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Lyapunov-based stability of delayed linear differential algebraic systems

Ann Al Sawoor , Miloud Sadkane *

Univ Brest, CNRS - UMR 6205, Laboratoire de Mathématiques de Bretagne Atlantique. 6, Av. Le Gorgeu. 29238 Brest Cedex 3, France

Abstract

The Lyapunov stability theorem for linear systems is extended to linear delay-differential algebraic systems of index one. In particular, bounds on the decay of the solution are established in terms of solvability of a certain Lyapunov-type matrix equation.

Key words: Linear differential algebraic systems, time delay, Lyapunov stability, decay rate

1. Introduction

It is known that if A is a stable matrix (i.e., all of its eigenvalues have negative real parts), then the linear ordinary differential system

$$\dot{x}(t) = Ax(t), \ t \geq 0 \tag{1}$$

is asymptotically stable, that is, $x(t) \to 0$ as $t \to +\infty$.
The Lyapunov theory can be used to inform about the quality of stability and to measure the decay rate of the solution (see, e.g., [8]). Indeed, under the assumption that A is stable, the Lyapunov matrix equation

$$C = -(A^T H + HA) \tag{2}$$

has the unique solution $H = \int_0^\infty e^{tA^T} C e^{tA} dt = H^T > 0$ (i.e., symmetric positive definite) for all matrices $C = C^T > 0$. Conversely, if the equation (2) is satisfied with some matrices $C = C^T > 0$ and $H = H^T > 0$, then A is stable and, using the quadratic function

$$v(x(t)) = x(t)^T H x(t), \tag{3}$$

* Corresponding author.
Email addresses: `Ann.AlSawoorMaskoni@univ-brest.fr` (Ann Al Sawoor), `miloud.sadkane@univ-brest.fr` (Miloud Sadkane).

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we obtain the bound (see, e.g., [9])

$$\|x(t)\| \leq \sqrt{\|H\|\,\|H^{-1}\|}\; e^{-\frac{\lambda_{\min}(C)}{2\|H\|}t}\; \|x(0)\|, \quad t \geq 0, \tag{4}$$

where $\|\ \|$ denotes the 2-norm for vectors and matrices and $\lambda_{\min}(C)$ denotes the smallest eigenvalue of C. See also [25] for a comparison with other bounds. Although this bound is not sharp, it measures the decay rate of the solution (because $\lambda_{\min}(C) > 0$) by involving quantities based on the computation of eigenvalues of symmetric (positive definite) matrices for which reliable algorithms are available [10]. Moreover, the norm of H can be used in preference to the spectral criterion "Re$(\lambda) < 0$ for all eigenvalues of A" for assessing the stability of A : the larger the norm of H, the less stable the matrix A. For related issues, see, e.g., [18,20,24].

Our goal in this note is to study to what extent a bound similar to (4) holds for linear differential algebraic systems with time delay, of the form

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bx(t-\tau), \quad t > 0, \\ x(t) = \psi(t), \quad -\tau \leq t \leq 0, \end{cases} \tag{5}$$

where E, A, B are real $n \times n$ matrices with E singular and τ is a fixed positive delay. Time delay systems arise generally in applications where a transport phenomenon occurs. A wide range of examples can be found in [7,15]. Solution theory for (5) is established for example in [2,11,22,21,23] and the references therein.

Stability analysis of systems of type (5) has been investigated by many authors; for example, in [5,6] the analysis is based on a careful choice of a Lyapunov-Krasovskii function (a generalization of (3)), and sufficient conditions for stability are given in terms of linear matrix inequalities. In [17], the stability is based on the computation of eigenvalues of certain matrix pencils. In [3], sufficient conditions for asymptotic stability are formulated with the help of the characteristic equation of the system. In [16], a spectrum-based approach is developed for the stability analysis and stabilization of systems described by delay differential algebraic equations. In [4], necessary and sufficient conditions for exponential stability of classes of systems of the form (5) are derived using the roots of the associated characteristic equation. In the present note, using a Lyapunov-Krasovskii type function, bounds on the decay of the solution, analogous to (4), are established in terms of solvability of a certain Lyapunov-type matrix equation.

2. Lyapunov-based stability analysis

We assume throughout this note that the pencil $\lambda E - A$ is regular (i.e., there exists λ such that $\det(\lambda E - A) \neq 0$) and of index 1 (this is the index of the nilpotent matrix in the Weierstraß canonical form of $\lambda E - A$, see, e.g., [19]). Since the matrices E and A are real, the real Weierstraß canonical form can be used to decompose E and A as

$$E = W \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} T, \quad A = W \begin{pmatrix} J & 0 \\ 0 & I_{n-r} \end{pmatrix} T, \tag{6}$$

where W and T are real nonsingular matrices, the symbol I_k denotes the identity matrix of order k, the matrix J is in real Jordan form (see, e.g., [12, Section 3.4]) and corresponds to the finite eigenvalues of the pencil $\lambda E - A$ and $r = \mathrm{rank}(E)$. Alternatively, the quasi-Weierstraß form or a block-diagonalization via the real generalized Schur form may also be used (see [1], [14]).

The spectral projection onto the right deflating subspace of $\lambda E - A$ corresponding to the finite eigenvalues is given by

$$\mathcal{P} = \frac{1}{2\pi i} \oint_{\Gamma} (\lambda E - A)^{-1} E \, d\lambda, \tag{7}$$

where Γ is a closed Jordan curve surrounding the finite eigenvalues. Using (6), we obtain

$$\mathcal{P} = T^{-1} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} T. \tag{8}$$

As a first step, a bound on $\mathcal{P}x(t)$ analogous to (4) is obtained in Theorem 2.1. Then, Theorem 2.2 extends the bounds to $x(t)$. In these theorems, the assumption that the pencil has index 1 plays an essential role (see the function $v(t, y(t))$ in the proof of Theorem 2.1 and the property (16)).
From (6) and (8) the system (5) can be written

$$\begin{cases} \hat{E}\dot{y}(t) = \hat{A}y(t) + \hat{B}y(t-\tau), \quad t > 0, \\ y(t) = \hat{\psi}(t), \quad -\tau \le t \le 0, \end{cases} \tag{9}$$

where

$$\begin{cases} \hat{E} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \ \hat{A} = \begin{pmatrix} J & 0 \\ 0 & I_{n-r} \end{pmatrix}, \ \hat{B} = W^{-1}BT^{-1}, \\ y(t) = Tx(t), \ \hat{\psi}(t) = T\psi(t). \end{cases} \tag{10}$$

With this notation, we obtain the following result.
Theorem 2.1 *Assume that there exists*

$$H = \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22} \end{pmatrix} \quad \textit{with } H_{11} = H_{11}^T > 0, \ \ H_{22} + H_{22}^T + \frac{1}{\tau} I_{n-r} < 0$$

such that

$$C = -\begin{pmatrix} \hat{A}^T H + H^T \hat{A} + \dfrac{1}{\tau} I_n & H^T \hat{B} \\ \hat{B}^T H & -\dfrac{1}{2\tau} I_n \end{pmatrix} = C^T > 0.$$

Then

(i) the pencil $\lambda E - A$ is stable (i.e., the eigenvalues of J lie in the open left-half plane).

(ii) The following bound holds for $t \ge 0$

$$\|\mathcal{P}x(t)\| \le \alpha \, e^{-\frac{1}{2}\frac{\beta}{\|H_{11}\|}t},$$

where

$$\alpha = \|T\| \|T^{-1}\| \sqrt{\|H_{11}^{-1}\| \ (\|H_{11}\| + \ln 2)} \max_{-\tau \le \nu \le 0} \|\mathcal{P}x(\nu)\|,$$

$$\beta = \min\left(\lambda_{\min}(C), \frac{\|H_{11}\|}{2\tau}\right).$$

Proof

(i) Let (λ, x_1) be an eigenpair of J and $\hat{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ and $x = \begin{pmatrix} \hat{x} \\ 0 \end{pmatrix}$ be respectively of size n and $2n$. Then, denoting by x^* the conjugate transpose of x, we obtain

$$-x^* C x = 2 x_1^* H_{11} x_1 \,\mathrm{Re}\lambda + \frac{\|x_1\|^2}{\tau}$$

which shows that $\mathrm{Re}(\lambda) < 0$.

(ii) Consider the Lyapunov-Krasovskii type function

$$v(t, y(t)) = y^T(t)\hat{E}Hy(t) + \int_{t-\tau}^{t} \frac{\|y(\nu)\|^2}{t-\nu+\tau}\, d\nu, \quad t \geq 0.$$

If we set $y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, then $y^T(t)\hat{E}Hy(t) = y_1^T(t)H_{11}y_1(t) \geq 0$ and

$$\begin{aligned}
\dot{v}(t, y(t)) &= y^T(t)H^T\hat{E}\dot{y}(t) + (\hat{E}\dot{y}(t))^T H y(t) + \frac{\|y(t)\|^2}{\tau} - \frac{\|y(t-\tau)\|^2}{2\tau} \\
&\quad - \int_{t-\tau}^{t} \frac{\|y(\nu)\|^2}{(t-\nu+\tau)^2}\, d\nu \\
&= y(t)^T H^T \left(\hat{A}y(t) + \hat{B}y(t-\tau)\right) + \left(\hat{A}y(t) + \hat{B}y(t-\tau)\right)^T H y(t) \\
&\quad + \frac{\|y(t)\|^2}{\tau} - \frac{\|y(t-\tau)\|^2}{2\tau} - \int_{t-\tau}^{t} \frac{\|y(\nu)\|^2}{(t-\nu+\tau)^2}\, d\nu \\
&= -\begin{pmatrix} y(t) \\ y(t-\tau) \end{pmatrix}^T C \begin{pmatrix} y(t) \\ y(t-\tau) \end{pmatrix} - \int_{t-\tau}^{t} \frac{\|y(\nu)\|^2}{(t-\nu+\tau)^2}\, d\nu \leq 0. \qquad (11)
\end{aligned}$$

Note that

$$\begin{aligned}
\begin{pmatrix} y(t) \\ y(t-\tau) \end{pmatrix}^T C \begin{pmatrix} y(t) \\ y(t-\tau) \end{pmatrix} &\geq \lambda_{\min}(C) \left\| \begin{pmatrix} y(t) \\ y(t-\tau) \end{pmatrix} \right\|^2 \geq \lambda_{\min}(C)\, \|y(t)\|^2 \\
&\geq \frac{\lambda_{\min}(C)}{\|H_{11}\|} y^T(t)\hat{E}Hy(t) \qquad (12)
\end{aligned}$$

$$t-\tau \leq \nu \leq t \Rightarrow (t-\nu+\tau)^2 \leq 2\tau(t-\nu+\tau). \qquad (13)$$

Using (11), (12) and (13) we obtain

$$\dot{v}(t, y(t)) \leq -\frac{\beta}{\|H_{11}\|} v(t, y(t)), \quad \beta = \min\left(\lambda_{\min}(C), \frac{\|H_{11}\|}{2\tau}\right).$$

Therefore

$$v(t, y(t)) \leq e^{-\frac{\beta}{\|H_{11}\|}t} v(0, y(0)).$$

Since

$$
\begin{aligned}
v(t,y(t)) &\geq y^{T}(t)\hat{E}Hy(t) = y_1^{T}(t)H_{11}y_1(t)\\
&\geq \lambda_{\min}(H_{11})\,\|y_1(t)\|^2 = \|H_{11}^{-1}\|^{-1}\,\|y_1(t)\|^2,\\
v(0,y(0)) &= y^{T}(0)\hat{E}Hy(0) + \int_{-\tau}^{0}\frac{\|y(\nu)\|^2}{-\nu+\tau}\,d\nu\\
&\leq \|\hat{E}H\|\|y(0)\|^2 + \max_{-\tau\leq\nu\leq 0}\|y(\nu)\|^2\,\ln 2\\
&\leq (\|H_{11}\| + \ln 2)\max_{-\tau\leq\nu\leq 0}\|y(\nu)\|^2,
\end{aligned}
$$

we deduce that

$$
\|y_1(t)\|^2 \leq \|H_{11}^{-1}\|\,(\|H_{11}\| + \ln 2)\max_{-\tau\leq\nu\leq 0}\|y(\nu)\|^2 e^{-\frac{\beta}{\|H_{11}\|}t}.
$$

The proof terminates by noticing that

$$
\|y_1(t)\| = \|T\mathcal{P}x(t)\| \text{ and } \|T^{-1}\|^{-1}\|\mathcal{P}x(t)\| \leq \|T\mathcal{P}x(t)\| \leq \|T\|\|\mathcal{P}x(t)\|.
$$

□

Remarks

- In Theorem 2.1, the connection between H and C is the analogue of equation (2). In particular, if $E = I$, $B = 0$, then the bound in (ii) reduces to (4).
- Similar to the ordinary case, the bound on $\|\mathcal{P}x(t)\|$ depends on the solvability of the Lyapunov-type equation involving C and H.
- In the expression of α, the coefficient $\ln 2$ results from our simple choice of the function v. Of course, more sophisticated choices can be considered.
- The factor α depends essentially on the condition number of T, which is an indicator of the quality of the spectral projection (8), and on $\|H_{11}\| = \|\hat{E}H\|$, which is an indicator of the stability of J and therefore of the pencil $\lambda E - A$. A large value of α can result in a transient growth of $\mathcal{P}x(t)$. The factor β depends on $\lambda_{\min}(C)$ and $\|H_{11}\|$ and is responsible for the decay and asymptotic behavior of $\mathcal{P}x(t)$.
- The assumption of the theorem results in constraints mainly on the matrix B. Indeed, by considering the Schur complement, the positive definiteness of C is equivalent to

$$
\hat{A}^{T}H + H^{T}\hat{A} + \frac{1}{\tau}I_n < 0, \tag{14}
$$

$$
\hat{B}^{T}H\left(\hat{A}^{T}H + H^{T}\hat{A} + \frac{1}{\tau}I_n\right)^{-1}H^{T}\hat{B} + \frac{1}{2\tau}I_n > 0. \tag{15}
$$

If we assume that the pencil $\lambda E - A$ is stable, then the condition (14) is easily satisfied. Indeed, by considering, for example, the matrix $K = \frac{c}{\tau}I_r$ of size $r \times r$ with $c > 1$, the Lyapunov equation $J^{T}H_{11} + H_{11}J + K = 0$ has a unique solution $H_{11} = H_{11}^{T} > 0$. Then

$$
\hat{A}^{T}H + H^{T}\hat{A} + \frac{1}{\tau}I_n = \begin{pmatrix} \frac{1-c}{\tau}I_r & 0\\ 0 & H_{22} + H_{22}^{T} + \frac{1}{\tau}I_{n-r}\end{pmatrix} < 0.
$$

In particular, the choice $H_{22} = -\frac{c}{2\tau}I_{n-r}$ leads to $\hat{A}^{T}H + H^{T}\hat{A} + \frac{1}{\tau}I_n = -\frac{c-1}{\tau}I_n < 0$. The condition (15) can be written $(W^{-1}BT^{-1})^{T}HH^{T}(W^{-1}BT^{-1}) - \frac{c-1}{2\tau^2}I_n < 0$ and will be satisfied if $\|B\| \leq (\|H\|\|T^{-1}\|\|W^{-1}\|)^{-1}\left(\frac{c-1}{2\tau^2}\right)^{\frac{1}{2}}$.

For example if $n = 4$, $A = \begin{pmatrix} J & 0 \\ 0 & I_2 \end{pmatrix}$, $J = \begin{pmatrix} -3 & 1 \\ 0 & -2 \end{pmatrix}$, $E = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}$, $\tau = 1$, then the equation $J^T H_{11} + H_{11} J + 2I_2 = 0$ has the unique solution $H_{11} = \begin{pmatrix} \frac{1}{3} & \frac{1}{15} \\ \frac{1}{15} & \frac{8}{15} \end{pmatrix}$. Taking $H_{22} = -I_2$ and letting $H = \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22} \end{pmatrix}$ leads to $A^T H + HA^T + \frac{1}{\tau} I_4 = -I_4$ and hence the condition (14) is satisfied. The condition (15) will be satisfied for any matrix B such $-2B^T H^2 B + I_4$ is positive definite, and, in particular if $\|B\| \le \frac{1}{\sqrt{2}\|H\|} = 0.70711$.

Under the assumption of Theorem 2.1, the pencil $\lambda E - A$ is stable and hence the matrix A is nonsingular. Multiplying equation (5) on the left by $(I_n - \mathcal{P})A^{-1}$ and noting that $(I_n - \mathcal{P})A^{-1}E = 0$, we obtain

$$0 = (I_n - \mathcal{P})x(t) + (I_n - \mathcal{P})A^{-1}Bx(t-\tau), \quad t > 0, \tag{16}$$

which will be used to derive a bound analogous to (4).

Theorem 2.2 *Under the assumption of Theorem 2.1, let $t = k\tau + \mu$, $0 \le \mu \le \tau$, $k \ge 0$. Then*

$$\|x(t)\| \le \alpha \sum_{j=0}^{k} \left(\|(I_n - \mathcal{P})A^{-1}B\| \, e^{\frac{\beta\tau}{2\|H_{11}\|}} \right)^j e^{-\frac{\beta}{2\|H_{11}\|}t} + \|(I_n - \mathcal{P})A^{-1}B\|^{k+1} \max_{-\tau \le \nu \le 0} \|x(\nu)\|,$$

where α and β are defined in Theorem 2.1.

Proof For $0 \le t \le \tau$, the equality (16) gives

$$\|(I_n - \mathcal{P})x(t)\| \le \|(I_n - \mathcal{P})A^{-1}B\| \max_{-\tau \le \nu \le 0} \|x(\nu)\| \tag{17}$$

and for $t = \tau + \mu$, $0 \le \mu \le \tau$, it gives

$$\|(I_n - \mathcal{P})x(t)\| \le \|(I_n - \mathcal{P})A^{-1}B\| \left[\|\mathcal{P}x(t-\tau)\| + \|(I_n - \mathcal{P})x(t-\tau)\| \right].$$

From Theorem 2.1 and the inequality (17) we obtain

$$\|\mathcal{P}x(t-\tau)\| \le \alpha e^{-\frac{\beta}{2\|H_{11}\|}\mu}, \quad \|(I_n - \mathcal{P})x(t-\tau)\| \le \|(I_n - \mathcal{P})A^{-1}B\| \max_{-\tau \le \nu \le 0} \|x(\nu)\|.$$

Therefore

$$\|(I_n - \mathcal{P})x(t)\| \le \alpha \|(I_n - \mathcal{P})A^{-1}B\| \, e^{-\frac{\beta}{2\|H_{11}\|}\mu} + \|(I_n - \mathcal{P})A^{-1}B\|^2 \max_{-\tau \le \nu \le 0} \|x(\nu)\|. \tag{18}$$

Continuing this way we easily obtain for $t = k\tau + \mu$, $0 \le \mu \le \tau$ a generalization of (17) and (18) as follows

$$\|(I_n - \mathcal{P})x(t)\| \le \alpha \sum_{j=1}^{k} \|(I_n - \mathcal{P})A^{-1}B\|^j \, e^{-\frac{\beta}{2\|H_{11}\|}(t-j\tau)} + \|(I_n - \mathcal{P})A^{-1}B\|^{k+1} \max_{-\tau \le \nu \le 0} \|x(\nu)\|.$$

Hence

$$
\begin{aligned}
\|x(t)\| &\le \|\mathcal{P}x(t)\| + \|(I_n-\mathcal{P})x(t)\| \\
&\le \alpha \sum_{j=0}^{k} \|(I_n-\mathcal{P})A^{-1}B\|^j \, e^{-\frac{\beta}{2\|H_{11}\|}(t-j\tau)} + \|(I_n-\mathcal{P})A^{-1}B\|^{k+1} \max_{-\tau\le\nu\le 0}\|x(\nu)\| \\
&= \alpha \sum_{j=0}^{k} \left(\|(I_n-\mathcal{P})A^{-1}B\| \, e^{\frac{\beta\tau}{2\|H_{11}\|}}\right)^j e^{-\frac{\beta}{2\|H_{11}\|}t} + \|(I_n-\mathcal{P})A^{-1}B\|^{k+1} \max_{-\tau\le\nu\le 0}\|x(\nu)\|.
\end{aligned}
$$

□

Corollary 1 *Under the assumptions of Theorem 2.1, let $t = k\tau+\mu$, $0\le\mu\le\tau$, $k\ge 0$.*

(i) If $\|(I_n-\mathcal{P})A^{-1}B\| \, e^{\frac{\beta\tau}{2\|H_{11}\|}} < 1$, then

$$
\|x(t)\| \le \left(\frac{\alpha}{1-\|(I_n-\mathcal{P})A^{-1}B\|e^{\frac{\beta\tau}{2\|H_{11}\|}}} + \max_{-\tau\le\nu\le 0}\|x(\nu)\|\right) e^{-\frac{\beta}{2\|H_{11}\|}t}.
$$

(ii) If $\|(I_n-\mathcal{P})A^{-1}B\| \, e^{\frac{\beta\tau}{2\|H_{11}\|}} = 1$, then

$$
\|x(t)\| \le \left(\alpha\left(\frac{t}{\tau}+1\right) + \max_{-\tau\le\nu\le 0}\|x(\nu)\|\right) e^{-\frac{\beta}{2\|H_{11}\|}t}.
$$

(iii) If $1 < \|(I_n-\mathcal{P})A^{-1}B\| \, e^{\frac{\beta\tau}{2\|H_{11}\|}}$ and $\|(I_n-\mathcal{P})A^{-1}B\| \le 1$, then

$$
\|x(t)\| \le \left(\frac{\alpha \, e^{\frac{\beta\tau}{2\|H_{11}\|}}}{\|(I_n-\mathcal{P})A^{-1}B\|e^{\frac{\beta\tau}{2\|H_{11}\|}}-1} + \max_{-\tau\le\nu\le 0}\|x(\nu)\|\right) e^{-\ln\left(\|(I_n-\mathcal{P})A^{-1}B\|^{-\frac{1}{\tau}}\right)t}.
$$

Proof For the three cases we use Theorem 2.2.

(i) If $\|(I_n-\mathcal{P})A^{-1}B\| \, e^{\frac{\beta\tau}{2\|H_{11}\|}} < 1$, then

$$
\|x(t)\| \le \alpha \sum_{j\ge 0} (\|(I_n-\mathcal{P})A^{-1}B\| \, e^{\frac{\beta\tau}{2\|H_{11}\|}})^j \, e^{-\frac{\beta}{2\|H_{11}\|}t} + e^{-\frac{\beta\tau(k+1)}{2\|H_{11}\|}} \max_{-\tau\le\nu\le 0}\|x(\nu)\|
$$

and the desired inequality follows from $(k+1)\tau \ge t$.

(ii) If $\|(I_n-\mathcal{P})A^{-1}B\| \, e^{\frac{\beta\tau}{2\|H_{11}\|}} = 1$, then

$$
\|x(t)\| \le \alpha(k+1) \, e^{-\frac{\beta}{2\|H_{11}\|}t} + e^{-\frac{\beta\tau(k+1)}{2\|H_{11}\|}} \max_{-\tau\le\nu\le 0}\|x(\nu)\|
$$

and the desired inequality follows from $t \le (k+1)\tau \le t+\tau$.

(iii) If $1 < \|(I_n-\mathcal{P})A^{-1}B\| \, e^{\frac{\beta\tau}{2\|H_{11}\|}}$ and $\|(I_n-\mathcal{P})A^{-1}B\| \le 1$, then

$$
\begin{aligned}
\|x(t)\| \le\ & \alpha \sum_{j=0}^{k} (\|(I_n-\mathcal{P})A^{-1}B\| \, e^{\frac{\beta\tau}{2\|H_{11}\|}})^{j-k} \, (\|(I_n-\mathcal{P})A^{-1}B\| \, e^{\frac{\beta\tau}{2\|H_{11}\|}})^k \, e^{-\frac{\beta}{2\|H_{11}\|}t} \\
&+ \|(I_n-\mathcal{P})A^{-1}B\|^{k+1} \max_{-\tau\le\nu\le 0}\|x(\nu)\| \\
\le\ & \alpha \sum_{j\ge 0} \left((\|(I_n-\mathcal{P})A^{-1}B\| \, e^{\frac{\beta\tau}{2\|H_{11}\|}})^{-1}\right)^j \, (\|(I_n-\mathcal{P})A^{-1}B\| \, e^{\frac{\beta\tau}{2\|H_{11}\|}})^k \, e^{-\frac{\beta}{2\|H_{11}\|}t} \\
&+ \|(I_n-\mathcal{P})A^{-1}B\|^{k+1} \max_{-\tau\le\nu\le 0}\|x(\nu)\| \\
\le\ & \left[\frac{\alpha \, e^{\frac{\beta\tau}{2\|H_{11}\|}}}{\|(I_n-\mathcal{P})A^{-1}B\|e^{\frac{\beta\tau}{2\|H_{11}\|}}-1} + \max_{-\tau\le\nu\le 0}\|x(\nu)\|\right] \|(I_n-\mathcal{P})A^{-1}B\|^{k+1}
\end{aligned}
$$

and the desired inequality follows again from $t \leq (k+1)\tau$.

□

As a consequence we have the following

Corollary 2 *Under the assumption of Theorem 2.1, the solution of (5) is asymptotically stable if* $\|(I_n - \mathcal{P})A^{-1}B\| < 1$ *and stable if* $\|(I_n - \mathcal{P})A^{-1}B\| = 1$.

In particular, when $E = I$ and hence $\mathcal{P} = I_n$, Corollary 2 implies that the solution of (5) is asymptotically stable, see also [13, Proposition 5.3].

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References

[1] T. Berger, A. Ilchmann, S. Trenn, The quasi-Weierstraß form for regular matrix pencils, Linear Algebra Appl. 436, 4052–4069 (2012).

[2] S.L. Campbell, Nonregular 2D descriptor delay systems, IMA J. Math. Cont. Inf. 12, 57–67 (1995).

[3] S.L. Campbell, V.H. Linh, Stability criteria for differential-algebraic equations with multiple delays and their numerical solutions, Appl. Math. Comput. 208, 397–415 (2009).

[4] N.H. Du, V.H. Linh, V. Mehrmann, D.D. Thuan, Stability and robust stability of linear time-invariant delay differential-algebraic equations, SIAM J. Matrix Anal. Appl. 34, 1631–1654 (2013).

[5] E. Fridman, New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems, Systems & Control Letters 43, 309–319 (2001).

[6] E. Fridman, Stability of linear descriptor systems with delay: A Lyapunov-based approach, J. Math. Anal. Appl. 273, 24–44 (2002).

[7] E. Fridman, Introduction to Time-Delay Systems, Analysis and Control, Springer International Publishing, 2014.

[8] Z. Gajič, M. T. J. Qureshi, Lyapunov Matrix Equation in System Stability and Control, Mineola, N. Y., Dover, 2008.

[9] S. K. Godunov, Modern Aspects of Linear Algebra, Translations of Mathematical Monographs, vol. 175, American Math. Society, Providence, RI, 1998.

[10] G. H. Golub, C. F. Van Loan, Matrix Computations, 3rd ed., The Johns Hopkins Univ. Press, Baltimore, MD, 1996.

[11] P. Ha, Analysis and Numerical solutions of Delay Differential-Algebraic Equations, Technische Universität Berlin, Dissertation, 2015. http://dx.doi.org/10.14279/depositionce-4385.

[12] R. A. Horn, C. R. Johnson, Matrix Analysis, 2nd Edition. Cambridge University Press, New York, 2013

[13] K. Gu, V.L. Kharitonov, J. Chen, Stability of Time-Delay Systems. Boston, MA : Birkhuser, 2003.

[14] B. Kågström, P. Poromaa, Computing eigenspaces with specified eigenvalues of a regular matrix pair (A, B) and condition estimation: theory, algorithms and software, Numerical Algorithms, 12, 369–407 (1996).

[15] W. Michiels, S.-I. Niculescu, Stability and Stabilization of Time-Delay Systems: An Eigenvalue-Based Approach, Advances in Design and Control, vol. 12, SIAM Publications, Philadelphia, 2007.

[16] W. Michiels, Spectrum-based stability analysis and stabilisation of systems described by delay differential algebraic equations. In: IET Control Theory Appl. 5, S. 1829–1842 (2011).

[17] S.-I. Niculescu, P. Fu, J. Chen, On the Stability of Linear Delay-Differential Algebraic Systems: Exact Conditions via Matrix Pencil Solutions, In: Proc. 45th IEEE Conf. Decision Control, 2006.

[18] M. Sadkane, R.B. Sidje, Efficient computation of the spectral projections of regular matrix pairs, J. Comput. Appl. Math., 298, 72–81 (2016).

[19] G.W. Stewart, J.-G. Sun, Matrix Perturbation Theory, Academic Press, San Diego, California, 1990.

[20] L.N. Trefethen, M. Embree, Spectra and pseudospectra: The Behavior of Nonnormal Ma- trices and Operators, Princeton University Press, Princeton, 2005.

[21] S. Trenn, B. Unger, Delay regularity of differential-algebraic equations, In: Proc. 58th IEEE Conf. Decision Control, 2019, Nice, France, S. 989–994 (2019).

[22] B. Unger, Discontinuity propagation in delay differential-algebraic equations, Electron. J. Linear Algebr. 34, 582–601 (2018).
[23] B. Unger, Well-Posedness and Realization Theory for Delay Differential-Algebraic Equations, Technische Universität Berlin, Dissertation, 2020. http://dx.doi.org/10.14279/depositonce-10707.
[24] C.F. Van Loan, How near is a stable matrix to an unstable matrix, Contemporary Mathematics, 47, 465–477 (1985).
[25] K. Veselić, Bounds for exponentially stable semigroups, Lin. Alg. Appl., 358, 309–333 (2003).