All bi-unitary perfect polynomials over F 2 only divisible by x, x + 1 and by Mersenne primes

Luis H. Gallardo -, Olivier Rahavandrainy

All bi-unitary perfect polynomials over $\mathbb{F}_2$ only divisible by x, $x+1$ and by Mersenne primes

Luis H. Gallardo - Olivier Rahavandrainy
Univ Brest, UMR CNRS 6205
Laboratoire de Mathématiques de Bretagne Atlantique
e-mail : luis.gallardo@univ-brest.fr - olivier.rahavandrainy@univ-brest.fr

April 27, 2022

a) Keywords: Bi-unitary divisors, Mersenne primes, finite fields

b) Mathematics Subject Classification (2010): 11T55, 11T06.

c) Corresponding author: Luis H. Gallardo

Abstract
We give all non splitting bi-unitary perfect polynomials over the prime field of two elements, which have only Mersenne polynomials as odd irreducible divisors.

1 Introduction

Let $S \in \mathbb{F}_2[x]$ be a nonzero polynomial. We say that S is odd if $\gcd(S, x(x+1)) = 1$, S is even if it is not odd. A *Mersenne* (*prime*) is a polynomial (irreducible) of the form $1 + x^a(x+1)^b$, with $\gcd(a, b) = 1$. A divisor D of S is called unitary if $\gcd(D, S/D) = 1$. We denote by $\gcd_u(S, T)$ the greatest common unitary divisor of S and T. A divisor D of S is called bi-unitary if $\gcd_u(D, S/D) = 1$.
We denote by $\sigma(S)$ (resp. $\sigma^*(S)$, $\sigma^{**}(S)$) the sum of all divisors (resp. unitary divisors, bi-unitary divisors) of S. The functions σ, σ^* and σ^{**} are all multiplicative. We say that a polynomial S is *perfect* (resp. *unitary perfect*, *bi-unitary perfect*) if $\sigma(S) = S$ (resp. $\sigma^*(S) = S$, $\sigma^{**}(S) = S$).
Finally, we say that a bi-unitary perfect polynomial is *indecomposable* if it is not a product of two coprime nonconstant bi-unitary perfect polynomials.
As usual, $\omega(S)$ designates the number of distinct irreducible factors of S.
Several studies are done about (unitary) perfect polynomials over $\mathbb{F}_2$. In particular, we gave ([3], [4], [5]) the list of them with $\omega(A) \leq 4$ and that of all which are divisible only by x, $x+1$ and by Mersenne primes ([6] and [8]).

We are interested in indecomposable bi-unitary perfect (i.b.u.p) polynomials (over $\mathbb{F}_2$) with only Mersenne primes as odd divisors and we get Theorem 1.1.

If A is a nonconstant b.u.p polynomial, then $x(x+1)$ divides A so that $\omega(A) \geq 2$ (see Lemma 2.1). Moreover, the only b.u.p polynomials over $\mathbb{F}_2$ with exactly two prime divisors are $x^2(x+1)^2$ and $x^{2^n-1}(x+1)^{2^n-1}$, for any nonnegative integer n (Lemma 2.1 and [1] Theorem 5).

Note that in the integer case, 6, 60 and 90 are the only b.u.p numbers ([9]).

In the rest of the paper, for $S \in \mathbb{F}_2[x]$, we denote by $\overline{S}$ (resp. S^*) the polynomial obtained from S with x replaced by $x+1$ (resp. the reciprocal of S): $\overline{S}(x) = S(x+1)$, $S^*(x) = x^{\deg(S)} \cdot S(x^{-1})$.
As usual, $\mathbb{N}$ (resp. $\mathbb{N}^*$) denotes the set of nonnegative integers (resp. of positive integers).

For $S, T \in \mathbb{F}_2[x]$ and $n \in \mathbb{N}^*$, we write: $S^n \| T$ if $S^n | T$ but $S^{n+1} \nmid T$.

We consider the following polynomials:

$M_1 = 1 + x + x^2 = \sigma(x^2),\ M_2 = 1 + x + x^3,\ M_3 = \overline{M_2} = 1 + x^2 + x^3,$
$M_4 = 1 + x + x^2 + x^3 + x^4 = \sigma(x^4), M_5 = \overline{M_4} = 1 + x^3 + x^4,$
$C_1 = x^3(x + 1)^4 M_1, C_2 = x^3(x + 1)^5 {M_1}^2, C_3 = x^4(x + 1)^4 {M_1}^2,$
$C_4 = x^6(x + 1)^6 {M_1}^2, C_5 = x^4(x + 1)^5 {M_1}^3, C_6 = x^7(x + 1)^8 M_5,$
$C_7 = x^7(x + 1)^9 {M_5}^2, C_8 = x^8(x + 1)^8 M_4 M_5, C_9 = x^8(x + 1)^9 M_4 {M_5}^2,$
$C_{10} = x^7(x + 1)^{10} {M_1}^2 M_5, C_{11} = x^7(x + 1)^{13} {M_2}^2 {M_3}^2,$
$C_{12} = x^9(x + 1)^9 {M_4}^2 {M_5}^2, C_{13} = x^{14}(x + 1)^{14} {M_2}^2 {M_3}^2,$
$C_{14} = x^8(x + 1)^{10} {M_1}^2 M_4 M_5, C_{15} = x^8(x + 1)^{12} {M_1}^2 M_2 M_3 M_5,$
$C_{16} = x^{10}(x + 1)^{13} {M_1}^2 {M_2}^2 {M_3}^2 M_4, C_{17} = x^{13}(x + 1)^{13} {M_1}^2 {M_2}^4 {M_3}^4 M_4 M_5,$
$C_{18} = x^{12}(x + 1)^{13} {M_1}^2 {M_2}^3 {M_3}^3, C_{19} = x^9(x + 1)^{13} {M_2}^2 {M_3}^2 {M_4}^2,$
$C_{20} = x^8(x + 1)^{13} {M_2}^2 {M_3}^2 M_4, C_{21} = x^9(x + 1)^{10} {M_1}^2 {M_4}^2 M_5,$
$C_{22} = x^7(x + 1)^{12} {M_1}^2 M_2 M_3, C_{23} = x^9(x + 1)^{12} {M_1}^2 M_2 M_3 {M_4}^2.$
The polynomials $M_1, \ldots, M_5$ are all Mersenne primes.

Theorem 1.1. *Let $A = x^a(x+1)^b {P_1}^{h_1} \cdots {P_r}^{h_r} \in \mathbb{F}_2[x]$ be such that the P_j's are Mersenne primes, $a, b, h_j \in \mathbb{N}$ and $\omega(A) \geq 3$. Then A is i.b.u.p if and only if $A, \overline{A} \in \{C_j : 1 \leq j \leq 23\}$.*

The polynomials $C_1, \ldots, C_{13}$ in Theorem 1.1 are already given in [1]. Our method consists in determining the possible irreducible divisors of such b.u.p polynomials and the upper bound of their exponents, without considering several distinct cases. We then use Maple computations to obtain our list.

2 Preliminaries

We need the following results. Some of them are obvious or (well) known, so we omit their proofs. We put $\mathcal{M} := \{M_1, M_2, M_3, M_4, M_5\}$.

Lemma 2.1. *If A is a nonconstant b.u.p polynomial over $\mathbb{F}_2$, then $x(x+1)$ divides A, so that A is even and $\omega(A) \geq 2$.*

Lemma 2.2. *If $A = A_1 A_2$ is b.u.p over $\mathbb{F}_2$ and if $\gcd(A_1, A_2) = 1$, then A_1 is b.u.p if and only if A_2 is b.u.p.*

Lemma 2.3. *If A is b.u.p over $\mathbb{F}_2$, then $\overline{A}$ is also b.u.p over $\mathbb{F}_2$.*

Lemma 2.4 is obtained from [7] (Lemma 2.6) and from Canaday's paper [2] (Lemmas 4, 5, 6, Theorem 8 and Corollary on page 728).

Lemma 2.4. *Let $P, Q \in \mathbb{F}_2[x]$ be odd and irreducible and let $n, m \in \mathbb{N}$.*
i) *If P is a Mersenne prime, then $\sigma(P^{2n})$ is odd and square-free.*
ii) *If P is a Mersenne prime and if $P = P^*$, then $P = M_1$ or $P = M_4$.*
iii) *If $\sigma(x^{2n}) = PQ$ and $\overline{P} = \sigma(x^{2m})$, then $n = 4$, $m = 1$ and $Q = P(x^3)$.*
iv) *If $\sigma(x^{2n})$ is only divisible by Mersenne primes, then $2n \in \{2, 4, 6\}$.*
v) *If $\sigma(x^{2r})$ is a Mersenne prime, then $2r \in \{2, 4\}$.*
vi) *If $\sigma(x^h) = \sigma((x+1)^h)$, then $h = 2^n - 2$, for some $n \in \mathbb{N}^*$.*
vii) *If $\sigma(P^{2n}) = Q^m$, then $m \in \{0, 1\}$.*

Lemma 2.5 ([8], Theorem 1.2)**.**
Let $M \in \mathcal{M}$ be such that $\sigma(M^{2m})$ (resp. $\sigma(M^{2m+1})$) has only Mersenne primes as odd divisors, then $2m = 2$ (resp. $2m + 1 = 3 \cdot 2^\alpha - 1$ for some $\alpha \in \mathbb{N}^*$) and $M \in \{M_2, M_3\}$.
All odd divisors of $\sigma(M^{2m})$ (resp. of $\sigma(M^{2m+1})$) lie in $\{M_1, M_4, M_5\}$.

Lemma 2.6. *Let T be an irreducible polynomial over $\mathbb{F}_2$ and $k, l \in \mathbb{N}^*$. Then, $\mathrm{gcd}_u(T^k, T^l) = 1$ (resp. T^k) if $k \neq l$ (resp. $k = l$).*
In particular, $\mathrm{gcd}_u(T^k, T^{2n-k}) = 1$ for $k \neq n$, $\mathrm{gcd}_u(T^k, T^{2n+1-k}) = 1$ for any $0 \leq k \leq 2n+1$.

Corollary 2.7. *Let $T \in \mathbb{F}_2[x]$ be irreducible. Then*
i) $\sigma^{**}(T^{2n}) = (1+T)\sigma(T^n)\sigma(T^{n-1}), \sigma^{**}(T^{2n+1}) = \sigma(T^{2n+1})$.
ii) For any $c \in \mathbb{N}$, T does not divide $\sigma^{**}(T^c)$.

Proof. i): $\sigma^{**}(T^{2n}) = 1 + T + \cdots + T^{n-1} + T^{n+1} + \cdots + T^{2n} = (1 + T^{n+1})\sigma(P^{n-1}) = (1+T)\sigma(T^n)\sigma(T^{n-1})$.
$\sigma^{**}(T^{2n+1}) = 1 + T + \cdots + T^{2n+1} = \sigma(T^{2n+1})$.
ii) follows from i). □

Corollary 2.8. *Let $T \in \mathbb{F}_2[x]$ be irreducible.*
i) If $a = 4r$, where $2r - 1$ is of the form $2^\alpha u - 1$, u odd, then
$\sigma^{**}(T^a) = (1+T)^{2^\alpha} \cdot \sigma(T^{2r}) \cdot (\sigma(T^{u-1}))^{2^\alpha}$ *and* $\gcd(\sigma(T^{2r}), \sigma(T^{u-1})) = 1$.
ii) If $a = 4r+2$, where $2r+1$ is of the form $2^\alpha u - 1$, u odd, then
$\sigma^{**}(T^a) = (1+T)^{2^\alpha} \cdot \sigma(T^{2r}) \cdot (\sigma(T^{u-1}))^{2^\alpha}$ *and* $\gcd(\sigma(T^{2r}), \sigma(T^{u-1})) = 1$.
iii) If $a = 2^\alpha u - 1$ is odd (with u odd), then $\sigma^{**}(T^a) = (1+T)^{2^\alpha - 1} \cdot (\sigma(T^{u-1}))^{2^\alpha}$.

We explicit the following formulas (useful for $T \in \{x, x+1\} \cup \mathcal{M}$).

$$\left\{ \begin{array}{l} \sigma^{**}(T^2) = (1+T)^2,\ \sigma^{**}(T^4) = (1+T)^2\sigma(T^2), \sigma^{**}(T^6) = (1+T)^4\sigma(T^2), \\ \sigma^{**}(T^8) = (1+T)^4\sigma(T^4),\ \sigma^{**}(T^{10}) = (1+T)^2(\sigma(T^2))^2\sigma(T^4), \\ \sigma^{**}(T^{12}) = (1+T)^2(\sigma(T^2))^2\sigma(T^6),\ \sigma^{**}(T^{14}) = (1+T)^8\sigma(T^6). \end{array} \right. \tag{1}$$

Corollary 2.9. *i) For any $j \leq 5$, neither M_2 nor M_3 divides $\sigma^{**}(M_j^{h_j})$.*
ii) $\sigma^{**}({M_2}^4) = x^2(x+1)^4 M_1 M_5$ and $\sigma^{**}({M_3}^4) = x^4(x+1)^2 M_1 M_4$.
iii) If $j \notin \{2,3\}$ and $r \geq 2$, then $\sigma^{**}({M_j}^{2r})$ has a non Mersenne prime divisor.
iv) If M_2 divides $\sigma^{**}(x^a)$, then $a \in \{12, 14, 7 \cdot 2^n - 1 : n \in \mathbb{N}^*\}$. In this case, M_3 also divides $\sigma^{**}(x^a)$.

Corollary 2.10. *i) The polynomial $\sigma^{**}(x^a)$ splits over $\mathbb{F}_2$ if and only if $a = 2$ or $a = 2^\alpha - 1$, for some $\alpha \in \mathbb{N}^*$.*
ii) Let $T \in \mathbb{F}_2[x]$ be odd and irreducible. Then $\sigma^{**}(T^c)$ splits over $\mathbb{F}_2$ if and only if T is a Mersenne prime and ($c = 2$ or $c = 2^\gamma - 1$ for some $\gamma \in \mathbb{N}^*$).

Lemma 2.11. *If $\sigma^{**}(x^{2m})$ has only Mersenne primes as odd divisors, then $2m \in \{4, 6, 8, 10, 12, 14\}$. In this case, all its divisors lie in $\mathcal{M}$.*

Proof. - Case 1: $2m = 4r$, with $r \geq 1$ and $2r - 1 = 2^\alpha u - 1$, u odd.
We obtain: $\sigma^{**}(x^{2m}) = (1+x)^{2^\alpha} \cdot \sigma(x^{2r}) \cdot (\sigma(x^{u-1}))^{2^\alpha}$
Therefore, $2r \in \{2, 4, 6\}$ and $u \in \{1, 3, 5, 7\}$, $2r = 2^\alpha u$. So, $2m \in \{4, 8, 12\}$.
- Case 2: $2m = 4r + 2$, with $r \geq 0$ and $2r + 1 = 2^\alpha u - 1$, u odd.
One has: $\sigma^{**}(x^{2m}) = (1+x)^{2^\alpha} \cdot \sigma(x^{2r}) \cdot (\sigma(x^{u-1}))^{2^\alpha}$.
Thus, $2r \in \{2, 4, 6\}$ and $u \in \{1, 3, 5, 7\}$, $2r = 2^\alpha u - 2$. So, $2m \in \{6, 10, 14\}$.
It remains to remark that $\sigma(x^2) = M_1$, $\sigma(x^4) = M_4$ and $\sigma(x^6) = M_2 M_3$. □

We get from Lemma 2.5 (with similar proofs):

Lemma 2.12. *If $\sigma^{**}(x^{2m+1})$ has only Mersenne primes as odd divisors, then $2m + 1 = 2^\alpha u - 1$ for some $\alpha \in \mathbb{N}^*$ and $u \in \{3, 5, 7\}$.*
In this case, all its odd divisors lie in $\mathcal{M}$.

Lemma 2.13. *Let $M \in \mathcal{M}$ such that $\sigma^{**}(M^{2m})$ has only Mersenne primes as odd divisors, then $2m \in \{4, 6\}$ and $M \in \{M_2, M_3\}$.*
In this case, all its divisors lie in $\{M_1, M_4, M_5\}$.

Lemma 2.14. *If $M \in \mathcal{M}$ and $\sigma^{**}(M^{2m+1})$ has only Mersenne primes as odd divisors, then $2m + 1 \in \{3 \cdot 2^\alpha - 1 : \alpha \in \mathbb{N}^*\}$ and $M \in \{M_2, M_3\}$.*
In this case, all its odd divisors lie in $\{M_1, M_4, M_5\}$.

Lemma 2.15.
If Q is a Mersenne prime divisor of $\sigma^{**}(A_1)$, then $Q \in \mathcal{M}$.

Proof. We apply Lemmas 2.11 and 2.13. If Q divides $\sigma^{**}(x^a)\sigma^{**}((x+1)^b)$, then $Q \in \mathcal{M}$. If Q divides $\sigma^{**}(P_i^{h_i})$ with $P_i \in \mathcal{M}$, then $P_i \in \{M_2, M_3\}$ and $Q \in \{M_1, M_4, M_5\}$. □

3 The proof of Theorem 1.1

Sufficiencies are obtained by direct computations. For the necessities, we shall apply Lemmas 2.11, 2.12, 2.13 and 2.14. We fix:

$A = x^a(x+1)^b \prod_{i \in I} P_i^{h_i} = A_1 A_2$, where $a, b, h_i \in \mathbb{N}$, P_i is a Mersenne prime,

$A_1 = x^a(x+1)^b \prod_{i=1}^{5} M_i^{h_i}$ and $A_2 = \prod_{P_i \notin \mathcal{M}} P_i^{h_i}$.

We suppose that A is i.b.u.p: $A_1 A_2 = A = \sigma^{**}(A) = \sigma^{**}(A_1)\sigma^{**}(A_2)$.

3.1 First reduction

Lemma 3.1. *For any $P_j \notin \mathcal{M}$, one has:* $\gcd(P_j^{h_j}, \sigma^{**}(A_1)) = 1$ *and* $h_j = 0$, *so that* $A = A_1$.

Proof. Any odd irreducible divisor of $\sigma^{**}(x^a)$ (resp. of $\sigma^{**}((x+1)^b)$, of $\sigma^{**}(M_i^{h_i})$, with $M_i \in \mathcal{M}$) must belong to $\mathcal{M}$. Thus, for all $P_j \notin \mathcal{M}$ and $M_i \in \mathcal{M}$, P_j divides neither $\sigma^{**}(x^a)$, $\sigma^{**}((x+1)^b)$ nor $\sigma^{**}(M_i^{h_i})$. Hence, $\gcd(P_j^{h_j}, \sigma^{**}(A_1)) = 1$.
Moreover, $P_j^{h_j}$ divides $\sigma^{**}(A_2)$ because it divides $A = \sigma^{**}(A) = \sigma^{**}(A_1)\sigma^{**}(A_2)$ and $\gcd(P_j^{h_j}, \sigma^{**}(A_1)) = 1$. Hence, A_2 divides $\sigma^{**}(A_2)$. So, A_2 is b.u.p and it is equal to 1, A being indecomposable. ☐

Corollary 3.2.
If A_1 is b.u.p, then $h_3 = h_2$, $h_2 \in \{0, 2, 4, 6, 2^n - 1, 3 \cdot 2^n - 1 : n \in \mathbb{N}^*\}$ and $h_i \in \{0, 2, 2^n - 1 : n \in \mathbb{N}^*\}$, for $i \in \{1, 4, 5\}$.

Proof. If M_2 (resp. M_3) divides $\sigma^{**}(A_1)$, then it divides $V = \sigma^{**}(x^a)\sigma^{**}((x+1)^b)$. Therefore, M_3 (resp. M_2) also divides V and $\sigma^{**}(A_1)$. Hence, $h_2 = h_3$. Suppose that $h_j \geq 1$. The polynomial $\sigma^{**}(M_j^{h_j})$ must factor in $\{x, x+1\} \cup \mathcal{M}$. Thus, if $j \notin \{2, 3\}$, then $h_j \in \{2, 2^n - 1 : n \in \mathbb{N}^*\}$. If $j \in \{2, 3\}$, then $h_j \in \{2, 4, 6\}$ or it is of the form $2^n u - 1$, where $n \geq 1$ and $u \in \{1, 3\}$. ☐

In the rest of the paper, we prove the following

Proposition 3.3. *If A_1 is b.u.p, then $A_1, \overline{A_1} \in \{C_1, \ldots, C_{23}\}$.*

3.2 Proof of Proposition 3.3

We write: $A_1 = x^a(x+1)^b M_1^{h_1} M_2^{h_2} M_3^{h_3} M_4^{h_4} M_5^{h_5}$.
Corollary 3.2 implies that for any i, $h_i \in \{0, 2, 4, 6, 2^n - 1, 3\cdot 2^n - 1 : n \in \mathbb{N}^*\}$.

Lemma 3.4. *For any $n, m \in \mathbb{N}^*$, $a \neq 2^n - 1$ or $b \neq 2^m - 1$.*

Proof. If $a = 2^n - 1$ and $b = 2^m - 1$ for some $n, m \geq 1$, then

$$x^a(x+1)^b M_1^{h_1} \cdots M_5^{h_5} = A_1 = \sigma^{**}(A_1) = (x+1)^a x^b \sigma^{**}(M_1^{h_1}) \cdots \sigma^{**}(M_5^{h_5}).$$

Thus, $a = b$ and $M_1^{h_1} \cdots M_5^{h_5}$ is b.u.p, which contradicts Lemma 2.1. $\square$

By direct computations (sketched in Section 3.3), we get Proposition 3.3 from Lemmas 3.5 and 3.6.
Set $K_1 = \{0, 1, 2, 3, 4, 5, 6, 7, 11, 23\}$ and $K_2 = \{0, 1, 2, 3, 4, 6, 7, 15\}$.

Lemma 3.5. *i) If a and b are both even, then $a, b \leq 14$ and $h_i \in K_1$.*
ii) If a is even and b odd, then $a \leq 14, b = 2^\beta v - 1$, with $\beta \leq 3, v \leq 7$, v odd and $h_i \in K_1$.
More precisely, $h_3 = h_2, h_2 \in \{0, 2, 4, 6\}$ and $h_1, h_4, h_5 \in \{0, 1, 2, 3, 7, 15\}$.

Proof. According to Corollary 3.2, it remains to give upper bounds for a, b and for h_i, if h_i is odd.
i): If a and b are both even, then a (resp. b) is of the form $4r$ or $4r + 2$, (resp. $4s$ or $4s + 2$). Thus, $\sigma(x^{2r})$ and $\sigma((x+1)^{2s})$ are both odd divisors of $\sigma^{**}(A_1) = A_1$. Hence, $2r, 2s \leq 6$ and $a, b \leq 14$.
If h_i is odd, then it is of the form $2^n u - 1$, with $u \in \{1, 3\}$. So, $\sigma^{**}({M_i}^{h_i}) = \sigma({M_i}^{h_i}) = (1+M_i)^{2^n - 1}(\sigma({M_i}^{u-1}))^{2^n}$. Thus, $2^n - 1 \leq a \leq 14$, by considering the exponents of x in A_1 and in $\sigma^{**}(A_1)$. We get $n \leq 3$ and $h_i \in K_1$.
ii): In this case, $2^\beta \leq a \leq 14$ so that $\beta \leq 3$. Moreover, $\sigma((x+1)^{v-1})$ lies in $\{1, M_1, M_2M_3, M_5\}$. We deduce that $v \leq 7$. As above, $h_i \in K_1$. $\square$

Lemma 3.6. *If a and b are both odd, then $a = 2^\alpha u - 1$, $b = 2^\beta v - 1$ with $u, v \leq 7$, u, v both odd, $(u, v) \neq (1, 1)$, $1 \leq \alpha, \beta \leq 3$ and $h_i \in K_2$.*

Proof. We give upper bounds for a, b and for h_i, if h_i is odd. One has:

$$\sigma^{**}(x^a) = (x+1)^{2^\alpha - 1}(\sigma(x^{u-1}))^{2^\alpha},\ \sigma^{**}((x+1)^b) = x^{2^\beta - 1}(\sigma((x+1)^{v-1}))^{2^\beta}.$$

Without loss of generality, we may suppose that $u \leq v$.
• If $u = 7$ or $v = 7$, then $h_2 \neq 0$ and $h_2 \in \{2^\alpha, 2^\beta, 2^\alpha + 2^\beta\}$ (compare h_2 with all possible exponents of M_2 in $\sigma^{**}(A_1)$). So, h_2 is even and thus $h_3 = h_2 \leq 6$, $\alpha, \beta \leq 2$ and $a, b \leq 27$.

Now, for $i \in \{1,4,5\}$ with h_i odd, one has: $h_i = 2^n - 1 \leq a \leq 27$, so that $n \leq 4$ and $h_i \in \{1,3,7,15\}$.
• If $u, v \leq 5$, then $h_3 = h_2 = 0$. For $j \in \{1,4,5\}$, M_j divides $\sigma^{**}(A_1)$ if and only if it divides $\sigma^{**}(x^a)\sigma^{**}((1+x)^b)$.
- The case $u = v = 1$ does not happen because A_1 does not split.
- If $u = 1$ and $v = 3$, then $h_1 = 2^\beta \leq 2$, $2^\alpha - 1 \leq b = 3 \cdot 2^\beta - 1$, $\beta = 1$, $\alpha \leq 2$.
- If $u = 1$ and $v = 5$, then $h_5 = 2^\beta \leq 2$, $2^\alpha - 1 \leq b = 5 \cdot 2^\beta - 1$, $\beta = 1$, $\alpha \leq 3$.
- If $u = v = 3$, then h_1 is even and $h_1 = 2^\alpha + 2^\beta \geq 4$, which is impossible.
- If $u = 3$ and $v = 5$, then $h_1 = 2^\alpha \leq 2$, $h_5 = 2^\beta \leq 2$, $\alpha = \beta = 1$.
- If $u = v = 5$, then $h_4 = 2^\alpha \leq 2$, $h_5 = 2^\beta \leq 2$ and $\alpha = \beta = 1$. $\square$

3.3 Maple Computations

According to Lemmas 3.5 and 3.6, we determine, in 4 parts, the set L of all 7-uples $[a, b, h_1, h_2, h_3, h_4, h_5]$ such that $a \leq b$. Then, we search $S = x^a(x+1)^b {M_1}^{h_1} {M_2}^{h_2} {M_3}^{h_3} {M_4}^{h_4} {M_5}^{h_5}$ satisfying: $\sigma^{**}(S) = S$.
The case where $a \geq b$ is obtained from the substitution: $x \longleftrightarrow x+1$.

1) If a and b are even, then $b \in \{0,2,4,6,8,10,12,14\}$ and $h_i \in K_1$. We get (after 6 mn) $\#L = 35000$ and $C_3, C_4, C_8, C_{13}, C_{14}, C_{15}$ as b.u.p polynomials.
2) If a is even and b odd, then $b \in \{0,1,3,5,7,9,11,13,19,23,27,39,55\}$,
$a \in \{0,2,4,6,8,10,12,14\}$ and $h_i \in K_1$.
We get (15 mn): $\#L = 70000$ and as b.u.p polynomials: $C_5, C_9, C_{16}, C_{18}, C_{20}$.
3) If a is odd and b even, then $a \in \{0,1,3,5,7,9,11,13,19,23,27,39,55\}$,
$b \in \{0,2,4,6,8,10,12,14\}$ and $h_i \in K_1$.
We get (5 mn): $\#L = 35000$ and $C_1, C_6, C_{10}, C_{21}, C_{22}, C_{23}$.
4) If a and b are both odd, then $a, b \leq 27$ and $h_i \in K_2$.
Moreover, $h_2 = h_3 = 0$ if $u, v \leq 5$.
We get (30 mn): $\#L = 97500$ and $C_2, C_7, C_{11}, C_{12}, C_{17}, C_{19}$.

The function σ^{**} is defined as Sigm2star

```
> Sigm2star1:=proc(S,a) if a=0 then 1;else if a mod 2 = 0
then n:=a/2:sig1:=sum(S^l,l=0..n):sig2:=sum(S^l,l=0..n-1):
Factor((1+S)*sig1*sig2) mod 2:
else Factor(sum(S^l,l=0..a)) mod 2:fi:fi:end:
> Sigm2star:=proc(S) P:=1:L:=Factors(S) mod 2:k:=nops(L[2]):
for j to k do S1:=L[2][j][1]:h1:=L[2][j][2]:
P:=P*Sigm2star1(S1,h1):od:P:end:
```

References

[1] J. T. B. Beard Jr, *Bi-Unitary Perfect polynomials over $GF(q)$*, Annali di Mat. Pura ed Appl. **149(1)** (1987), 61–68.

[2] E. F. Canaday, *The sum of the divisors of a polynomial*, Duke Math. J. **8** (1941), 721–737.

[3] L. H. Gallardo, O. Rahavandrainy, *There is no odd perfect polynomial over $\mathbb{F}_2$ with four prime factors*, Port. Math. (N.S.) **66(2)** (2009), 131–145.

[4] L. H. Gallardo, O. Rahavandrainy, *Even perfect polynomials over $\mathbb{F}_2$ with four prime factors*, Intern. J. of Pure and Applied Math. **52(2)** (2009), 301–314.

[5] L. H. Gallardo, O. Rahavandrainy, *All unitary perfect polynomials over $\mathbb{F}_2$ with at most four distinct irreducible factors*, Journ. of Symb. Comput. **47** (2012), 492–502.

[6] L. H. Gallardo, O. Rahavandrainy, *On even (unitary) perfect polynomials over $\mathbb{F}_2$* , Finite Fields Appl. **18** (2012), 920–932.

[7] L. H. Gallardo, O. Rahavandrainy, *Characterization of Sporadic perfect polynomials over $\mathbb{F}_2$*, Funct. et Approx. **55(1)** (2016), 7–21.

[8] L. H. Gallardo, O. Rahavandrainy, *All even (unitary) perfect polynomials over $\mathbb{F}_2$ with only Mersenne primes as odd divisors*, arXiv:2202.06357 (2022).

[9] Charles R. Wall, *Bi-Unitary Perfect numbers*, Proceedings of the AMS **33, N. 1** (1972), 39–42.