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Gilles Burel

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Gilles BUREL
LEST, Université de Bretagne Occidentale
BP 809, 29285 Brest cedex, France
Gilles.Burel@univ-brest.fr http://www.univ-brest.fr/lest/tst/

Abstract : — Multi-Input Multi-Output (MIMO) transmission has been a topic of great interest for a few years due to the huge spectral efficiency gain it can provide over rich scattering transmission channels, such as indoor (e.g. wireless local area networks) or urban outdoor (e.g. mobile wireless communications). MIMO transmission channels are usually modelled by random matrices. In this paper, we use results from random matrices theory to derive the statistics of the smallest singular value of a MIMO channel. Indeed, the smallest singular value is of crucial importance for the performances of the transmission system because it determines the minimum distance between the received vectors.

Key-Words : — MIMO, Wireless Digital Transmissions, Random matrices, Wishart matrices, Eigenvalues, Singular Values, Statistics.

1 Introduction

Recent research [2] has shown that very high spectral efficiency can be obtained over rich scattering wireless channels by using multielement antenna arrays at both transmitter and receiver (i.e. MIMO: Multi-Input, Multi-Output transmitters). For instance, an algorithm, now known as BLAST (Bell Laboratories Layered Space-Time), has been proposed: initial laboratory results [3] have shown that spectral efficiencies as high as 20 bits/s/Hz can be obtained. This spectral efficiency is far above the efficiency provided by single antenna transmission systems. The principle of MIMO transmission is as follows: n_T digital transmitters (for instance, QAM transmitters) operate co-channel at symbol rate $1/T$ with synchronized symbol timing. n_R digital receivers ($n_R \geq n_T$) also operate co-channel, with synchronized timing. An algorithm (e.g. [3]) is used to estimate the transmitted symbols from the components of the received mixture. During the last few years, many algorithms have been proposed and evaluated for MIMO transmission systems, and these systems appear as good candidates to improve the transmission efficiency in contexts such as indoor (wireless local areas networks) or urban mobile wireless communications.

When the bandwidth is narrow, the MIMO channel is modelled by an $n_R \times n_T$ random matrix H, and the received vector y (dimension n_R) is given by the equation below:

$$y = Hx + n \tag{1}$$

where n is the noise vector (dimension n_R) and x the transmitted vector (dimension n_T). When the bandwidth is large, Orthogonal Frequency Division Multiplexing (OFDM) [4] can be used to divide the large bandwidth into narrow ones [1]. In each sub-band, the model above is used.

The most widely used model for indoor or urban channels is the Rayleigh model [8]: the entries of H are independent identically distributed circular complex Gaussian random variables with zero mean and variance σ^2.

While a huge amount of work concerning algorithms for MIMO receivers [2] and precoders [9][10], as well as space-time coding and decoding, has been published for a few years, there has been, surprisingly, few works concerning the theoretical statistical study of MIMO channels [11]. However, such works are essential because most MIMO channels are basically random channels. Indeed, MIMO transmission systems are systems with transmit and receive diversity and they are subject to random fading.

The few statistical results available in the literature about MIMO transmission systems are either asymptotic results (i.e. valid for large values of n_R and n_T) and/or results concerning channel capacity. Despite the great theoretical interest of such results, it is clear that

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actual MIMO transmission systems are and will be of limited size (i.e. values of n_R and n_T typically below ten), that is values for which asymptotic results are of little use. Furthermore, in an actual MIMO transmission system, the theoretical capacity is never achieved because it would require a practically unfeasible coding.

In this paper, after explaining that the performances of a MIMO transmission are strongly linked to the smallest singular value of the channel matrix, we use results from random matrices theory to derive the cumulative distribution function of the smallest singular value of H. The originality of this paper is twofold:

- The results obtained are valid for any number of antennas (not only for large numbers);
- Instead of considering channel capacity, we consider the minimum distance between the received vectors. This criterion is more relevant than capacity for actual realizations of MIMO systems.

The paper is organized as follows. In Section 2, we recall a few mathematical results about the Gamma function. In Section 3, we explain why the smallest singular value of matrix H is important to characterize the performances of a MIMO channel. Then, in Section 4, we derive the statistics of the smallest singular value of H. Finally, experimental results are provided in Section 5 to illustrate the approach.

2 A few mathematical recalls about the Gamma function

In this paper, we will obtain mathematical results based on the Gamma function. We recall that the Gamma function is defined as below, for real and strictly positive values of p:

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt \tag{2}$$

This function is widely used in statistics. For example it is used to derive the mathematical expressions of chi-square distributions ([8] p. 42). In fact, the Gamma function interpolates the factorial function, and when p is a positive integer, we have:

$$\Gamma(p) = (p-1)! \tag{3}$$

The incomplete Gamma function is defined as follows:

$$\Gamma_a(p) = \frac{1}{\Gamma(p)} \int_0^a t^{p-1} e^{-t} dt \tag{4}$$

In this paper, in order to simplify the mathematical expressions, we define two additional functions, $\overline{\Gamma}_a(p)$ and $\widetilde{\Gamma}_a(p)$:

$$\overline{\Gamma}_a(p) = 1 - \Gamma_a(p) \tag{5}$$

$$\widetilde{\Gamma}_a(p) = \int_a^\infty t^{p-1} e^{-t} dt \tag{6}$$

$$= \Gamma(p) - \int_0^a t^{p-1} e^{-t} dt \tag{7}$$

$$= \Gamma(p)\left(1 - \Gamma_a(p)\right) \tag{8}$$

$$= \Gamma(p)\overline{\Gamma}_a(p) \tag{9}$$

Please note that the Gamma function, as well as the incomplete Gamma function, are implemented in most scientific software, such as Matlab. Hence, the results obtained in this paper can be easily programmed. Another result that may be useful for people who do not own such software is the expression of the incomplete Gamma function when p is a positive integer:

$$\Gamma_a(p) = 1 - e^{-a} \sum_{k=0}^{p-1} \frac{a^k}{k!} \tag{10}$$

3 Why is the smallest singular value of a MIMO channel so important?

Let us consider a MIMO transmission channel with n_T transmit antennas and n_R receive antennas ($n_R \geq n_T$). The channel is modelled by an $n_R \times n_T$ random matrix H mentioned in the introduction (see Eq. 1). The singular value decomposition (SVD) of matrix H is ([6] p. 76):

$$H = U \Lambda V^* \tag{11}$$

where U is an $(n_R \times n_R)$ unitary matrix, V an $(n_T \times n_T)$ unitary matrix, V^* its conjugate transpose, and Λ the $(n_R \times n_T)$ matrix below:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_{n_T} \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \tag{12}$$

$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n_T} \geq 0$ are the singular values of H. The smallest singular value is $\lambda_{\min} = \lambda_{n_T}$. From matrix algebra, we know that, for any vector x, we have:

$$\|Hx\| \geq \lambda_{\min} \|x\| \quad (13)$$

Let us note $\mathcal{S} = \{s_i\}$ the set of all possible transmitted vectors (i.e. multidimensional constellation). For example, for a BPSK (Binary Phase Shift Keying) signalling and $n_T = 2$ transmitters, we have:

$$\mathcal{S} = \left\{ \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ +1 \end{pmatrix}, \begin{pmatrix} +1 \\ -1 \end{pmatrix}, \begin{pmatrix} +1 \\ +1 \end{pmatrix} \right\} \quad (14)$$

The noise-free received vectors belong to the set $\mathcal{R} = \{r_i\}$ where $r_i = Hs_i$. The probability of error on the receiver side is strongly linked to the minimum distance $d_{\min}$ between the elements of $\mathcal{R}$. Indeed, the maximum likelihood receiver searches the element of $\mathcal{R}$ which is the closest to the actual received vector y. Then, for a symbol error to occur, a necessary condition is that the norm of the noise vector goes above half the minimum distance (i.e. $\|n\| \geq d_{\min}/2$).

The minimum distance is:

$$d_{\min} = \min_{i \neq j} \|r_i - r_j\| \quad (15)$$

$$= \min_{i \neq j} \left\| H \left(s_i - s_j\right) \right\| \quad (16)$$

But, as mentioned above (Eq. 13), we know that:

$$\left\| H \left(s_i - s_j\right) \right\| \geq \lambda_{\min} \left\| s_i - s_j \right\| \quad (17)$$

Hence:

$$d_{\min} \geq \lambda_{\min} \min_{i \neq j} \left\| s_i - s_j \right\| \quad (18)$$

Therefore, if we note d_0 the minimum distance between the elements of $\mathcal{S}$, we have:

$$d_{\min} \geq \lambda_{\min} d_0 \quad (19)$$

where $\lambda_{\min}$ is the smallest singular value of H. This equation shows that a large value of $\lambda_{\min}$ guarantees a large value of $d_{\min}$, and, as a consequence, a low probability of error. Hence, knowing the statistical distribution of $\lambda_{\min}$ is of great importance to characterize a MIMO transmission system.

4 Statistical analysis

The MIMO channel is characterized by an $n_R \times n_T$ random matrix H. The entries of H are independent identically distributed circular complex Gaussian random variables with zero mean and variance σ^2. In the sequel, we will consider $\sigma^2 = 1$. This does not imply any loss of generality, because σ (or σ^2) would just act as a factor in the equations. It will be reintroduced in the final result.

The SVD of matrix H is given by equation 11. Let us consider the $n_T \times n_T$ matrix W below:

$$W = H^* H \quad (20)$$

Since $W = V \left(\Lambda^* \Lambda\right) V^*$, the singular values of H are the square roots of the eigenvalues of W. From random matrices theory, we know that W follows a Wishart distribution. Let us note $n_S = n_R - n_T$. The probability density function of the *unordered* eigenvalues $\mu_1, \ldots, \mu_{n_T}$ is [5][7]:

$$p(\mu_1, \cdots, \mu_{n_T}) = \alpha_{n_T n_S} \left(\prod_{i=1}^{n_T} \mu_i^{n_S} e^{-\mu_i} \right) \prod_{1 \leq i < j \leq n_T} \left(\mu_i - \mu_j\right)^2 \quad (21)$$

where $\alpha_{n_T n_S}$ is a normalization factor. Our objective, here, is to determine the statistics of the smallest eigenvalue $\mu_{\min}$ from which we will then derive the statistics of the smallest channel singular value $\lambda_{\min} = \sqrt{\mu_{\min}}$. However, obtaining the probability density function (pdf) of $\mu_{\min}$ from equation 21 is not trivial because this equation is valid for *unordered* eigenvalues only. The approach we propose to achieve this task is based on:

- The use of the cumulative distribution function (cdf);
- The exploitation of determinants and of their properties;
- The use of the incomplete Gamma function.

First of all, let us use a determinant to express the pdf of the unordered eigenvalues. From equation 21 we can write:

$$p(\mu_1, \cdots, \mu_{n_T}) = \alpha_{n_T n_S} \left(\prod_{i=1}^{n_T} \mu_i^{n_S} e^{-\mu_i} \right) (\det \Theta)^2 \quad (22)$$

where

$$\Theta = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mu_1 & \mu_2 & \cdots & \mu_{n_T} \\ \vdots & \vdots & & \vdots \\ \mu_1^{n_T - 1} & \mu_2^{n_T - 1} & \cdots & \mu_{n_T}^{n_T - 1} \end{pmatrix} \quad (23)$$

Let us note P_{n_T} the set of permutations of $[0, 1, ..., n_T - 1]$ and $k = \left[k_1, ..., k_{n_T}\right]$ an element of P_{n_T}. We recall that the signature $\varepsilon(k)$ of a permutation k is -1 or $+1$ depending on whether the number of transpositions that compose the permutation is odd or even. From the definition of the determinant, we have:

$$\det \Theta = \sum_{k \in P_{n_T}} \varepsilon(k) \prod_{i=1}^{n_T} \mu_i^{k_i} \tag{24}$$

Then:

$$(\det \Theta)^2 = \sum_{k,l \in P_{n_T}} \varepsilon(k)\varepsilon(l) \prod_{i=1}^{n_T} \mu_i^{k_i+l_i} \tag{25}$$

and, finally, using equation 22:

$$p(\mu_1, \cdots, \mu_{n_T}) = \alpha_{n_T n_S} \sum_{k,l \in P_{n_T}} \varepsilon(k)\varepsilon(l) \prod_{i=1}^{n_T} \mu_i^{n_S+k_i+l_i} e^{-\mu_i} \tag{26}$$

We will now use this result to express the cumulative distribution function (cdf) of $\mu_{\min}$. Indeed, the probability that $\mu_{\min}$ is lower than a is $P\left(\mu_{\min} < a\right) = 1 - P\left(\mu_{\min} \geq a\right)$ and we can write:

$$\begin{aligned}
&P\left(\mu_{\min} \geq a\right) \\
&= \int_a^{\infty} \cdots \int_a^{\infty} p(\mu_1, \cdots, \mu_{n_T}) d\mu_1 \cdots d\mu_{n_T} &(27)\\
&= \alpha_{n_T n_S} \sum_{k,l \in P_{n_T}} \varepsilon(k)\varepsilon(l) \prod_{i=1}^{n_T} \int_a^{\infty} \mu_i^{n_S+k_i+l_i} e^{-\mu_i} d\mu_i &(28)\\
&= \alpha_{n_T n_S} \sum_{k \in P_{n_T}} \varepsilon(k) \sum_{l \in P_{n_T}} \varepsilon(l) \prod_{i=1}^{n_T} \widetilde{\Gamma}_a(n_S + k_i + l_i + 1) &(29)\\
&= \alpha_{n_T n_S} \sum_{k \in P_{n_T}} \varepsilon(k) \det\left(A_{n_T,n_S,a,k}\right) &(30)\\
&= n_T!\, \alpha_{n_T n_S} \left|\det(A_{n_T,n_S,a})\right| &(31)
\end{aligned}$$

where $A_{n_T,n_S,a,k}$ and $A_{n_T,n_S,a} = A_{n_T,n_S,a,[0,\ldots,n_T-1]}$ are the $(n_T \times n_T)$ matrices below:

$$A_{n_T,n_S,a,k} = \begin{pmatrix} \widetilde{\Gamma}_a(n_S + k_1 + 1) & \cdots & \widetilde{\Gamma}_a(n_S + k_{n_T} + 1) \\ \widetilde{\Gamma}_a(n_S + k_1 + 2) & \cdots & \widetilde{\Gamma}_a(n_S + k_{n_T} + 2) \\ \vdots & & \vdots \\ \widetilde{\Gamma}_a(n_R + k_1) & \cdots & \widetilde{\Gamma}_a(n_R + k_{n_T}) \end{pmatrix} \tag{32}$$

and

$$A_{n_T,n_S,a} = \begin{pmatrix} \widetilde{\Gamma}_a(n_S + 1) & \cdots & \widetilde{\Gamma}_a(n_R) \\ \widetilde{\Gamma}_a(n_S + 2) & \cdots & \widetilde{\Gamma}_a(n_R + 1) \\ \vdots & & \vdots \\ \widetilde{\Gamma}_a(n_R) & \cdots & \widetilde{\Gamma}_a(n_R + n_T - 1) \end{pmatrix} \tag{33}$$

Since $P\left(\mu_{\min} \geq 0\right) = 1$, we have:

$$\alpha_{n_T n_S}^{-1} = n_T! \left|\det(A_{n_T,n_S,0})\right| \tag{34}$$

Finally, we obtain the cumulative distribution function of the smallest eigenvalue:

$$P\left(\mu_{\min} < a\right) = 1 - \left|\frac{\det(A_{n_T,n_S,a})}{\det(A_{n_T,n_S,0})}\right| \tag{35}$$

and the cdf of the smallest singular value:

$$\begin{aligned}
P\left(\lambda_{\min} < a\right) &= P(\lambda_{\min}^2 < a^2) &(36)\\
&= P\left(\mu_{\min} < a^2\right) &(37)\\
&= 1 - \left|\frac{\det(A_{n_T,n_S,a^2})}{\det(A_{n_T,n_S,0})}\right| &(38)
\end{aligned}$$

If the variance of the elements of matrix H is σ^2 instead of 1, we have:

$$P\left(\lambda_{\min} < a\right) = 1 - \left|\frac{\det(A_{n_T,n_S,(a\sigma)^2})}{\det(A_{n_T,n_S,0})}\right| \tag{39}$$

5 Illustration

5.1 Verification of the theoretical results

First of all, let us check our theoretical results. Figure 1 shows the cumulative distribution function (cdf) of the smallest singular value of a MIMO transmission system with $n_T = 6$ transmit antennas and $n_R = 10$ receive antennas. Here, and in the next subsections, the variance of each element of matrix H is $\sigma^2 = 1$. One curve is the theoretical cdf (Eq. 39), and the other is the estimation of the cdf obtained with 300 random channels. When the number of random channels used for estimation is increased, the estimated cdf converges to the theoretical cdf provided by Eq. 39. This confirms the validity of our theoretical results.

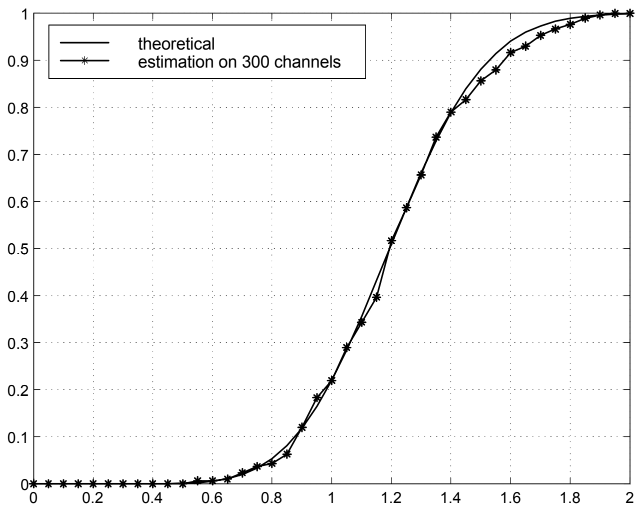

Figure 1: Theoretical and estimated cdf ($n_T = 6$, $n_R = 10$)

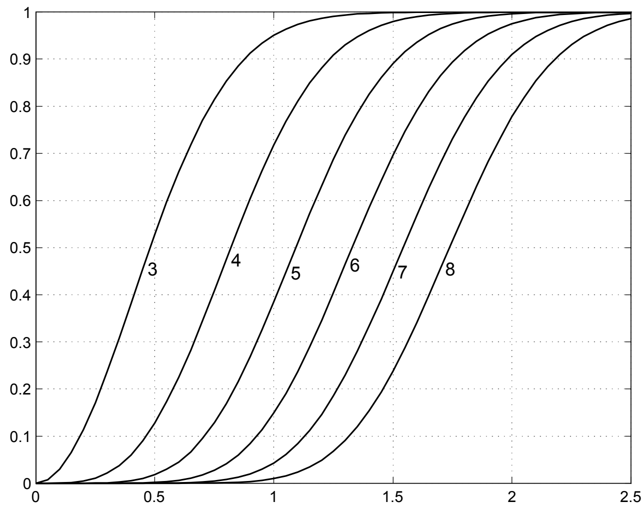

Figure 2: cdf for $n_T = 3$ transmit antennas. The number of receive antennas is $n_R = 3$ to $n_R = 8$ and is mentioned near each cdf

5.2 Impact of the number of receive antennas on the cdf of the smallest singular value

Now, let us have a look on the impact of the number of receive antennas on the cdf of the smallest singular value. For example, let us consider a MIMO transmission system with $n_T = 3$ transmit antennas. Figure 2 shows the cdf obtained when the number of received antennas is increased from $n_R = 3$ to $n_R = 8$ (the figure is obtained using equation 39). When the number of receive antennas is small, the probability to obtain a low value of $\lambda_{\min}$ (and, therefore, the probability to obtain a low value of $d_{\min}$ and a large Bit Error Rate) is not negligible. For example, when $n_R = 3$, the probability to have $\lambda_{\min}$ below 0.2 is equal to 11%. Adding one additional receive antenna ($n_R = 4$) is sufficient to reduce this probability to 0.5%.

Let us take another example. If we want $P(\lambda_{\min} < 1)$ below 10%, figure 2 shows that we must use a MIMO system with, at least, $n_R = 7$ antennas. Hence, these results are useful to help to determine the number of antennas in a MIMO system.

5.3 Determination of the number of receive antennas

Let us take a last example to illustrate a possible exploitation of our theoretical results. Consider a MIMO transmission system with $n_T = 3$ transmit antennas and a QPSK signalling. The noise variance on any receive antenna is $\sigma_n^2 = 2.5 \times 10^{-3}$, that is a standard deviation $\sigma_n = 0.05$. We want to guarantee, on the receiver side, a minimum distance equal to four times the noise standard deviation. By "guarantee", we mean that we want the probability $P\left(d_{\min} < 4\sigma_n\right)$ below 10^{-5}.

Figure 3 shows, with logarithmic scale, the cdf of the smallest singular value for a MIMO transmission system with $n_T = 3$ transmit antennas and $n_R = 3, 4, ..., 8$ receive antennas (the figure is obtained using equation 39).

If the QPSK symbols are $\pm 1 \pm i$, the minimum distance between the possible transmitted vectors is $d_0 = 2$. Using equation 19, we will try to obtain $P\left(d_0 \lambda_{\min} < 4\sigma_n\right) < 10^{-5}$, that is $P\left(\lambda_{\min} < 0.1\right) < 10^{-5}$. Figure 3 clearly shows that the number of receive antennas must be, at least, $n_R = 5$.

6 Conclusion

In this paper, we have used results from random matrices theory to derive the statistics of the smallest singular value of Multiple-Input Multiple-Output transmission channels. The smallest singular value is, indeed, of crucial importance because it is linked to the minimum distance between the received vectors. The obtained results do not need iterative algorithms and are easy to program, since they require only a few matrices manipulations. We have shown that these results can be used to help determining the number of antennas in a MIMO system. This exploitation of the results is, of course, not restrictive. For example, they could be used, also, to provide a good estimation of the Bit Error Rate with-

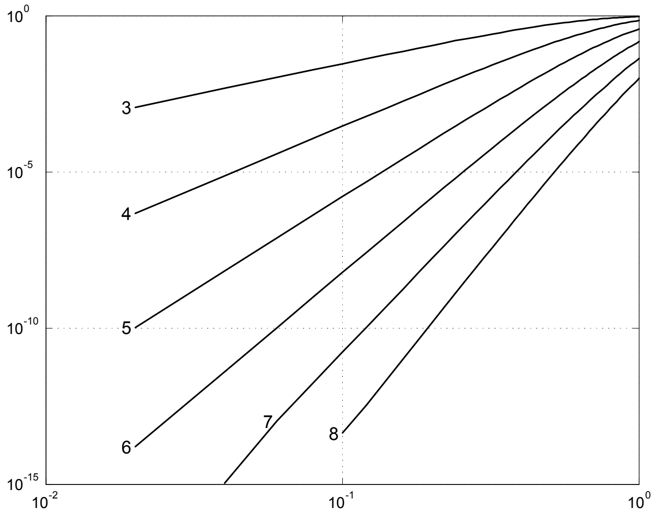

Figure 3: cdf for $n_T = 3$ transmit antennas (logarithmic scale). The number of receive antennas is $n_R = 3$ to $n_R = 8$ and is mentioned near each cdf

out the need of simulation.

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