers, when they exist, see [6].

Conditions such as (H) and (S) can be rephrased in the context of optimal control, providing Lipschitz regularity of minimizers and boundedness of optimal controls (*cf.* [5, 7, 8, 19]).

We study here the problem of finding a *uniform* Lipschitz constant for minimizers of a Bolza type control problem of the form

$$\min J_t(y,u) := \int_t^T \Lambda(s,y(s),u(s))\,\mathrm{d}s + g(y(T)) \tag{P$_{t,x}$}$$

Subject to:

$$\begin{cases} y \in \mathrm{W}^{1,1}([t,T];\mathbb{R}^n) \\ y' = b(y)u \text{ a.e. } s \in [t,T],\ y(t) = x \\ u(s) \in \mathcal{U} \text{ a.e. } s \in [t,T],\ y(s) \in \mathcal{S}\ \forall s \in [t,T], \end{cases} \tag{D}$$

as the initial time t and point x vary on compact sets. A motivation is the study of the regularity of the value function, when one can assume the existence of an optimal pair for any initial data. This existence hypothesis on minimizers is widespread in the literature and becomes a starting point to derive properties on the value function, see for instance Dal Maso–Frankowska [17] in the autonomous and superlinear case of the Calculus of Variations. In the real valued case our main result, Theorem 4.1 below, states that if Λ satisfies Condition (S) and a growth condition of type (H), then the minimizers of $(P_{t,x})$ are locally equi-Lipschitz whenever $t < T, x \in \mathbb{R}^n$. Furthermore, if one knows an a priori upper bound of the integral term $\int_t^T \Lambda(s,y_*(s),u_*(s))\,\mathrm{d}s$ along the minimizers, a common Lipschitz rank may be explicitly written. We shall consider also the case of the extended valued Lagrangians: in this case some further assumptions, namely the lower semicontinuity of $\Lambda(s,y,u)$ with respect to (s,u) and a topological property of the effective domain of Λ, are needed in order to prove our regularity result on minimizers.

The growth condition introduced in Section 3.4 represents a violation of the (DBR) condition for big values of the velocity; it coincides with Clarke's original one [13] when the compact set is reduced to a single initial datum (t_0,x_0) and the Lagrangian is convex in the velocity variable. The Lipschitz regularity of the (local) minimizers for fixed initial time and data under this kind of growth condition was studied in [5]. The novelty here is the fact that we can obtain uniform estimates of the Lipschitz constant when the initial data vary in a neighbourhood of a given (t_0,x_0) with $t_0 < T$.

As a byproduct of our formulation, the growth condition (Condition (G), see Sect. 3.2) introduced by Cellina–Treu–Zagatti in [11], and studied in [9, 10, 18], becomes a particular case of the class of the growth condition of type (H) considered here.

An equi-Lipschitz minimizers regularity was recently established in [19] under the additional assumption that $0 < r \mapsto \Lambda(s,y,ru)$ is convex for all u (called 'radial convexity'); in our paper we consider problems which may not be necessarily radially convex. Moreover, differently from [19], minimizers are allowed to be just local ones, in the sense of the absolutely continuous norm. The fundamental tool in the proof of Theorem 4.1 is the Du Bois-Reymond condition established in Theorem 3.1 of [5].

As an application, we extend the local Lipschitz regularity of the value function formulated in [17] in the framework of autonomous and superlinear Lagrangians to the nonautonomous ones under the slower growth condition of type (H).

2. Preliminaries

2.1. Basic setting and notation

Let $t < T$ and $x \in \mathbb{R}^n$. We consider the Bolza type **optimal control problem** $(\mathrm{P}_{t,x})$-(D) above with the following basic assumptions.

Basic Assumptions and Notation. The following conditions hold $(n, m \geq 1)$.

- $t_0 < T$ are given real numbers, and $t \in [t_0, T]$.
- The **Lagrangian** $\Lambda : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$, $(s, y, u) \mapsto \Lambda(s, y, u)$ is Lebesgue – Borel measurable (*i.e.*, measurable with respect to the $\mathcal{L}([t_0, T]) \times \mathcal{B}_{\mathbb{R}^n \times \mathbb{R}^m}$ measurable sets).
- **(Linear growth from below)** There are $\alpha > 0$ and $d \geq 0$ satisfying, for a.e. $s \in [t_0, T]$ and every $y \in \mathbb{R}^n, u \in \mathcal{U}$,
$$\Lambda(s, y, u) \geq \alpha|u| - d. \tag{L}$$
- $b : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^m)$ (the space of linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m$) is a Borel measurable function such that, for some $\theta \geq 0$,
$$|b(y)| \leq \theta(1 + |y|). \tag{2.1}$$
  We refer to $y' = b(y)u$ as the **controlled differential equation**. It will be useful to recall that, if $y' = b(y)u, y(t) = x$ for some integrable function $u(\cdot)$ on $[t, T]$ then, owing to Gronwall's lemma (see [15], Thm. 6.41), the following estimate holds:
$$\forall s \in [t, T] \quad |y(s) - x| \leq (|x| + 1)e^{\theta \int_t^T |u(\tau)|\, \mathrm{d}\tau}. \tag{2.2}$$
- The **control** $u : [t, T] \mapsto \mathbb{R}^m$ is measurable.
- The **state constraint** set $\mathcal{S}$ is a nonempty subset of $\mathbb{R}^n$;
- The **control set** $\mathcal{U} \subset \mathbb{R}^m$ is a cone, *i.e.* if $u \in \mathcal{U}$ then $\lambda u \in \mathcal{U}$ whenever $\lambda > 0$.
- The **cost function** $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is not identically equal to $+\infty$.
- The **effective domain** of Λ, is given by
$$\mathrm{Dom}(\Lambda) := \{(s, y, u) :\, \Lambda(s, y, u) < +\infty\}.$$

An **admissible pair** for $(\mathrm{P}_{t,x})$ is a pair of functions $(y, u) : [t, T] \to \mathbb{R}^n \times \mathbb{R}^m$ with y absolutely continuous u measurable, (y, u) satisfying (D) and such that $J_t(y, u) < +\infty$. We assume henceforth that, for each $t \in [t_0, T]$ and $x \in \mathcal{S}$, there exists at least an admissible pair for $(\mathrm{P}_{t,x})$.

Notice, that in the particular case where the function $b \equiv 1$ in the controlled differential equation, then $(\mathrm{P}_{t,x})$ becomes a problem of the **Calculus of Variations**.
If $z \in \mathbb{R}^k$ we shall denote by $B_r^k(z)$ (simply B_r^k if $z = 0$) the closed ball of center z and radius r in $\mathbb{R}^k$. The norm in L^1 is denoted by $\|\cdot\|_1$, and the norm in L^∞ by $\|\cdot\|_\infty$.

2.2. Condition (S)

We will consider the following local Lipschitz condition on the Lagrangian Λ with respect to the time variable.

Condition (S). There are $\kappa, A \geq 0, \gamma \in L^1([t_0,T])$, $\varepsilon_* > 0$ satisfying, for a.e. $s \in [t_0,T]$

$$|\Lambda(s_2,y,u) - \Lambda(s_1,y,u)| \leq \big(\kappa\Lambda(s,y,u) + A|u| + \gamma(s)\big)\,|s_2 - s_1| \tag{2.3}$$

whenever $s_1, s_2 \in [s-\varepsilon_*, s+\varepsilon_*] \cap [t_0,T]$, $y \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, are such that $(s_1,y,u),(s_2,y,u) \in \mathrm{Dom}(\Lambda)$.

Remark 2.1. Notice that, when (y_*,u_*) is an admissible pair for $(\mathrm{P}_{t,x})$, then the growth condition (L) implies that $u_* \in L^1([t,T])$ and thus the function

$$\kappa\Lambda(s,y_*(s),u_*(s)) + A|u_*(s)| + \gamma(s) \in L^1([t,T]).$$

Condition (S) is satisfied if $\Lambda(s,y,u) = \Lambda(y,u)$ is *autonomous*. Indeed in that case (2.3) holds with $\kappa = A = 0$,$\gamma \equiv 0$ and $\varepsilon_* = T$.

3. Growth conditions

The definitions and results in this section are similar to those ones which have been introduced in some recent papers (see [5, 6, 19]). There are however some differences: the present definition of Condition $(\mathrm{H}^\delta_B(\chi))$ is more general than the corresponding growth condition used in [5, 6], and we do not require, as in [19], that the Lagrangian is radially convex in the control variable. Therefore, the detailed proofs of the properties displayed in this section are reported below for the convenience of the reader.

3.1. Partial derivatives and subgradients

In what follows we often deal with subdifferentials in the sense of convex analysis.

Notation. If $(s,y,u) \in \mathrm{Dom}(\Lambda)$, we shall denote by

– $\partial_\mu \left(\Lambda\Big(s,y,\dfrac{u}{\mu}\Big)\mu\right)_{\mu=1}$ the **convex subdifferential** of the map

$$0 < \mu \mapsto \Lambda\Big(s,y,\frac{u}{\mu}\Big)\mu$$

at $\mu = 1$;

– $\partial_r\Lambda\big(s,y,ru\big)_{r=1}$ the **convex subdifferential** of the map

$$0 < r \mapsto \Lambda(s,y,ru)$$

at $r = 1$;

– $\nabla_u\Lambda(s,y,u)$ the **gradient** of $\Lambda(s,y,\cdot)$ at u. If $\Lambda(s,y,\cdot)$ is differentiable then the (classical) directional derivative of Λ w.r.t. the vector u is written $D_u\Lambda(s,y,u) = u \cdot \nabla_u\Lambda(s,y,u)$.

Remark 3.1. Let $(s,y,u) \in \mathrm{Dom}(\Lambda)$. A simple change of variable $r = \dfrac{1}{\mu}$ shows that

$$p \in \partial_\mu\Big(\Lambda\Big(s,y,\frac{u}{\mu}\Big)\mu\Big)_{\mu=1} \Leftrightarrow \Lambda(s,y,u) - p \in \partial_r\Lambda\big(s,y,ru\big)_{r=1}.$$

The growth assumptions introduced below involve some uniform limits.

3.2. The growth Condition (G)

The growth Condition (G) was thoroughly studied by Cellina and his school for autonomous Lagrangians of the Calculus of Variations that are smooth or convex in the velocity variable. The extension to the radial convex case, recalled here, was considered in [18] in the autonomous case and was subsequently generalized to the nonautonomous case in [4, 5].

Growth Condition (G). We say that Λ satisfies (G) if, for all $K \geq 0$,

$$\lim_{\substack{|u|\to+\infty \\ (s,y,u)\in \mathrm{Dom}(\Lambda),\, u\in\mathcal{U} \\ P(s,y,u)\in\partial_\mu(\Lambda(s,y,\frac{u}{\mu})\mu)_{\mu=1}\neq\emptyset}} P(s,y,u) = -\infty \text{ unif. } |y| \leq K, \tag{3.1}$$

meaning that for all $M \in \mathbb{R}$ there exists $R > 0$ such that $P(s,y,u) \leq M$ for all $(s,y,u) \in \mathrm{Dom}(\Lambda)$ whenever $\partial_\mu(\Lambda(s,y,\frac{u}{\mu})\mu)_{\mu=1}$ is no empty, $|y| \leq K$, $u \in \mathcal{U}$, $|u| \geq R$.

Remark 3.2. 1. If $u \mapsto \Lambda(s,y,u)$ is differentiable, (3.1) becomes

$$\lim_{\substack{|u|\to+\infty \\ (s,y,u)\in \mathrm{Dom}(\Lambda),\, u\in\mathcal{U} \\ \partial_r\Lambda(s,y,ru)_{r=1}\neq\emptyset}} \Lambda(s,y,u) - u\cdot\nabla_u\Lambda(s,y,u) = -\infty \text{ unif. } |y| \leq K.$$

Superlinearity plays a key role in Tonelli's existence theorem. It has been widely used as a sufficient condition for Lipschitz regularity of minimizers.

Superlinearity. For every $K \geq 0$ there exists $\Theta_K :]-\infty,+\infty[\to\mathbb{R}$ such that, for a.e. $s \in [t_0,T]$ and every $y \in \mathbb{R}^n, |y| \leq K$, $u \in \mathcal{U}$,

$$\Lambda(s,y,u) \geq \Theta_K(|u|), \quad \lim_{r\to+\infty}\frac{\Theta_K(r)}{r} = +\infty. \tag{G_Θ}$$

Superlinearity, together with some local boundedness condition, implies the validity of the growth Condition (G). We refer to Proposition 2 and Remark 11 of [6] for the proof of the following result.

Proposition 3.3 (**Superlinearity** $\Rightarrow$ (G))**.** *Let Λ be superlinear and assume that for every $K \geq 0$ there is $r_K > 0$ such that $\Lambda(s,y,u)$ is bounded when $s \in [t_0,T], |y| \leq K, u \in \mathcal{U}$, $|u| = r_K$. Then Λ satisfies Assumption* (G).

3.3. Assumptions on Dom(Λ) and distance-like functions

We assume that for a.e. $s \in [t_0,T]$ and every $y \in \mathbb{R}^n$ the set

$$\{u \in \mathbb{R}^m : (s,y,u) \in \mathrm{Dom}(\Lambda)\}$$

is **strictly star-shaped in the variable u w.r.t. the origin**, *i.e.*,

$$\Lambda(s,y,u) < +\infty,\ 0 < r \leq 1 \Rightarrow \Lambda(s,y,ru) < +\infty.$$

Definition 3.4 (u-distance, ∞-distance, Euclidean distance)**.**

– We shall denote by dist_e the usual **Euclidean distance** in $[t_0,T]\times\mathbb{R}^n\times\mathbb{R}^m$.

– The **infinity distance-like** dist_∞ is defined for all $\omega_i = (s_i, z_i, v_i) \in [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m$ $(i = 1, 2)$,

$$\text{dist}_\infty(\omega_1, \omega_2) = \begin{cases} +\infty & \text{if } \omega_1 \neq \omega_2 \\ 0 & \text{if } \omega_1 = \omega_2. \end{cases}$$

– The u**-distance-like** is the function defined on the *pairs of points* $\omega_1 = (s_1, z_1, v_1), \omega_2 = (s_2, z_2, v_2) \in [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m$ such that $(s_1, z_1) = (s_2, z_2)$ by

$$\text{dist}_u(\omega_1, \omega_2) = |v_2 - v_1|.$$

If $\chi \in \{e, u, \infty\}$ and $(s, z, v) \in \text{Dom}(\Lambda)$ we set $\text{dist}_\chi((s, z, v), \text{Dom}(\Lambda)^c)$ to be equal to

$$\inf\{\text{dist}_\chi((s, z, v), \omega) : \omega \in \text{Dom}(\Lambda)^c\}.$$

Remark 3.5. Differently from the Euclidean distance, the infinity distance-like and the u-distance-like are not metrics on $[t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m$. Indeed both can take the value $+\infty$ and dist_u is defined just on a strict subset of pairs of $[t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m$. We point out, however, that as well as dist_e and dist_∞, dist_u satisfy the triangular inequality among triples of points that have the same first two coordinates: if $\omega_i := (s, z, v_i) \in [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m$ $(i = 1, 2, 3)$ and $\chi \in \{e, u, \infty\}$ then

$$\text{dist}_\chi(\omega_1, \omega_3) \leq \text{dist}_\chi(\omega_1, \omega_2) + \text{dist}_\chi(\omega_2, \omega_3).$$

Definition 3.6 (Well-inside $\text{Dom}(\Lambda)$ for $\text{dist}_\chi, \chi \in \{e, u, \infty\}$)**.** We say that a subset E of $\text{Dom}(\Lambda)$ is **well-inside** $\text{Dom}(\Lambda)$ w.r.t. $\text{dist}_\chi(\chi \in \{e, u, \infty\})$ if it is contained in $\{(s, y, v) \in \text{Dom}(\Lambda) : \text{dist}_\chi((s, y, v), \text{Dom}(\Lambda)^c) \geq \rho\}$, for a suitable $\rho > 0$.

– If $\chi = e$ the above means that for all $(s, y, v) \in E$, the open ball of radius ρ in $[t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m$ and center in (s, y, v) is contained in $\text{Dom}(\Lambda)$;
– If $\chi = u$ the above means that

$$(s, y, v) \in E,\ 0 < r < \rho \Rightarrow (s, y, v + rv) \in \text{Dom}(\Lambda).$$

– If $\chi = \infty$ the above means simply that $E \subset \text{Dom}(\Lambda)$.

Remark 3.7. Notice that, if $\omega := (s, y, u) \in \text{Dom}(\Lambda)$ and $F := \text{Dom}(\Lambda)^c$, then

$$\text{dist}_e(\omega, F) \leq \text{dist}_u(\omega, F) \leq \text{dist}_\infty(\omega, F).$$

Thus, if $\mathcal{M}_\chi$ is the class of sets that are well inside $\text{Dom}(\Lambda)$ w.r.t. dist_χ we have

$$\mathcal{M}_e \subset \mathcal{M}_u \subset \mathcal{M}_\infty. \tag{3.2}$$

Example 3.8. Let Λ be autonomous and $\text{Dom}(\Lambda) = \{(y, u) \in \mathbb{R}^2 : |y| \leq 1\}$. Then the set $\{(y, u) \in \mathbb{R}^2 : |y| \leq 1, |u| \leq 1\}$ is well-inside $\text{Dom}(\Lambda)$ w.r.t. to d_u but not w.r.t. d_e.

3.4. Growth condition $(\text{H}^{\boldsymbol{\delta}}_{\boldsymbol{B}}(\boldsymbol{\chi}))$

Let $\delta \in [t_0, T[$. The number B represents an upper bound of the integral term in $(\text{P}_{t,x})$ for a prescribed family of admissible pairs, with initial time t varying in $[t_0, \delta]$. The following quantities $c_t(B)$ and $\Phi(B)$ will play a role in the proof of the main results.

Definition 3.9 ($c_t(B)$ **and** $\Phi(B)$)**.** Let $t \in [t_0, T[$, $B \geq 0$ and assume the linear growth from below (L). Set

$$c_t(B) := \frac{B + d(T-t)}{\alpha\,(T-t)}.$$

Moreover, if Condition (S) holds, we define

$$\Phi(B) := \begin{cases} 0 \text{ if } \Lambda \text{ is autonomous,} \\ \kappa B + \dfrac{A}{\alpha}(B + d\,(T - t_0)) + \|\gamma\|_1 \text{ otherwise.} \end{cases}$$

Remark 3.10. Notice that, in Definition 3.9, $c_t(B) \leq c_\delta(B)$ for all $t \in [t_0, \delta]$.

The next result highlights the roles of $\Phi(B)$ and $c_t(\mathrm{B})$, we refer to Proposition 4.10 of [19] for a proof.

Proposition 3.11 (**The role of** $\phi(\mathrm{B})$ **and** $c_t(\mathrm{B})$)**.** *Assume the linear growth from below* (L) *and the validity of Condition* (S)*. Let* $t \in [t_0, T[$*,* $x \in \mathbb{R}^n$*, and take an admissible pair* (y, u) *for* $(\mathrm{P}_{t,x})$ *with* $\displaystyle\int_t^T \Lambda(s, y(s), u(s))\,\mathrm{d}s \leq B$ *for some* $B \geq 0$*. Then*

1.
$$\int_t^T |u(s)|\,\mathrm{d}s \leq \frac{B + d(T-t)}{\alpha} = (T-t)c_t(B).$$

2. *For every* $c > c_t(B)$ *the set* $\{s \in [t, T] : |u(s)| \leq c\}$ *is non negligible.*
3. $\displaystyle\int_t^T \left\{\kappa\Lambda(s, y(s), u(s)) + A|u(s)| + \gamma(s)\right\}\mathrm{d}s \leq \Phi(B).$

Given $B \geq 0$ and $\delta \in [t_0, T[$, the growth Condition ($\mathrm{H}^\delta_B(\chi)$) below requires the validity of Condition (S), unless Λ is autonomous. It will be applied when B is an upper bound for the values of a given set of admissible pairs for problems $(\mathrm{P}_{t,x})$ as $t \in [t_0, \delta]$.
Below, taking the inf/sup where $P(s, y, u) \in \partial_\mu(\Lambda(s, y, \frac{u}{\mu})\mu)_{\mu=1} \neq \emptyset$ means that we consider just those points (s, y, u) such that $\partial_\mu(\Lambda(s, y, \frac{u}{\mu})\mu)_{\mu=1} \neq \emptyset$.

Growth Condition ($\mathbf{H}^\delta_B(\chi)$**).** Assume that Λ satisfies Condition (S) and let $\chi \in \{e, u, \infty\}$. Let $B \geq 0$ and $\delta \in [t_0, T[$. We say that Λ satisfies ($\mathrm{H}^\delta_B(\chi)$) if for all $K \geq 0$, there are $\overline{\nu} > 0$ and $c > c_\delta(B)$ satisfying, for all $\rho > 0$,

$$\sup_{\substack{s\in[t_0,T], |y|\leq K \\ |u|\geq\overline{\nu}, u\in\mathcal{U} \\ \Lambda(s,y,u)<+\infty \\ P(s,y,u)\in\partial_\mu(\Lambda(s,y,\frac{u}{\mu})\mu)_{\mu=1}\neq\emptyset}} \{P(s, y, u)\} + \Phi(B) < \inf_{\substack{s\in[t_0,T], |y|\leq K \\ |u|\leq c, u\in\mathcal{U} \\ \mathrm{dist}_\chi((s,y,u),\mathrm{Dom}(\Lambda)^c)\geq\rho \\ P(s,y,u)\in\partial_\mu(\Lambda(s,y,\frac{u}{\mu})\mu)_{\mu=1}\neq\emptyset}} P(s, y, u). \tag{3.3}$$

Remark 3.12. Condition ($\mathrm{H}^\delta_B(\chi)$) below is a refinement of Condition (H) of [13], introduced by Clarke, who first thoroughly began the investigation on existence and regularity under such a kind of weak growth condition. Condition (H) of [13] corresponds to Condition ($\mathrm{H}^\delta_B(\chi)$) with $\delta = t_0$, $\chi = \infty$. It was subsequently considered in [6] to derive Lipschitz regularity of minimizers for a given initial datum (t, x). Here we are interested in investigating the uniformity of the Lipschitz constant of the minimizers as the initial data (t, x) vary. Moreover, allowing the cases when $\chi = e$ or $\chi = u$, we enlarge the class of Lagrangians that satisfy ($\mathrm{H}^\delta_B(\chi)$). Notice, in view of (3.2), that from (3.3) we have

$$(\mathrm{H}_B^\delta(\infty)) \Rightarrow (\mathrm{H}_B^\delta(u)) \Rightarrow (\mathrm{H}_B^\delta(e)). \tag{3.4}$$

We refer to Example 4.18 of [19] for a Lagrangian that satisfies $(\mathrm{H}_B^\delta(e))$ but not $(\mathrm{H}_B^\delta(\infty))$. Notice that, if Λ is autonomous, Condition (3.3) is much simpler and does not depend anymore on B, since $\Phi(B) = 0$.

Remark 3.13. 1. The validity of Condition $(\mathrm{H}_B^\delta(\chi))$ implies that the right side of inequality (3.3) is not equal to $-\infty$.
2. If $u \mapsto \Lambda(s, y, u)$ is differentiable, (3.3) may be rewritten as

$$\sup_{\substack{s\in[t_0,T],|y|\le K\\ |u|\ge\overline{\nu},u\in\mathcal{U}\\ \Lambda(s,y,u)<+\infty\\ \partial_\mu(\Lambda(s,y,\frac{u}{\mu})\mu)_{\mu=1}\neq\emptyset}} \{\Lambda(s,y,u) - u\cdot\nabla_u\Lambda(s,y,u)\} + \Phi(B) \;<\; \inf_{\substack{s\in[t_0,T],|y|\le K\\ |u|\le c,u\in\mathcal{U}\\ \mathrm{dist}_\chi((s,y,u),\mathrm{Dom}(\Lambda)^c)\ge\rho\\ \partial_\mu(\Lambda(s,y,\frac{u}{\mu})\mu)_{\mu=1}\neq\emptyset}} \{\Lambda(s,y,u) - u\cdot\nabla_u\Lambda(s,y,u)\}.$$

Remark 3.14 (Interpretation of (G) and of $(\mathrm{H}_B^\delta(\chi))$). Consider for simplicity a Lagrangian $\Lambda(u)$ of the variable u. Let $\Lambda(u) < +\infty$ and assume that

$$P(u) \in \partial_\mu\Big(\Lambda\Big(\frac{u}{\mu}\Big)\mu\Big)_{\mu=1} \neq \emptyset.$$

Then $P(u) = \Lambda(u) - Q(u)$ for some $Q(u) \in \partial_r\Lambda(ru)_{r=1}$. Notice that

$$\Lambda(ru) \ge \phi_u(r) := \Lambda(u) + Q(u)(r-1) \quad \forall r > 0.$$

The value $\phi_u(0) = P(u) := \Lambda(u) - Q(u)$ represents the intersection of the "tangent" line $z = \phi_u(r)$ to $0 < r \mapsto \Lambda(ru)$ at $r = 1$ with the z axis.

Condition (G) thus means that the ordinate $P(u)$ of the above intersection point tends to $-\infty$ as $|u|$ goes to ∞.

Condition $(\mathrm{H}_B^\delta(\chi))$ means that there is a gap of at least $\Phi(B)$ between the above points as $|u| \ge \overline{\nu}$ and when evaluated at u such that $|u| \le c$, more precisely that (see Fig. 1)

$$\sup_{|u|\ge\overline{\nu}} P(u) + \Phi(B) < \inf_{|u|\le c} P(u).$$

The validity of Condition $(\mathrm{H}_B^\delta(\chi))$ implies that the infimum (resp. the sup) involved in (3.3) is not equal to $-\infty$ (resp. $+\infty$). These facts, actually, occur quite often, independently of Condition $(\mathrm{H}_B^\delta(\chi))$: their validity is actually a slow growth Condition, it was introduced and named (M_B^δ) in [19]. Claim 2) of Proposition 3.15 improves the sufficient condition formulated in Proposition 4.24 of [19].

Proposition 3.15. *Let $K \ge 0$. The following implications hold:*

1. *Assume that Λ is bounded on the bounded sets that are* ***well-inside*** $\mathrm{Dom}(\Lambda)$ ***w.r.t.*** $\mathrm{dist}_\chi (\chi \in \{e, u\})$. *Then for any $c, \rho > 0$,*

$$-\infty < \inf_{\substack{s\in[t_0,T],|y|\le K\\ |u|\le c,u\in\mathcal{U}\\ \mathrm{dist}_\chi((s,y,u),\mathrm{Dom}(\Lambda)^c)\ge\rho\\ P(s,y,u)\in\partial_\mu(\Lambda(s,y,\frac{u}{\mu})\mu)_{\mu=1}\neq\emptyset}} P(s,y,u). \tag{3.5}$$

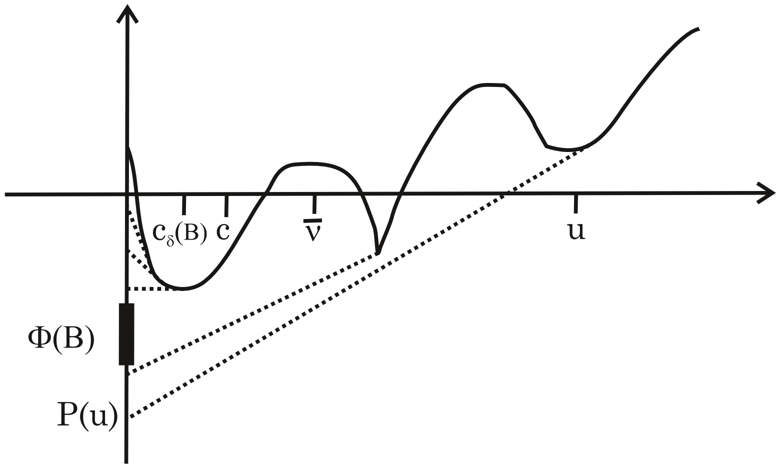

FIGURE 1. Condition ($\mathrm{H}_B^{\delta}(e)$).

2. *Assume that there is $\nu > 0$ such that*

$$\Lambda \textit{ is bounded on } ([t_0, T] \times B_K^n \times B_\nu^m) \cap \mathrm{Dom}(\Lambda). \tag{$\mathcal{B}$}$$

Then

$$\sup_{\substack{s\in[t_0,T],|y|\le K\\ |u|\ge\nu,u\in\mathcal{U}\\ \Lambda(s,y,u)<+\infty\\ P(s,y,u)\in\partial_\mu(\Lambda(s,y,\frac{u}{\mu})\mu)_{\mu=1}\neq\emptyset}} P(s,y,u) < +\infty. \tag{3.6}$$

Proof. 1) Fix $c, \rho > 0$. It is not restrictive to assume that $\partial_\mu(\Lambda(s, y, \frac{u}{\mu})\mu)_{\mu=1} \neq \emptyset$ for some $(s, y, u) \in \mathrm{Dom}(\Lambda)$, $\mathrm{dist}_\chi((s, y, u), \mathrm{Dom}(\Lambda)^c) \ge \rho$. It follows from Remark 3.1 that

$$\inf_{\substack{s\in[t_0,T],|y|\le K\\ |u|\le c,u\in\mathcal{U}\\ \mathrm{dist}_\chi((s,y,u),\mathrm{Dom}(\Lambda)^c)\ge\rho\\ P(s,y,u)\in\partial_\mu(\Lambda(s,y,\frac{u}{\mu})\mu)_{\mu=1}\neq\emptyset}} P(s,y,u) = \inf_{\substack{s\in[t_0,T],|y|\le K\\ |u|\le c,u\in\mathcal{U}\\ \mathrm{dist}_\chi((s,y,u),\mathrm{Dom}(\Lambda)^c)\ge\rho\\ Q(s,y,u)\in\partial_r(\Lambda(s,y,ru)\mu)_{r=1}\neq\emptyset}} \{\Lambda(s, y, u) - Q(s, y, u)\}.$$

The claim follows directly from Lemma 3.17.
2) Let $(s, y, u) \in \mathrm{Dom}(\Lambda)$ with $|y| \le K$ and $|u| \ge \nu, u \in \mathcal{U}$. Assume that $P(s, y, u) \in \partial_\mu(\Lambda(s, y, \frac{u}{\mu})\mu)_{\mu=1} \neq \emptyset$. Then $P(s, y, u) = \Lambda(s, y, u) - Q(s, y, u)$ for some $Q(s, y, u) \in \partial_r(\Lambda(s, y, ru))_{r=1}$ (Rem. 3.1). The assumption that $\mathrm{Dom}(\Lambda)$ is star-shaped in the control variable implies that $\left(s, y, \nu\dfrac{u}{|u|}\right) \in \mathrm{Dom}(\Lambda)$ and thus

$$\Lambda\Big(s, y, \nu\frac{u}{|u|}\Big) - \Lambda(s, y, u) \ge Q(s, y, u)\Big(\frac{\nu}{|u|} - 1\Big),$$

from which we deduce that

$$\Lambda(s, y, u) - Q(s, y, u) \le \Lambda\Big(s, y, \nu\frac{u}{|u|}\Big) - \frac{\nu}{|u|}Q(s, y, u). \tag{3.7}$$

The assumptions imply that

$$\Lambda\Big(s, y, \nu\frac{u}{|u|}\Big) \le C_1(K,\nu) \tag{3.8}$$

for some constant $C_1(K,\nu)$ depending only on K,ν.

We now provide un upper bound for $-Q(s,y,u)$. The assumption that $\mathrm{Dom}(\Lambda)$ is strictly star-shaped in the control variable implies that $\Big(s, y, \frac{\nu}{2}\frac{u}{|u|}\Big) \in \mathrm{Dom}(\Lambda)$ and thus

$$\Lambda\Big(s, y, \frac{\nu}{2}\frac{u}{|u|}\Big) - \Lambda(s,y,u) \ge Q(s,y,u)\left(\frac{\nu}{2|u|} - 1\right),$$

so that the linear growth hypothesis (L) gives

$$\begin{aligned} -Q(s,y,u) &\le \frac{1}{\left(1-\dfrac{\nu}{2|u|}\right)}\left[\Lambda\Big(s, y, \frac{\nu}{2}\frac{u}{|u|}\Big) - \Lambda(s,y,u)\right] \\ &\le 2\left[\Lambda\Big(s, y, \frac{\nu}{2}\frac{u}{|u|}\Big) + d\right] \le C_2(K,\nu) \end{aligned} \tag{3.9}$$

for some constant $C_2(K,\nu)$ depending only on K and ν. It follows from (3.8)–(3.9) that the right side of (3.7) is bounded above by a constant depending only on K and ν. □

Remark 3.16. Assumption ($\mathcal{B}$) in Proposition 3.15 is a known sufficient condition for the nonoccurrence of the Lavrentiev gap for positive autonomous Lagrangians of the Calculus of Variations (see [1], Assump. (B)). Unsurprisingly, the more recent Conditions (3.5)–(3.6) play a role in the avoidance of the Lavrentiev phenomenon (see [19]).

Lemma 3.17 (**Bound of $\partial_r\Lambda(s,y,ru)_{r=1}$ on bounded sets**)**.** *Assume that* $\Lambda(s,y,u)$ *is* ***bounded on the bounded subsets that are well-inside*** $\mathrm{Dom}(\Lambda)$ ***w.r.t.*** $\mathrm{dist}_\chi(\chi\in\{e,u\})$***.*** *Let*

$$\Sigma := \{(s,y,u)\in \mathrm{Dom}(\Lambda) : \partial_r\Lambda(s,y,ru)_{r=1} \ne \emptyset\},$$

and Q be any function satisfying $Q(s,y,u)\in\partial_r\Lambda(s,y,ru)_{r=1}$ for every $(s,y,u)\in\Sigma$. Then Q is bounded on the bounded sets of Σ that are well-inside $\mathrm{Dom}(\Lambda)$ *w.r.t.* dist_χ.

Proof. Let $(s,y,u)\in\mathrm{Dom}(\Lambda)$ and $Q(s,y,u)\in\partial_r\Lambda(s,y,ru)_{r=1}\ne\emptyset$. Suppose that, for some $C>0, \rho>0$, $|y|+|u|\le C$ and

$$\mathrm{dist}_\chi((s,y,u),\mathrm{Dom}(\Lambda)^c)\ge\rho.$$

The triangular inequality (see Rem. 3.5) implies that

$$\mathrm{dist}_\chi\left(\Big(s, y, u+\frac{\rho}{2C}u\Big), \mathrm{Dom}(\Lambda)^c\right) \ge \frac{\rho}{2}.$$

Assuming that

$$\partial_r\Lambda(s,y,ru)_{r=1}\ne\emptyset$$

we obtain

$$\Lambda\left(s,y,u+\frac{\rho}{2C}u\right)-\Lambda(s,y,u)\geq\frac{\rho}{2C}Q(s,y,u).$$

The boundedness assumption of Λ implies that $Q(s,y,u)$ is bounded above by a constant depending only on C and ρ. Similarly, from

$$\Lambda\left(s,y,u-\frac{\rho}{2C}u\right)-\Lambda(s,y,u)\geq-\frac{\rho}{2C}Q(s,y,u),$$

we deduce a lower bound for Q. □

The fact that the validity of Condition (G) implies that of Condition ($\mathrm{H}^{\delta}_{B}(\chi)$) was proved in [6] for real valued Lagrangians and in Proposition 4.21 of [19] under the additional assumption that $0<r\mapsto\Lambda(s,y,ru)$ is convex. Actually, the result holds true in greater generality.

Proposition 3.18 (**(G) implies ($\mathrm{H}^{\delta}_{B}(\chi)$) for all** B,δ)**.** *Assume that Λ satisfies Condition* (S) *and that Λ is bounded on the bounded subsets that are well-inside* $\mathrm{Dom}(\Lambda)$ *w.r.t.* $\mathrm{dist}_{\chi}(\chi\in\{e,u\})$*. If Λ satisfies Condition* (G) *then Λ satisfies Hypothesis* ($\mathrm{H}^{\delta}_{B}(\chi)$)*, whatever are the choices of $\delta\in[t_0,T[$, $c>0$ and $B\geq 0$.*

Proof. Take any $K\geq 0$. Assume that

$$\lim_{\substack{|u|\to+\infty\\(s,y,u)\in\mathrm{Dom}(\Lambda),\,u\in\mathcal{U}\\Q(s,y,u)\in\partial_r(\Lambda(s,y,r\,u))_{r=1}\neq\emptyset}}\{\Lambda(s,y,u)-Q(s,y,u)\}=-\infty\quad\text{unif. }|y|\leq K.$$

Then we obtain

$$\lim_{\nu\to+\infty}\sup_{\substack{s\in[t_0,T]\\|u|\geq\nu,u\in\mathcal{U}\\\Lambda(s,y,u)<+\infty\\Q(s,y,u)\in\partial_r(\Lambda(s,y,r\,u))_{r=1}\neq\emptyset}}\{\Lambda(s,y,u)-Q(s,y,u)\}=-\infty\text{ unif. }|y|\leq K.$$

It follows from 1) of Proposition 3.15 that Condition ($\mathrm{H}^{\delta}_{B}(\chi)$) is valid, for any choice of $B,c>0$, $\delta\in[t_0,T[$. □

Remark 3.19. In Proposition 3.18, the assumption that Λ is bounded on bounded sets that are well-inside $\mathrm{Dom}(\Lambda)$ is not a merely technical hypothesis (see [19], Ex. 4.25).

4. Uniform regularity for optimal pairs

We say that (y_*,u_*) is a $W^{1,1}$-weak local optimal pair for ($\mathrm{P}_{t,x}$) if there is $\varepsilon>0$ such that $J_t(y_*,u_*)\leq J_t(y,u)$ for any admissible pair (y,u) such that $\|y-y_*\|_1+\|y'-y'_*\|_1\leq\varepsilon$. In Theorem 4.2 of [5] it is shown that, if (y_*,u_*) is a $W^{1,1}$-weak local optimal pair for ($\mathrm{P}_{t,x}$) and Condition ($\mathrm{H}^{t_0}_{J_t(y_*,u_*)}(e)$) holds, then u_* is bounded and y_* has a finite Lipschitz rank. We give here a sufficient condition under which the above bounds are uniform as the initial time t varies in an interval $[t_0,\delta]$ ($\delta\in[t_0,T[$) and the initial point x varies in a compact set.

Theorem 4.1 ($L^{\infty}-$ **uniform boundedness for optimal controls and equi-Lipschitz rank of minimizers).** *Assume that Λ takes values in $\mathbb{R}$ and satisfies Assumption* (S)*. Fix $\delta\in[t_0,T[$, $\delta_*\geq 0$ and $x_*\in\mathbb{R}^n$. Let (y_*,u_*) be a $W^{1,1}$-weak local optimal pair for* ($\mathrm{P}_{t,x}$) *where $t\in[t_0,\delta]$, $x\in B^n_{\delta_*}(x_*)$, and let $B\geq 0$ be such that*

$$\int_t^T\Lambda(s,y_*(s),u_*(s))\,\mathrm{d}s\leq B.$$

Assume that Λ *satisfies the growth condition* $(\mathrm{H}^{\delta}_{B}(e))$. *Then* u_* *is bounded and* y_* *is Lipschitz with bounds and ranks depending only on* δ, B, δ_*, x_*.

The same conclusion is valid when Λ *takes values in* $\mathbb{R}\cup\{+\infty\}$, *provided that we impose also the following assumptions:*

a) $(s,u)\mapsto\Lambda(s,y,u)$ *is lower semicontinuous for every* y;
b) For every $(s,y,u)\in\mathrm{Dom}(\Lambda)$, *the set* $\{\lambda>0:\ \Lambda(s,y,\lambda u)<+\infty\}$ *is open;*
c) For some $\chi\in\{e,u,\infty\}$, Λ *satisfies the growth condition* $(\mathrm{H}^{\delta}_{B}(\chi))$ *and for a.e.* $s\in[t,T]$, $\{(s,y_*(s),u_*(s))\}$ *is well-inside* $\mathrm{Dom}(\Lambda)$ *w.r.t.* dist_χ, *i.e.,*

$$\exists\rho_s>0\ \mathrm{dist}_\chi((s,y_*(s),u_*(s)),\mathrm{Dom}(\Lambda)^c)\geq\rho_s\qquad a.e.\ s\in[t_0,T].\tag{W_χ}$$

Remark 4.2. When Λ is an extended valued function, in Theorem 4.1 we impose the additional assumptions a), b) and c). Condition c) is employed in the proof of Theorem 4.1 (for the extended valued case) to take advantage of the information provided by 'inf'-term in (3.3) of the growth Condition $(\mathrm{H}^{\delta}_{B}(\chi))$, while assumptions a) and b) are used just to ensure the validity of the Du Bois-Reymond condition ([6], Thm. 2). Therefore, a) and b) can be removed and the regularity properties of Theorem 4.1 remain valid provided that the Du Bois-Reymond condition ([6], Thm. 2) is in force. This is the case, for instance, when Λ is the indicator function of a (bounded) control set U (*cf.* [6], Rem. 4).

Remark 4.3. In the case of an extended valued Lagrangian, the choice of $\chi\in\{e,u,\infty\}$ in Theorem 4.1 depends on the validity of both conditions $(\mathrm{H}^{\delta}_{B}(\chi))$ and (W_χ) of Condition c) of Theorem 4.1. Now, it appears that the best choice of χ for the first condition may be the worse for the second one, and vice versa. Indeed, from (3.4),

$$(\mathrm{H}^{\delta}_{B}(\infty))\Rightarrow(\mathrm{H}^{\delta}_{B}(u))\Rightarrow(\mathrm{H}^{\delta}_{B}(e)),$$

whereas

$$(\mathrm{W}_e)\Rightarrow(\mathrm{W}_u)\Rightarrow(\mathrm{W}_\infty).$$

Thus, if $(\mathrm{H}^{\delta}_{B}(e))$ or (W_∞) are not fulfilled, Theorem 4.1 is not applicable. Otherwise, one has to find a trade-off for a common value of $\chi\in\{e,u,\infty\}$ in order to satisfy both the conditions.

Remark 4.4. Let χ be as in Hypothesis $(\mathrm{H}^{\delta}_{B}(\chi))$. Then, in Theorem 4.1:

– If $\chi=u$ then (W_χ) of Assumption c) follows from Assumption b).
– If $\chi=\infty$ then (W_χ) of Assumption c) is always satisfied.
– If $\chi=e$ Assumptions b) and (W_χ) of c) are fulfilled if $\mathrm{Dom}(\Lambda)$ is open in $[t_0,T]\times\mathbb{R}^n\times\mathbb{R}^m$.
– The validity of (W_χ) of Assumption c) is ensured once, for all $s\in[t_0,T]$,

$$\lim_{\mathrm{dist}_\chi((s,z,v),\mathrm{Dom}(\Lambda)^c)\to0}\Lambda(s,z,v)=+\infty,\tag{4.1}$$

uniformly w.r.t. z in compact sets, *i.e.*, if for all compact $K\subset\mathbb{R}^n$ and $M\geq0$ there is $\rho>0$ such that, for all $(s,z,v)\in\mathrm{Dom}(\Lambda)$ with $z\in K$,

$$\mathrm{dist}_\chi\left((s,z,v),\mathrm{Dom}(\Lambda)^c\right)\leq\rho\Rightarrow\Lambda(s,z,v)\geq M.$$

Condition (4.1) has been used in the assumptions ([19], Thm. 5.1) among the sufficient conditions for the existence of minimizing sequences with equi-bounded controls.

Proof of Theorem 4.1. Let α, d be as in (L) and (y_*, u_*) be a $W^{1,1}$-weak local optimal pair for $(\mathrm{P}_{t,x})$. From Point 1 of Proposition 3.11 we have

$$\int_t^T |u_*|\,\mathrm{d}s \le \frac{B+d(T-t)}{\alpha} \le R = R(B) := \frac{B+d(T-t_0)}{\alpha}. \tag{4.2}$$

Claim: There is $K := K(\delta, B, \delta_*, x_*)$ such that $|y_*(s)| \le K$ for every $s \in [t,T]$. Indeed, for a.e. $s \in [t,T]$,

$$|y_*'(s)| \le \theta(1+|y_*(s)|)|u_*(s)|.$$

Since $y_*(t) = x$, (2.2) and (4.2) imply that, for all $s \in [t,T]$,

$$\begin{aligned}|y_*(s) - x| &\le \int_t^s \exp\left(\theta \int_\tau^s |u_*(r)|\,dr\right)\theta|u_*(\tau)|(|x|+1)\,\mathrm{d}\tau \\ &\le (|x|+1)(e^{R\theta}-1),\end{aligned}$$

so that

$$|y_*(s)| \le |x| + (|x|+1)(e^{R\theta}-1) \le e^{R\theta}(|x_*|+\delta_*+1),$$

where in the latter we used the fact that $x \in B_{\delta_*}(x)$. The claim follows from the fact that R depends on B, with

$$K = e^{R\theta}(|x_*|+\delta_*+1).$$

We prove the result in the extended valued case since when Λ is real valued the analysis is simpler: we just take $\chi = e$ and hypotheses a), b), c) are not necessary anymore. Assumptions a), b) imply that the Lagrangian Λ satisfies ([5], Hypothesis $(\mathrm{S}^\infty_{(y_*,u_*)})$). The optimal pair (y_*, u_*) satisfies the Du Bois-Reymond – Erdmann condition formulated in Theorem 3.1 of [5]. In particular

$$\partial_\mu\left(\Lambda\Big(s, y_*(s), \frac{u_*(s)}{\mu}\Big)\mu\right)_{\mu=1} \ne \emptyset \qquad \text{a.e. } s \in [t,T]$$

and there is an absolutely continuous function $p \in W^{1,1}([t,T])$ such that

$$p(s) \in \partial_\mu\left(\Lambda\Big(s, y_*(s), \frac{u_*(s)}{\mu}\Big)\mu\right)_{\mu=1} \qquad \text{a.e. } s \in [t,T],$$

$$|p'(s)| \le \kappa\Lambda(s, y_*(s), u_*(s)) + A|u_*(s)| + \gamma(s) \quad \text{a.e. } s \in [t,T]. \tag{4.3}$$

We consider $P(s,z,v) \in \partial_\mu\Big(\Lambda\Big(s,z,\dfrac{v}{\mu}\Big)\mu\Big)_{\mu=1}$ such that

$$p(s) = P(s, y_*(s), u_*(s)) \quad \text{a.e. } s \in [t,T].$$

Remark 3.10 tells us that the parameter c in Condition $(\mathrm{H}^\delta_B(\chi))$ satisfies $c > c_t(B)$. It follows from Claim 2 of Proposition 3.11 that there is a non negligible set of $\tau \in [t,T]$ satisfying $|u_*(\tau)| < c$ and $p(\tau) = P(\tau, y_*(\tau), u_*(\tau))$.

We fix such a τ and set $\rho := \mathrm{dist}_\chi((\tau, y_*(\tau), u_*(\tau)), \mathrm{Dom}(\Lambda)^c)$; notice that Assumption c) implies that $\rho > 0$. Let $\overline{\nu}$ be such that (3.3) holds. We have

$$P(s, y_*(s), u_*(s)) = p(\tau) + \int_\tau^s p'(\sigma)\,\mathrm{d}\sigma \qquad \text{a.e. } s \in [t, T]. \tag{4.4}$$

It follows from (4.3) and (4.4) that for a.e. $s \in [t, T]$ we have

$$\begin{aligned} p(\tau) &= P(s, y_*(s), u_*(s)) - \int_\tau^s p'(\sigma)\,\mathrm{d}\sigma \\ &\le P(\sigma, y_*(s), u_*(s)) + \int_\tau^s [\kappa\Lambda(\sigma, y_*, u_*) + A|u_*| + \gamma]\,\mathrm{d}\sigma. \end{aligned}$$

Assume that there is a non negligible subset F of $[t, T]$ such that $|u_*| > \overline{\nu}$ on F. By taking $s \in F$ we deduce that

$$\begin{aligned} p(\tau) &\le \sup_{\substack{s\in[t_0,T],|z|\le K \\ |v|\ge\overline{\nu}, v\in\mathcal{U} \\ \Lambda(s,z,v)<+\infty \\ \partial_\mu(\Lambda(s,z,\frac{v}{\mu})\mu)_{\mu=1}\neq\emptyset}} \{P(s, z, v)\} + \left|\int_\tau^s \kappa\Lambda(\sigma, y_*(\sigma), u_*) + A|u_*| + \gamma\,\mathrm{d}\sigma\right| \\ &\le \sup_{\substack{s\in[t_0,T],|z|\le K \\ |v|\ge\overline{\nu}, v\in\mathcal{U} \\ \Lambda(s,z,v)<+\infty \\ \partial_\mu(\Lambda(s,z,\frac{v}{\mu})\mu)_{\mu=1}\neq\emptyset}} \{P(s, z, v)\} + \Phi(B), \end{aligned} \tag{4.5}$$

where the last inequality is justified by Claim 3 of Proposition 3.11. Now,

$$p(\tau) = P(\tau, y_*(\tau), u_*(\tau)) \ge \inf_{\substack{s\in[t_0,T],|z|\le K \\ |v|\le c, v\in\mathcal{U}, \\ \mathrm{dist}_\chi((s,z,v),\mathrm{Dom}(\Lambda)^c)\ge\rho \\ \partial_\mu(\Lambda(s,z,\frac{v}{\mu})\mu)_{\mu=1}\neq\emptyset}} P(s, z, v). \tag{4.6}$$

Therefore (4.5) and (4.6) imply that

$$\sup_{\substack{s\in[t_0,T],|z|\le K \\ |v|\ge\overline{\nu}, v\in\mathcal{U},\, \Lambda(s,z,v)<+\infty \\ \partial_\mu(\Lambda(s,z,\frac{v}{\mu})\mu)_{\mu=1}\neq\emptyset}} \{P(s, z, v)\} + \Phi(B) \ge \inf_{\substack{s\in[t_0,T],|z|\le K \\ |v|\le c, v\in\mathcal{U}, \\ \mathrm{dist}_\chi((s,z,v),\mathrm{Dom}(\Lambda)^c)\ge\rho \\ \partial_\mu(\Lambda(s,z,\frac{v}{\mu})\mu)_{\mu=1}\neq\emptyset}} P(s, z, v),$$

contradicting (3.3). It follows that $|u_*| \le \overline{\nu}$ a.e. on $[t, T]$. The Lipschitzianity of y_* and the uniformity of its rank follows from (2.1). □

Remark 4.5. The proof of Theorem 4.1 shows that if Λ is real valued then a uniform bound for the optimal control u_* satisfying the conditions of the claim is given by any $\overline{\nu} > 0$ satisfying one of the assumptions of Condition $(\mathrm{H}_B^\delta(\chi))$, with $K = e^{R\theta}(|x_*| + \delta_* + 1)$ and $R = \dfrac{B + d(T - t_0)}{\alpha}$.

One of the assumptions of Theorem 4.1 is the existence of an upper bound B for the cost of the optimal pairs. Such a bound exists and can be explicitly computed for some classes of problems, *e.g.*, for finite valued Lagrangians of the Calculus of Variations, or if the cost function g is real valued. Corollary 4.6 below extends

([17], Prop. 3.3) in various directions: Nonautonomous Lagrangians, weaker growths than superlinearity, optimal control problems more general than problems of the Calculus of Variations, no convexity in the velocity variable.

Corollary 4.6 (**The Calculus of Variations or real valued final cost** g)**.** *Assume that* Λ *is* ***real valued****, satisfies Assumption* (S) *and is bounded on bounded sets. Suppose, moreover, that* g *is bounded from below and that one of the following assumptions holds:*

i) $b = 1$ *in the controlled differential equation,* $\mathcal{S}$ *is convex and* $\mathcal{U} = \mathbb{R}^n$*;*
ii) The cost function g *is real valued, locally bounded and* $0 \in \mathcal{U}$*;*
iii) b *is Lipschitz continuous, the cost function* g *is real valued, locally bounded and* $\mathcal{S} = \mathbb{R}^n$*.*

Let $\delta \in [t_0, T[$*,* $\delta_* \geq 0$*,* $x_* \in \mathbb{R}^n$*. Let* $t \in [t_0, \delta]$*,* $x \in B_{\delta_*}(x_*)$ *and* (y_*, u_*) *be optimal for* $(P_{t,x})$*.*

1. *There is* $B \in \mathbb{R}$ *such that*

$$B \geq \int_t^T \Lambda(s, y_*(s), u_*(s))\,\mathrm{d}s.$$

2. *Assume that* Λ *satisfies* $(\mathrm{H}_B^\delta(e))$*. Then,* u_* *is uniformly bounded and* y_* *is uniformly Lipschitz as* $t \in [t_0, \delta], x \in B_{\delta_*}(x_*)$*.*

Proof. In view of Theorem 4.1, Claim 2 is an immediate consequence of Claim 1.
If *i*) or *ii*) hold, Claim 1 follows from Lemma 5.3 of [19]. Assume that *iii*) holds. Let Υ be a bound from below for g. Consider $\overline{u} \in \mathcal{U}$ and let $\overline{y}$ be the solution to

$$y' = b(y)\overline{u}, \quad \overline{y}(t) = x.$$

Then

$$J_t(y_*, u_*) \leq J_t(\overline{y}, \overline{u}) = \int_t^T \Lambda(s, \overline{y}(s), \overline{u})\,\mathrm{d}s + g(\overline{y}(T))$$

is bounded above by a constant $\overline{B}$ that does not depend on $x \in B^n_{\delta_*}(x_*)$ and on $t \in [t_0, \delta]$, owing to standard a priori boundedness properties of trajectories. Therefore

$$\int_t^T \Lambda(s, y_*(s), y_*'(s))\,\mathrm{d}s \leq \overline{B} - g(y_*(T)) \leq \overline{B} - \Upsilon =: B.$$

□

5. Lipschitz continuity of the value function

We consider here problem $(\mathrm{P}_{t,x})$ in the framework of the **Calculus of Variations**, *i.e.*, with $b \equiv 1$ in (D). The **value function** $V(t, x)$ associated with problem $(\mathrm{P}_{t,x})$ is the function defined by

$$\forall t \in [t_0, T], \forall x \in \mathbb{R}^n \qquad V(t, x) = \inf\,(\mathrm{P}_{t,x}).$$

In this section we shall assume that Λ is **real valued** and **bounded on bounded sets**: since g is not identically $+\infty$ it follows that $V(t, x) < +\infty$ for every (t, x). The next result extends to the nonautonomous case ([17], Cor. 3.4), formulated there for autonomous and superlinear Lagrangians.

Corollary 5.1 (**Local Lipschitz continuity of the value function in the Calculus of Variations setting**)**.** *Suppose that* Λ *is* ***real valued, bounded on bounded sets****, satisfies Assumption* (S) *and, moreover,*

$b=1$, $\mathcal{S}=\mathcal{U}=\mathbb{R}^n$. *Let* $\delta\in]t_0,T[$, $x_*\in\mathbb{R}^n,\delta_*>0$. *We suppose that* $(\mathrm{P}_{t,x})$ *admits at least an optimal pair for each* $t\in[t_0,\delta],x\in B_{\delta_*}(x_*)$. *Assume, moreover, the validity of at least one of the following conditions:*

1. Λ *satisfies* $(\mathrm{H}^{\delta}_B(e))$, *where* $B\geq 0$ *is such that, for any* $t\in[t_0,\delta]$, $x\in B_{\delta_*}(x_*)$ *and optimal pair* (y_*,u_*) *for* $(\mathrm{P}_{t,x})$,
$$B\geq\int_t^T\Lambda(s,y_*(s),u_*(s))\,\mathrm{d}s; \tag{5.1}$$
2. Λ *satisfies* (G).

Then the value function $V(t,x)$ *is locally Lipschitz on* $[t_0,\delta[\times B_{\delta_*}(x_*)$.

Remark 5.2. If the cost function g is bounded from below, then the existence of B in (5.1) is ensured by Corollary 4.6 (Case i)).

Remark 5.3. Sufficient conditions for the existence of a minimizer under the slow growth condition of type (H), required in Corollary 5.1, are provided in [13, 19].

Example 5.4. The result covers Lagrangians of slow growth not considered in previous literature concerning the regularity of the value function. Consider, for instance, $\Lambda(s,y,u):=L(y,u)=a(y)\sqrt{1+|u|^2}\,(y,u\in\mathbb{R})$ with a lower semicontinuous and $1\leq a\leq 2$, $\mathcal{S}=\mathcal{U}=\mathbb{R}$; let g be continuous on $\mathbb{R}$ (this example is inspired by Example 2.4.2 of [13]). Clearly L is not superlinear and, since $L(y,u)\geq|u|$, it satisfies (L). Fix $t_0=0,T=1$, $\delta\in[0,1[$, $x_*\in\mathbb{R},\delta_*>0$, $t\in[0,\delta]$ and $x\in[x_*-\delta_*,x_*+\delta_*]$. Let y_* be a minimizer for $(\mathrm{P}_{t,x})$; its existence follows from Theorem 2 of [13]. Claim 1 of Corollary 4.6 says that there is $B\geq 0$ depending only on δ and x_*,δ_* such that

$$B\geq\int_t^1 L(y_*(s),y_*'(s))\,\mathrm{d}s.$$

Now, denoting by L_u the partial derivative of L with respect to u, we have

$$L(y,u)-uL_u(y,u)=\frac{a(y)}{\sqrt{1+u^2}},$$

so that, for any $\nu,c>0$,

$$\sup_{|u|\geq\nu}\{L(y,u)-uL_u(y,u)\}=\frac{a(y)}{\sqrt{1+\nu^2}}\leq\frac{2}{\sqrt{1+\nu^2}},$$

$$\inf_{|u|\leq c}\{L(y,u)-uL_u(y,u)\}=\frac{a(y)}{\sqrt{1+c^2}}\geq\frac{1}{\sqrt{1+c^2}}.$$

Since $\displaystyle\lim_{\nu\to+\infty}\frac{1}{\sqrt{1+\nu^2}}=0$, then L satisfies $(\mathrm{H}^{\delta}_B(e))$. Corollary 5.1 shows that the value function $V(t,x)$ is locally Lipschitz in $[0,\delta]\times[x_*-\delta_*,x_*+\delta_*]$.

Proof of Corollary 5.1. Assume the validity of Condition 1. Let $t_*\in[t_0,\delta[,x\in B_{\delta_*}(x_*)$ and fix $\varepsilon>0$ such that

$$0<\varepsilon<\min\{\delta-t_*,\delta_*\}.$$

Take $t_0 \le t_1, t_2 \in [t_* - \varepsilon/5, t_* + \varepsilon/5]$ and $x_1, x_2 \in B^n_{\varepsilon/5}(x_*)$ with either $t_2 \neq t_1$ or $x_2 \neq x_1$. Set $\Delta := |t_2 - t_1| + |x_2 - x_1|$. Notice that

$$t_0 \le t_1 < t_1 + \Delta \le t_* + \varepsilon \le \delta, \quad t_0 \le t_2 \le t_1 + \Delta \le \delta.$$

Since $\inf(\mathrm{P}_{t_2,x_2})$ is attained, let $y_2 \in W^{1,1}([t_2, T]; \mathbb{R}^n)$ be such that

$$y_2(t_2) = x_2, \quad J_{t_2}(y_2, y_2') = V(t_2, x_2).$$

Theorem 4.1 shows that every minimizer y for $(\mathrm{P}_{t,x})$ is such that $||y||_\infty,\ ||y'||_\infty \le K$, where the constant K depends only on δ, δ_* and x_*.

Let

$$u := \frac{y_2(t_1 + \Delta) - x_1}{\Delta}.$$

The choice of ε yields

$$\begin{aligned}
|u| &\le \frac{|y_2(t_1+\Delta) - y_2(t_2)|}{\Delta} + \frac{|y_2(t_2) - x_1|}{\Delta} \\
&\le \frac{|y_2(t_1+\Delta) - y_2(t_2)|}{\Delta} + \frac{|x_2 - x_1|}{\Delta} \\
&\le K\frac{|t_1 + \Delta - t_2|}{\Delta} + \frac{|x_2 - x_1|}{\Delta} \le K\frac{|\Delta| + |t_2 - t_1|}{\Delta} + 1 \le 2K + 1.
\end{aligned}$$

We consider now the competitor z for (P_{t_1,x_1}) given by

$$z(s) := \begin{cases} x_1 + (s - t_1)u & t_1 \le s \le t_1 + \Delta, \\ y_2(s) & t_1 + \Delta \le s \le T. \end{cases}$$

Since $z(T) = y_2(T)$ we get

$$\begin{aligned}
V(t_1, x_1) &\le \int_{t_1}^{t_1+\Delta} \Lambda(s, z, z')\,\mathrm{d}s + \int_{t_1+\Delta}^{T} \Lambda(s, y_2, y_2')\,\mathrm{d}s + g(y_2(T)) \\
&\le \int_{t_1}^{t_1+\Delta} \Lambda(s, z, z')\,\mathrm{d}s + V(t_2, x_2) - \int_{t_2}^{t_1+\Delta} \Lambda(s, y_2, y_2')\,\mathrm{d}s.
\end{aligned} \tag{5.2}$$

Since $0 \le \Delta \le 4\varepsilon/5$ for all $s \in [t_1, t_1 + \Delta]$ we obtain

$$|z(s)| \le |x_1| + \Delta|u| \le |x_*| + \varepsilon/5 + 4(2K+1)\varepsilon/5, \quad |z'(s)| \le |u| \le 2K + 1,$$

so that, given that Λ is bounded on bounded sets,

$$\left| \int_{t_1}^{t_1+\Delta} \Lambda(s, z, z')\,\mathrm{d}s \right| \le C\Delta = 2C(|t_2 - t_1| + |x_2 - x_1|),$$

for some positive constant C which depends only on δ, δ_* and x_*. Moreover, as observed above, from Theorem 4.1 we obtain that $||y_2||_\infty,\ ||y_2'||_\infty \le K$, and thus, using the fact that $|t_2 - t_1| + \Delta \le 2\Delta$ (we can take the same constant C previously employed)

$$\left|\int_{t_2}^{t_1+\Delta} \Lambda(s, y_2, y_2')\,\mathrm{d}s\right| \le C|t_1+\Delta-t_2| \le 2C(|t_2-t_1|+|x_2-x_1|).$$

It follows from (5.2) that

$$V(t_1,x_1) - V(t_2,x_2) \le 4C(|t_2-t_1|+|x_2-x_1|).$$

Exchanging the roles of (t_1,x_1) and (t_2,x_2) we arrive at

$$|V(t_1,x_1) - V(t_2,x_2)| \le 4C(|t_2-t_1|+|x_2-x_1|),$$

which proves the claim. The result under Condition 2 follows immediately as a consequence of Proposition 3.18. □

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