Theoretical Results for Fast Determination of the Number of Antennas in MIMO Transmission Systems

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ABSTRACT
Multi-Input Multi-Output transmission has been a topic of great interest for the last few years due to the huge spectral efficiency gain it can provide over rich scattering transmission channels, such as indoor (for wireless local area networks) or urban outdoor (for mobile wireless communications). However, since each additional antenna has a cost and requires additional circuits for processing, one of the first question people in charge of realizing actual systems (and not only simulations) usually ask is: what is the required number of antennas? In this paper, we use random matrices theory to derive a theoretical approximation of the error rate with respect to the number of antennas. Thanks to this result, long simulations can be avoided and the required number of antennas can be easily determined.

KEY WORDS
Wireless Digital Transmissions, MIMO, Error Rate, Number of antennas, Random matrices.

1 Introduction

Recent research [1] has shown that very high spectral efficiency can be obtained over rich scattering wireless channels by using multielement antenna arrays at both transmitter and receiver (i.e. MIMO: Multi-Input, Multi-Output transmitters). The principle of MIMO transmission is as follows: n_T digital transmitters (for instance, QAM transmitters) operate co-channel at a given symbol rate with synchronized symbol timing. n_R digital receivers ($n_R \geq n_T$) also operate co-channel, with synchronized timing. On the receiver side, an optimal Maximum-Likelihood algorithm (or a suboptimal, but faster one, such as [2]) is used to estimate the transmitted symbols from the components of the received mixture.

When the bandwidth is narrow, the MIMO channel is modelled by an $n_R \times n_T$ random matrix H, and the received vector y (dimension n_R) is given by the equation below:

$$y = Hx + n \tag{1}$$

where n is the noise vector (dimension n_R) and x the transmitted vector (dimension n_T). The noise covariance matrix is $\sigma^2 I_{n_R}$.

When the bandwidth is large, OFDM can be used to divide the large bandwidth into narrow ones [3], and the model above is used in each subband. Typical applications of MIMO systems are indoor (wireless local areas networks) or urban mobile wireless communications. Using MIMO, spectral efficiencies far above the efficiency provided by single antenna transmission systems can be obtained (e.g. 20 bits/s/Hz [2]). The most widely used model for indoor or urban channels is the Rayleigh model [4]: the entries of H are independent identically distributed circular complex gaussian random variables with zero mean and unit variance. Please note that considering unit variance does not imply any loss of generality because multiplying y by any normalization constant does not modify the error rate. Similarly, the total transmit power is usually normalized to one (i.e. $E\left\{\|x\|^2\right\} = 1$). In this paper, we will also use these usual normalization hypotheses.

A problem that people in charge of designing MIMO systems are faced to is the determination of the number of antennas. Each additional receive antenna improves performances (in terms of error rate), but increases the cost and the complexity of the system. A MIMO system can be simulated, but simulations are extremely long, especially when the number of antennas becomes large and/or when the signal to noise ratio is high.

In this paper, we use random matrices theory to derive a theoretical estimation of the error rate. The probability of error we consider is the probability of vector error, that is the probability that the vector estimated by the Maximum Likelihood receiver is wrong. Thanks to this result long and complex simulations can be avoided.

Application of random matrices theory to MIMO channels still represents a little part of scientific work concerning MIMO transmissions, while a few very recent works (such as [5] for theoretical prediction of capacity), begin to be published. The major part of work in the MIMO transmissions domain is dedicated to space-time coding and receiver or precoder algorithms [6][7]. However, theoretical study of MIMO channels from the random matrices theory point of view is of crucial importance and can provide a lot of new results, because MIMO channel are basically multidimensional random channels.

The paper is organized as follows. In Section 2, we recall a few mathematical results about chi-square distributions and the Gamma function. In Section 3, we explain why the minimum distance between noise-free received vectors is important to characterize the performances of a MIMO channel. Then, in Section 4, we derive the statistical distribution of the minimum distance, and in Section 5 we provide a theoretical approximation of the error rate for channels with a given minimum distance. Finally, in Section 6 we combine results from Sections 4 and 5 to provide a theoretical estimation of the error rate, and we illustrate these results in Section 7.

People who are interested in using the results and not in the proofs can go directly to Section 6 (equation 48).

2 Mathematical recalls

The Gamma function $\Gamma(p)$ and the incomplete Gamma function $\Gamma_a(p)$ are defined below:

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt \tag{2}$$

$$= (p-1)! \qquad \text{(if } p \text{ is a positive integer)} \tag{3}$$

$$\Gamma_a(p) = \frac{1}{\Gamma(p)} \int_0^a t^{p-1} e^{-t} dt \tag{4}$$

$$= 1 - e^{-a} \sum_{k=0}^{p-1} \frac{a^k}{k!} \quad \text{(if } p \text{ is a positive integer)} \tag{5}$$

The probability density function (pdf) of the sum of the squares of m_r independent identically distributed (i.i.d.) real gaussian random variables with zero mean and variance σ_r^2 is a chi-square distribution with m_r degrees of freedom ([4] p. 42):

$$p_r(t) = \frac{1}{\left(2\sigma_r^2\right)^{m_r/2} \Gamma(m_r/2)} t^{m_r/2-1} e^{-t/2\sigma_r^2} \tag{6}$$

It follows that the pdf of the sum of the square moduli of m i.i.d. circular complex gaussian random variables with zero mean and variance σ_c^2 is

$$p_c(t) = \frac{1}{\sigma_c^{2m} \Gamma(m)} t^{m-1} e^{-t/\sigma_c^2} \tag{7}$$

because it is the sum of the squares of $m_r = 2m$ i.i.d. real gaussian random variables with zero mean and variance $\sigma_r^2 = \sigma_c^2/2$. Its cumulative distribution function (cdf) is:

$$F_c(a) = \int_0^a p_c(t) dt \tag{8}$$

$$= \frac{1}{\sigma_c^{2m} \Gamma(m)} \int_0^a t^{m-1} e^{-t/\sigma_c^2} dt \tag{9}$$

$$= \frac{1}{\Gamma(m)} \int_0^a \left(\frac{t}{\sigma_c^2}\right)^{m-1} e^{-t/\sigma_c^2} d\frac{t}{\sigma_c^2} \tag{10}$$

$$= \frac{1}{\Gamma(m)} \int_0^{a/\sigma_c^2} v^{m-1} e^{-v} dv \tag{11}$$

$$= \Gamma_{a/\sigma_c^2}(m) \tag{12}$$

3 Definition of the minimum distance

Let us consider a MIMO transmission channel with n_T transmit antennas and n_R receive antennas ($n_R \geq n_T$), and let us note M the number of symbols in the basic constellation (for instance, $M = 4$ for a QPSK modulation). The channel is modelled by an $n_R \times n_T$ random matrix H mentioned in the introduction (see Eq. 1). We note $\mathcal{S} = \{s_p,\ p = 1, ..., M^{n_T}\}$ the set of all possible transmitted vectors (i.e. multidimensional constellation). As usual, the vectors s_p are assumed to be normalized in order to have a total transmit power equal to 1. That is:

$$\frac{1}{M^{n_T}} \sum_{p=1}^{M^{n_T}} \|s_p\|^2 = 1 \tag{13}$$

For example, for a BPSK (Binary Phase Shift Keying) signalling and $n_T = 2$ transmitters, we have:

$$\mathcal{S} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ +1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} +1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} +1 \\ +1 \end{pmatrix} \right\} \tag{14}$$

From $\mathcal{S}$, we define the set of difference vectors $\widetilde{\mathcal{S}}$. This set contains all the vectors $s_p - s_q$, with $p \neq q$. Hence, $\widetilde{\mathcal{S}} = \{\widetilde{s}_k,\ k = 1, ..., N\}$ with $N = M^{n_T}(M^{n_T} - 1)$ and any $\widetilde{s}_k$ is the difference between two vectors of $\mathcal{S}$.

Let us note d_0 the minimum distance between the elements of $\mathcal{S}$. It is equal to the minimum norm of the vectors of $\widetilde{\mathcal{S}}$. For instance, for the BPSK example mentioned above, we have $d_0 = \sqrt{2}$.

The noise-free received vectors belong to the set $\mathcal{R} = \{r_p,\ p = 1, ..., M^{n_T}\}$ where $r_p = Hs_p$. The probability of error is strongly linked to the minimum distance $d_{\min}$ between the elements of $\mathcal{R}$. Indeed, the maximum likelihood receiver searches the element of $\mathcal{R}$ which is the closest to the actual received vector y. If $d_{\min}$ is small, some vectors of $\mathcal{R}$ are close together and a small noise is sufficient to cause an error. The minimum distance is:

$$d_{\min} = \min_{p \neq q} \|r_p - r_q\| \tag{15}$$

$$= \min_{p \neq q} \left\|H\left(s_p - s_q\right)\right\| \tag{16}$$

$$= \min_k \|H\widetilde{s}_k\| \tag{17}$$

Actual computation of $d_{\min}$ can be time-consuming. Indeed, if we consider, for example, a QAM-16 constellation ($M = 16$), and $n_T = 6$ transmitters, the number of possible noise-free received vectors r_p is $16^6 \simeq 1.7 \times 10^7$ and the number of difference vectors $r_p - r_q$ to consider is approximately the square of this value, that is 2.8×10^{14}.

However, one can notice that the vectors $\widetilde{s}_k$ which are the most likely to provide the minimum distance are vectors whose norm is d_0. Since each transmitter uses the same constellation, it is obvious that, whatever the constellation is, the minimum of $\|s_p - s_q\|$ is provided by vectors s_p and s_q who differ by one component only. Hence, $\|\widetilde{s}_k\| = d_0$ implies that the elements of $\widetilde{s}_k$ are equal to zero, except one element whose value will be noted $d_0 e^{i\theta}$. Then, the general form of a vector $\widetilde{s}_k$ such that $\|\widetilde{s}_k\| = d_0$ is:

$$\widetilde{s}_k = d_0 e^{i\theta} u_m \tag{18}$$

where u_m is the unit vector with the m^{th} component equal to 1 and all other elements equal to 0:

$$u_m = (\underbrace{0 \quad \cdots \quad 0}_{m-1} \quad 1 \quad 0 \quad \cdots \quad 0\ )^T \tag{19}$$

Possible values of θ depend on the basic constellation. For example, for the BPSK case mentioned above, $\theta \in \{0, \pi\}$. For a QPSK, $\theta \in \{0, \pi/2, \pi, 3\pi/2\}$. Finally, since θ has no impact on the norm of $\|H\widetilde{s}_k\|$, we can write:

$$d_{\min} \simeq \min \{d_0 \|Hu_m\|,\ m = 1, ..., n_T\} \tag{20}$$

which is equivalent to:

$$d_{\min} \simeq d_0 \min_{m=1,...,n_T} \|h_m\| \tag{21}$$

where h_m is the m^{th} column of H. This formula is important because it shows that the statistical distribution of $d_{\min}$ is determined by the statistical distribution of the norms of the columns of H.

In the following, we will note m_0 the value of m which provides the minimum distance in equation 21 (hence, $d_{\min} = d_0 \|h_{m_0}\|$). We will also note $\overline{h}_{m_0}$ the normalized vector h_{m_0}, that is $h_{m_0} / \|h_{m_0}\|$.

4 Cumulative distribution function of the minimum distance

A large value of $d_{\min}$ guarantees a low probability of error. Hence, knowing the statistical distribution of $d_{\min}$ is of great importance to characterize a MIMO transmission system and predict its error rate. In this section, we use Equation 21 to derive the statistical distribution of $d_{\min}$.

$\|h_m\|^2$ is the sum of n_R square moduli of complex i.i.d. circular gaussian random variables with variances 1. Hence, it is chi-square distributed and, using equation 7, its pdf is:

$$p_h(t) = \frac{1}{(n_R - 1)!} t^{n_R - 1} e^{-t} \tag{22}$$

Using equation 12, we can see that its cdf is:

$$F_h(u) = P\left(\|h_m\|^2 < u\right) \tag{23}$$
$$= \Gamma_u(n_R) \tag{24}$$

Hence, the cdf of $d_{\min}$ is:

$$F(a) = P(d_{\min} < a) \tag{25}$$
$$= 1 - \prod_{m=1}^{n_T} P\left(d_0^2 \|h_m\|^2 > a^2\right) \tag{26}$$
$$= 1 - \left(1 - F_h\left((a/d_0)^2\right)\right)^{n_T} \tag{27}$$

that is:

$$\boxed{F(a) = 1 - \left(1 - \Gamma_{(a/d_0)^2}(n_R)\right)^{n_T}} \tag{28}$$

where we recall that d_0 is the minimum distance between the possible transmitted vectors.

5 Probability of error for a given minimum distance

In the sequel, $P_e(d_{\min})$ stands for the probability of error, given that the minimum distance is $d_{\min}$.

Let us consider a noise-free received vector Hs_q. As already mentioned, its nearest neighbors (i.e. neighbors at distance $d_{\min}$) are likely to be vectors Hs_p, such that $s_p - s_q = d_0 e^{i\theta} u_{m_0}$. Most errors are due to confusion between the received noisy vector $Hs_q + n$ and one of its nearest neighbors Hs_p. Such an error occurs when:

$$\|(Hs_q + n) - Hs_p\| < \|(Hs_q + n) - Hs_q\| \tag{29}$$

That is:

$$2\,\mathrm{Re}\left\{n^* H\left(s_p - s_q\right)\right\} > \left\|H\left(s_p - s_q\right)\right\|^2 \tag{30}$$

where n^* is the conjugate transpose of the noise vector. We have (for notations used, see Section 3):

$$H\left(s_p - s_q\right) = H d_0 e^{i\theta} u_{m_0} \tag{31}$$
$$= d_0 e^{i\theta} h_{m_0} \tag{32}$$
$$= d_{\min} e^{i\theta} \overline{h}_{m_0} \tag{33}$$

Inserting (33) in (30), we obtain:

$$\mathrm{Re}\left\{e^{i\theta} n^* \overline{h}_{m_0}\right\} > d_{\min}/2 \tag{34}$$

$n^* \overline{h}_{m_0}$ is a circular complex random variable with zero mean and variance σ^2, which will be noted ν. Then, we have:

$$\mathrm{Re}\left\{e^{i\theta} \nu\right\} > d_{\min}/2 \tag{35}$$

If θ can take only one value for a given s_p (this is the case for BPSK), the probability of error is then:

$$P_{e,\inf}(d_{\min}) = P\left(\mathrm{Re}\left\{e^{i\theta}\nu\right\} > d_{\min}/2\right) \tag{36}$$

$$= \frac{1}{2}\,\mathrm{erf}c\left(\frac{d_{\min}}{2\sigma}\right) \tag{37}$$

If θ can take more than one value for a given s_p (this is the case for all constellation other than BPSK), the expression above is a lower bound.

An upper bound is provided by evaluating the probability of the event below:

$$|\nu| > d_{\min}/2 \tag{38}$$

Since ν is a complex circular gaussian random variable with variance σ^2, the cdf of its square modulus is (see Eq. 12 and 5 with $m = 1$):

$$F_{|\nu|^2}(u) = P\left(|\nu|^2 < u\right) \tag{39}$$

$$= \Gamma_{u/\sigma^2}(1) \tag{40}$$

$$= 1 - e^{-u/\sigma^2} \tag{41}$$

The upper bound is then:

$$P_{e,\sup}(d_{\min}) = P\left(|\nu|^2 > (d_{\min}/2)^2\right) \tag{42}$$

$$= 1 - F_{|\nu|^2}\left((d_{\min}/2)^2\right) \tag{43}$$

$$= \exp\left(-\left(\frac{d_{\min}}{2\sigma}\right)^2\right) \tag{44}$$

A good compromise, which provides good results for usual constellations (except BPSK) is:

$$P_e(d_{\min}) = \mathrm{erf}c\left(\frac{d_{\min}}{2\sigma}\right) \tag{45}$$

6 Theoretical estimation of the probability of error

The probability of error can be written:

$$P_e = \int_0^{\infty} p(d_{\min}) P_e(d_{\min}) d d_{\min} \tag{46}$$

where $P_e(d_{\min})$ is the average probability of error for channels whose minimum distance is $d_{\min}$, and $p(d_{\min})$ is the probability density function (pdf) of $d_{\min}$. Using integration by parts, we can write:

$$P_e = [F(d_{\min})P_e(d_{\min})]_0^{\infty} - \int_0^{\infty} F(d_{\min})P_e'(d_{\min}) d d_{\min} \tag{47}$$

where $P_e'(d_{\min})$ the derivative of $P_e(d_{\min})$ with respect to $d_{\min}$ and $F(d_{\min})$ the cumulative distribution function of $d_{\min}$. Since $F(0) = 0$ and $P_e(\infty) = 0$, we have:

$$\boxed{P_e = -\int_0^{\infty} F(d_{\min})P_e'(d_{\min}) d d_{\min}} \tag{48}$$

This integral can be computed numerically. $F(d_{\min})$ is provided by equation 28. $P_e'(d_{\min})$ is given by one of the three equations below (respectively for the lower bound, upper bound, and good compromise):

$$P_{e,\inf}'(d_{\min}) = -\frac{1}{2\sigma\sqrt{\pi}}\exp\left(-\left(\frac{d_{\min}}{2\sigma}\right)^2\right) \tag{49}$$

$$P_{e,\sup}'(d_{\min}) = -\frac{d_{\min}}{2\sigma^2}\exp\left(-\left(\frac{d_{\min}}{2\sigma}\right)^2\right) \tag{50}$$

$$P_e'(d_{\min}) = -\frac{1}{\sigma\sqrt{\pi}}\exp\left(-\left(\frac{d_{\min}}{2\sigma}\right)^2\right) \tag{51}$$

7 Illustration

First of all, let us check our theoretical results. Consider a MIMO transmission system with $n_T = 2$ transmit antennas, $n_R = 3$ receive antennas and BPSK signalling. Figure 1 shows the experimental and theoretical estimations of the probability of error with respect to the *snr*, which is defined as $snr = -20\log_{10}(\sigma)$. We recall that the average total transmit power is normalized to 1, as well as the variance of the elements of matrix H.

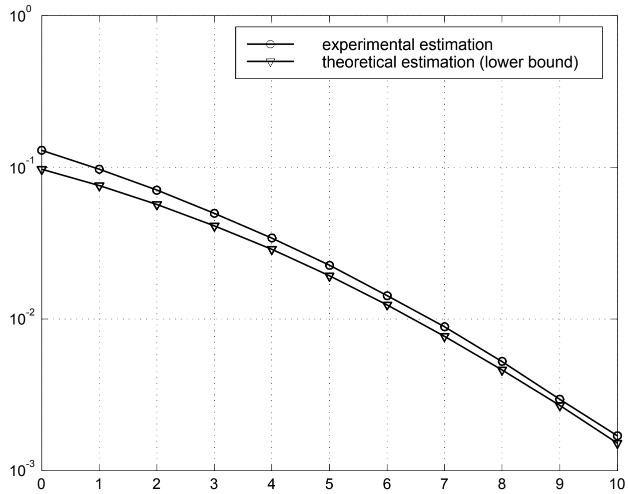

Figure 1. Comparison of the experimental and theoretical estimations of the probability of error ($n_T = 2$, $n_R = 3$). The horizontal axis is *snr*

The theoretical estimation is provided by equation 48 (with equations 28 and 49). The experimental estimation is obtained by averaging over 1000 random matrices H, and 200000 random noise vectors for each H.

Figure 2 shows the theoretical estimation of the probability of error with respect to snr, for $n_T = 3$ transmit antennas and a number of receive antennas varying from

3 to 7 (n_R is mentioned near each curve). These curves are obtained using equation 48 (with equations 28 and 51). This figure is obtained in less than one second with a non optimized Matlab program, while weeks of computations are required to obtain the same results by simulation.

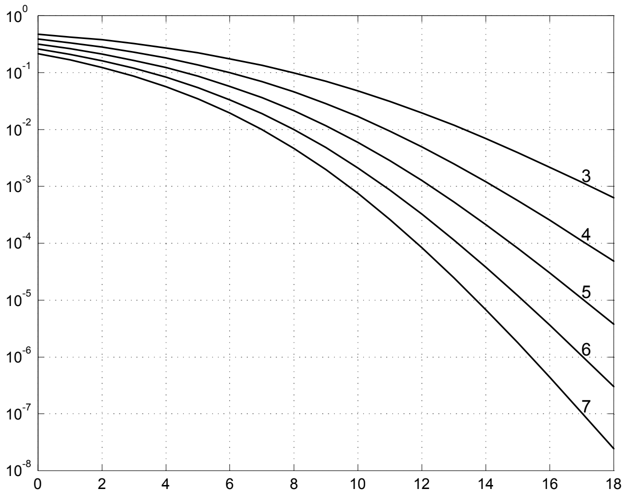

Figure 2. Probability of error with respect to *snr*, for $n_T = 3$ transmit antennas and $n_R = 3$ to 7 receive antennas

We can use these results to determine the required number of antennas. For example, if a probability of error lower than 10^{-4} at $snr = 14dB$ is required, Fig. 2 shows that at least $n_R = 6$ receive antennas are required.

Figure 3 shows the error curves for $n_T = 6$ transmit antennas and a number of receive antennas between 6 and 10. Here again, the figure is obtained is less than one second. In this case, due to the number of transmit antennas, obtaining the same result by simulation, with a sufficient accuracy, is not feasible in reasonable time, unless, maybe, sophisticated importance sampling techniques are used.

Here, if a probability of error lower than 10^{-4} at $snr = 14dB$ is required, Fig. 3 shows that at least $n_R = 9$ receive antennas are required.

8 Conclusion

In this paper we have developed a fast and efficient method to predict the error rate in MIMO transmission systems. These results can be used to determine the required number of antennas in the MIMO system. The method is extremely easy to implement, because it requires only equation 48 with Eq. 28 and 51 (or, alternatively, Eq. 49 or 50).

Using random matrices theory, we derived the statistical distribution of the minimum distance in a MIMO system, from which we then obtained a theoretical expression of the probability of error. Thanks to this result, computationally intensive simulations can be avoided. Furthermore, the theoretical formulae point out the important system parameters, and could be used for further developments.

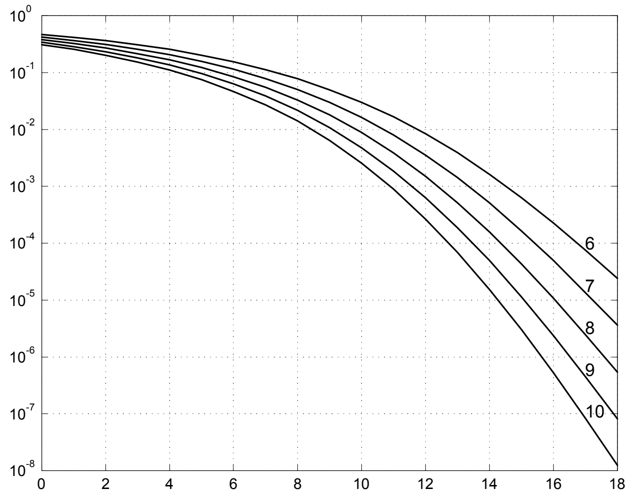

Figure 3. Probability of error with respect to *snr*, for $n_T = 6$ transmit antennas and $n_R = 6$ to 10 receive antennas

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