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Abstract

The paper describes a method for estimation of the orientation of 3D objects without point correspondence information. It is based on decomposition of the object onto a basis of spherical harmonics. Tensors are obtained, and their normalization provides the orientation of the object. Theoretical and experimental results show that the approach is more accurate than the classical method based on the diagonalisation of the inertia matrix. Fast registration of 3D objects is a problem of practical interest in domains such as robotics and medical imaging, where it helps to compare multimodal data.

Keywords

3D Registration, 3D Orientation, Spherical Harmonics, Tensors

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1 Introduction

There is a recent interest in the analysis of 3D images, thanks to the development of 3D sensors and 3D reconstruction techniques, such as scanners, laser rangefinders, computed tomography, cineangiography, and nuclear magnetic resonance imaging. A great deal of work has been devoted to the reconstruction of 3D models from sensors which provide partial 3D information only, such as cineangiography [11], or laser rangefinders. For instance, in [11], the 3D coordinates of coronary artery bifurcation points are obtained by cineangiography. Then, the coefficients of a parametric model of the left ventricle are computed by an optimization algorithm. In [1] and [7], 3D models are recovered from range images, using parametric functions such as Superquadrics. The parameters are tuned by an iterative optimization algorithm. An initial estimate of the rotation of the object is required for the minimization of the cost function: it is obtained by determining the axes of inertia. Another approach to model reconstruction from 3D measurements has been proposed in [5]: thanks to the decoupling of the degrees of freedom, the fitting solution has a simple, closed-form solution.

Given a 3D representation of an object, it is often useful to determine its orientation with respect to a model. For instance, in the medical domain, registration of 3D objects help to compare data taken at different times, or acquired using different sensors. Registration may also be useful for object recognition, while there exists an alternate approach based on the derivation of invariants ([5] [6], for instance). Methods for estimating the orientation of 3D objects have largely focused on polyhedral models [4], and numerous methods need point correspondence information [9] [10]. Another kind of approach is based on the minimization of a distance between the objects to register, with respect to a set of parameters modelizing the 3D transformation. Such approaches avoid the need of correspondence information, and may modelize non-rigid transformations, but they are computationally intensive. The use of genetic algorithms has been proposed recently to speed up the algorithm [3]. Finally, when only rough estimation of the orientation is required, methods based on the diagonalisation of the inertia matrix are used [2].

In this paper, a method which is not restricted to polyhedral objects, and which does not need point correspondence information is proposed. The method is based on the properties of tensors. In this sense, it has some similarities with methods based on the inertia matrix, because the inertia matrix is a second order tensor. However, the method is more accurate than the inertia-based approach, because it uses tensors of rank 1 instead of 2, and also because it avoids multiplications by high powers of the coordinates, as will be proved in section 4. The method is fast because the 3D transformation is computed directly, without iterative search. The 3D object is decomposed onto a basis of spherical harmonics, wherefrom tensors are obtained.

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The normalization of these tensors determines the orientation of the object with respect to a standard position. The input of the method is a 3D representation of the surface of the object. For instance, in the medical domain, such information can be easily derived from scanner data.

The paper is organized as follows. In section 2, the principle and the interest of the representation of a 3D object in the basis of spherical harmonics are presented. Then, the determination of the 3D transformation is explained in section 3. Theoretical results that show the link with the inertia-based method are derived in section 4. Finally, experimental results on a problem of registration of vertebrae are provided in section 5.

In the sequel, we will use the following notations:

- $i = \sqrt{-1}$
- $\mathcal{C}$: the set of complex numbers
- $\mathcal{F_S}$: the space of differentiable functions from $[0,\pi[\text{x}[0,2\pi]$ to $\mathcal{C}$, with finite energy (i.e. $\int d\phi \int \sin\theta \, d\theta |\Psi(\theta,\phi)|^2 < \infty$)
- R^T: the transpose of matrix R
- $\langle u|v\rangle$: the scalar product of vectors u and v
- z^*: the conjugate of a complex number z

2 Decomposition onto the basis of spherical harmonics

To each 3D object, we associate a function $|\Psi\rangle$ of $\mathcal{F_S}$ such that $\Psi(\theta,\phi)$ is the distance between the center of gravity of the object and the farthest point of the object in the direction (θ,ϕ). This kind of representation is usual for 3D objects (see [2], for instance). The conventions for the spherical coordinates are given on figure 1. We have:

$$x = r\,\sin\theta\,\cos\phi \qquad y = r\,\sin\theta\,\sin\phi \qquad z = r\,\cos\theta \tag{1}$$

with $r \geq 0,\ \ 0 \leq \theta \leq \pi,$ and $0 \leq \phi < 2\pi$

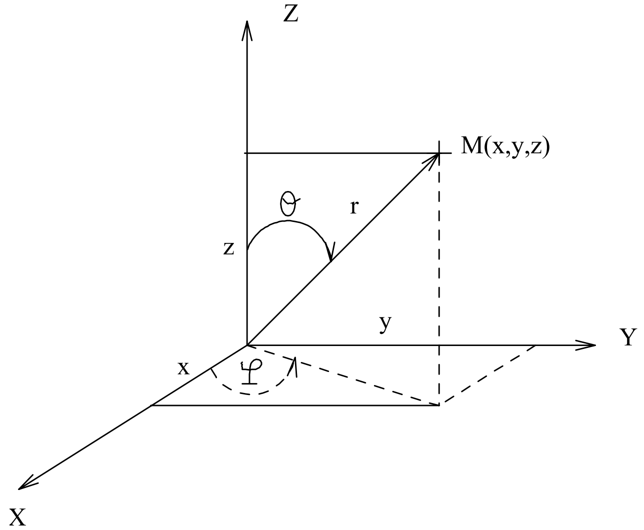

Figure 1: Conventions for the spherical coordinates

Our objective is to obtain tensors of rank 1. Since the dimension of $\mathcal{F}_{\mathcal{S}}$ is infinite, we want to find subspaces of finite dimension which are globally invariant by rotation. Using group theory, it can be proved [12] that the best decomposition of $\mathcal{F}_{\mathcal{S}}$ into invariant subspaces is obtained when the spherical harmonics basis is used.

The spherical harmonics are functions of $\mathcal{F}_{\mathcal{S}}$ which can be computed by recurrence equations (appendix 1). The set of spherical harmonics $\{|Y_{lm}\rangle;\ l = 0, ..., \infty;\ m = -l, ..., l\}$ is an orthonormal basis of $\mathcal{F}_{\mathcal{S}}$. Hence, any function $\Psi(\theta, \phi)$ can be described by its coordinates in this basis:

$$c_l^m = \langle Y_{lm}|\Psi\rangle = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \; d\theta \; Y_{lm}^*(\theta, \phi) \; \Psi(\theta, \phi) \tag{2}$$

The effect of a rotation of the object on these coordinates is given by [12]:

$$\begin{pmatrix} \tilde{c}_0^0 \\ \tilde{c}_1^{-1} \\ \tilde{c}_1^0 \\ \tilde{c}_1^1 \\ \tilde{c}_2^{-2} \\ \tilde{c}_2^{-1} \\ \tilde{c}_2^0 \\ \tilde{c}_2^1 \\ \tilde{c}_2^2 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} D_0 & & & \\ & \begin{pmatrix} D_1 \end{pmatrix} & & \\ & & \begin{pmatrix} D_2 \end{pmatrix} & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} c_0^0 \\ c_1^{-1} \\ c_1^0 \\ c_1^1 \\ c_2^{-2} \\ c_2^{-1} \\ c_2^0 \\ c_2^1 \\ c_2^2 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \tag{3}$$

This equation shows the interest of reasoning in the basis of spherical harmonics instead of the canonical basis: $\mathcal{F}_S$ is decomposed into a direct sum of orthogonal subspaces which are globally invariant by rotation:

- $\mathcal{E}_0$ whose basis is $\{|Y_{00}\rangle\}$
- $\mathcal{E}_1$ whose basis is $\{|Y_{1,-1}\rangle, |Y_{10}\rangle, |Y_{11}\rangle\}$
- $\mathcal{E}_2$ whose basis is $\{|Y_{2,-2}\rangle, |Y_{2,-1}\rangle, |Y_{20}\rangle, |Y_{21}\rangle, |Y_{22}\rangle\}$
- etc

One can prove, using group theory [12], that is is impossible to find a basis in which the rotation operator takes a simpler form. Figure 2 shows the module of some spherical harmonics as a function of the spherical coordinates (θ, ϕ).

Let us define a rotation by Euler angles. A rotation of the coordinates system (x,y,z) is decomposed into 3 elementary rotations: a rotation α around z, which transforms y into u, followed by a rotation β $(0 \leq \beta < \pi)$ around u, which transforms z into Z, and finally a rotation γ around Z. The effect of such a rotation on the c_l^m is [12]:

$$c_l^m(\alpha, \beta, \gamma) = \sum_n D_{mn}^l(\alpha, \beta, \gamma) c_l^n \tag{4}$$

with

$$D_{mn}^l(\alpha, \beta, \gamma) = e^{-i\gamma n}.d_{mn}^l(\beta).e^{-i\alpha m} \tag{5}$$

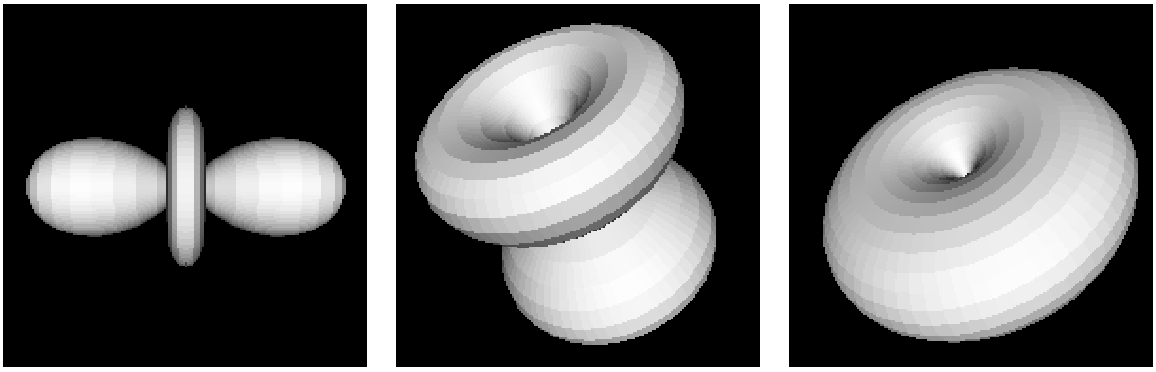

Figure 2: Module of the spherical harmonics Y20, Y21, Y22

and

$$d^l_{mn}(\beta) = \sum_{t=max(0,n-m)}^{min(l+n,l-m)} (-1)^t \frac{\sqrt{(l+n)!(l-n)!(l+m)!(l-m)!}}{(l+n-t)!(l-m-t)!(t+m-n)!t!} \left(\cos\frac{\beta}{2}\right)^{(2l+n-m-2t)} \left(\sin\frac{\beta}{2}\right)^{(2t+m-n)} \tag{6}$$

Symmetry properties can be used to reduce the computations:

$$d^l_{mn}(\beta) = d^l_{-n,-m}(\beta) \qquad \text{and} \qquad d^l_{mn}(\beta) = (-1)^{m+n} d^l_{nm}(\beta) \tag{7}$$

Using basic properties of the trigonometric functions, it can be proved that $d^l_{mn}(\beta)$ is a polynomial of degree l in $(\cos\beta, \sin\beta)$.

Let us note $c = \cos\beta$ and $s = \sin\beta$. In $\mathcal{E}_1$ we have:

$$d^1(\beta) = \begin{pmatrix} \frac{1+c}{2} & -\frac{s}{\sqrt{2}} & \frac{1-c}{2} \\ \frac{s}{\sqrt{2}} & c & -\frac{s}{\sqrt{2}} \\ \frac{1-c}{2} & \frac{s}{\sqrt{2}} & \frac{1+c}{2} \end{pmatrix} \tag{8}$$

In $\mathcal{E}_2$ we have:

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$$d^2(\beta) = \begin{pmatrix} (\frac{1+c}{2})^2 & -\frac{(1+c)}{2}s & \frac{\sqrt{6}}{4}s^2 & -\frac{(1-c)}{2}s & (\frac{1-c}{2})^2 \\ \frac{(1+c)}{2}s & \frac{(1+c)}{2}(2c-1) & -\sqrt{\frac{3}{2}}sc & \frac{(1-c)}{2}(2c+1) & -\frac{(1-c)}{2}s \\ \frac{\sqrt{6}}{4}s^2 & \sqrt{\frac{3}{2}}sc & \frac{3}{2}c^2 - \frac{1}{2} & -\sqrt{\frac{3}{2}}sc & \frac{\sqrt{6}}{4}s^2 \\ \frac{(1-c)}{2}s & \frac{(1-c)}{2}(2c+1) & \sqrt{\frac{3}{2}}sc & \frac{(1+c)}{2}(2c-1) & -\frac{(1+c)}{2}s \\ (\frac{1-c}{2})^2 & \frac{(1-c)}{2}s & \frac{\sqrt{6}}{4}s^2 & \frac{(1+c)}{2}s & (\frac{1+c}{2})^2 \end{pmatrix} \tag{9}$$

3 Determination of the orientation

3.1 Principle of the approach

In the following, we propose a method which determines the rotation that brings the object to a standard orientation characterized by constraints on the tensors c_l^m. Using basic properties of the spherical harmonics, one can prove that $c_l^{-m} = (-1)^m (c_l^m)^*$. Hence, we will consider coefficients with $m \geq 0$ only.

Since we have 3 degrees of fredom, we can try to cancel one complex coefficient plus one imaginary part. Further constraints on the signs of some coefficients will be added to avoid residual ambiguities.

There are many variants of the method, according to the constraints which are chosen. When the constraints are defined in $\mathcal{E}_1$ or $\mathcal{E}_2$, explicit equations can be written and the angles can be easily computed using trigonometric functions. These cases are detailed in sections 3.2 and 3.3. If the constraints are defined in a subspace of higher order, the explicit equations are more difficult to derive. In such cases, exhaustive search of the (α, β) couple that cancels a given coefficient may be required (this search is very fast when l is lower than ten). Then, γ is directly determined in order to cancel the imaginary part of one coefficient. It is also possible to get a first estimation using constraints in a low order $\mathcal{E}_l$, and then to perform fine tuning around the estimated values using a higher order $\mathcal{E}_l$.

The extension to registration is straightforward. Let us note P_i the coordinates of a point P of the object when the object is in orientation i. If R_{ji} denotes the rotation which brings the object from orientation i to orientation j, we have

$$P_j = R_{ji} P_i \tag{10}$$

Hence:

$$R_{21} = R_{02}^T R_{01} \tag{11}$$

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where position 0 is a reference position characterized by constraints on the tensors.

3.2 Cancellation of a coefficient of $\mathcal{E}_1$

We will try to determine the rotation which yields to:

$$\begin{cases} c_1^1(\alpha,\beta,\gamma) & = 0 \\ c_2^1(\alpha,\beta,\gamma) & \textit{real and positive} \end{cases} \tag{12}$$

A rotation α leaves c_1^0 unchanged, and transforms c_1^1 into $c_1^1(\alpha) = e^{-i\alpha}c_1^1$. This coefficient becomes real and positive if α is given by:

$$\alpha = Arg(c_1^1) \tag{13}$$

Then, a rotation β produces:

$$c_1^1(\alpha,\beta) = \cos\beta\, c_1^1(\alpha) + \frac{\sin\beta}{\sqrt{2}} c_1^0(\alpha) \tag{14}$$

This coefficient will be null if:

$$\tan\beta = -\sqrt{2}\frac{c_1^1(\alpha)}{c_1^0(\alpha)} \tag{15}$$

Since β is restricted to $[0,\pi[$, the solution is unique. After the rotations α and β, c_1^1 is cancelled. Hence, a rotation γ has no influence in $\mathcal{E}_1$. However, in $\mathcal{E}_2$ we have:

$$c_2^1(\alpha,\beta,\gamma) = e^{-i\gamma}c_2^1(\alpha,\beta) \tag{16}$$

This coefficient will be real and positive if γ is given by:

$$\gamma = Arg(c_2^1(\alpha,\beta)) \tag{17}$$

The discussion above shows how to take profit of a particular set of constraints. Obviously, other sets of constraints could be chosen. In practice, we usually prefer to take profit of constraints in $\mathcal{E}_2$, because they provide more accurate results: details are given in the next section.

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3.3 Cancellation of a coefficient of $\mathcal{E}_2$

We will try to determine the rotation which yields to:

$$\begin{cases} c_2^1(\alpha,\beta,\gamma) = 0 \\ c_2^2(\alpha,\beta,\gamma) \text{ real, positive and maximal} \\ Re\{c_1^1(\alpha,\beta,\gamma)\} \geq 0 \text{ and } Im\{c_1^1(\alpha,\beta,\gamma)\} \geq 0 \end{cases} \tag{18}$$

The interest of positivity and maximality constraints will appear later: they avoid residual ambiguities.

3.3.1 Determination of α and β

In $\mathcal{E}_2$, the effect of a rotation α is given by $c_2^m(\alpha) = e^{-im\alpha}c_2^m$. Let us note:

$$c_2^0(\alpha) = a_0 \qquad c_2^1(\alpha) = a_1 + ib_1 \qquad c_2^2(\alpha) = a_2 + ib_2 \tag{19}$$

Then, according to equation (9) the effect of a rotation β is given by:

$$c_2^1(\alpha,\beta) = -A\,\sin(2\beta) + a_1\cos(2\beta) + i(b_1\cos\beta - b_2\sin\beta) \tag{20}$$

where $A = (\frac{a_2}{2} - \frac{1}{2}\sqrt{\frac{3}{2}}a_0)$. To cancel $c_2^1(\alpha,\beta)$, we must have:

$$\frac{2tan\beta}{1 - tan^2\beta} = \frac{a_1}{A} \qquad \text{and} \qquad tan(\beta) = \frac{b_1}{b_2} \tag{21}$$

By replacing the second equation in the first one, and assuming $Ab_2 \neq 0$ and $b_1^2 \neq b_2^2$, we get $\mathcal{F}(\alpha) = 0$, where:

$$\mathcal{F}(\alpha) = a_1(b_2^2 - b_1^2) - b_1b_2(a_2 - \sqrt{\frac{3}{2}}a_0) \tag{22}$$

Therefore, α must be a solution of $\mathcal{F}(\alpha) = 0$. The development of equation (22) shows that

$$F(\alpha) = \rho_3\cos(3\alpha + \phi_3) + \rho_1\cos(\alpha + \phi_1) \tag{23}$$

where ρ_1, ρ_3, ϕ_1, ϕ_3 are real numbers. Hence, the number of solutions in the interval $[0,\pi[$ is always comprised between 1 and 3. These solutions can be found by any zero-finding method. Once α is determined, β is given by the second equation of (21). Finally, (α,β) which produces the largest value of $|c_2^2(\alpha,\beta)|$ is kept.

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3.3.2 Determination of γ

A rotation $\gamma \in [0, \pi[$ produces: $c_2^2(\alpha, \beta, \gamma) = e^{-2i\gamma} c_2^2(\alpha, \beta)$. We obtain $c_2^2(\alpha, \beta, \gamma)$ real and positive if:

$$\gamma = \frac{1}{2} Arg(c_2^2(\alpha, \beta)) \quad (24)$$

Until now, we restricted α and γ to $[0, \pi[$. It can be easily proved that, when this restriction is cancelled, we get 3 new candidates: the possible solutions are $\{(\alpha, \beta, \gamma), (\alpha, \beta, \gamma + \pi), (\alpha + \pi, \pi - \beta, \gamma), (\alpha + \pi, \pi - \beta, \gamma + \pi)\}$. The constraint on the sign of the real and imaginary parts of $c_1^1(\alpha, \beta, \gamma)$ determines the solution to keep. In fact, the signs of the real and imaginary parts of any $c_l^1(\alpha, \beta, \gamma)$ with $l \neq 2$ could be constrained for this determination.

4 Link with the inertia matrix

4.1 Principle of inertia-based methods

Let us note I_i the inertia matrix when the object is in orientation i. Using the definition of the inertia matrix, and equation (10), we get:

$$\begin{aligned} I_2 &= E\{P_2 P_2^T\} && (25) \\ &= R_{02}^T I_0 R_{02} && (26) \end{aligned}$$

Usually, the reference orientation is the orientation which produces a diagonal inertia matrix (i.e. alignment with the principal axes). A matrix diagonalisation method (such as Jacobi method, for instance), provides:

$$I_2 = U \Lambda U^T \quad (27)$$

where Λ is a diagonal matrix (the elements of the diagonal are the eigenvalues), and U is a matrix whose columns are the eigenvectors. The eigenvalues are ordered by decreasing magnitude, and the eigenvectors are unitary. Hence, identifying (27) and (26), we get:

$$R_{02} = M\, U^T \quad (28)$$

where M is a diagonal matrix whose diagonal elements belong to $\{-1, +1\}$. The reason of this indetermination lies in the fact that the sign of the eigenvectors cannot be determined. Since R_{02} is a rotation matrix, its determinant must be positive, hence there remain 4 possible solutions. It is impossible to suppress this indetermination using the inertia matrix only. Usually, it is assumed that the rotation is small, in order to suppress the ambiguity: the smallest rotation (among the 4 possibilities) is kept.

When the goal is registration, R_{01} is obtained by the same procedure, and then R_{21} is computed using equation (11).

4.2 Expression of the inertia matrix using Ostrogradsky theorem

In order to show the link between the spherical harmonics approach and the inertia approach, we use Ostrogradsky theorem to derive a surface expression of the moments. According to Ostrogradsky theorem, we have:

$$\int\int\int_{vol} (div\vec{V})d\tau = \int\int_{surf} \vec{V}.\vec{n}.dS \tag{29}$$

where $d\tau$ and dS are the elementary volume and surface, and $\vec{n}$ is the local normal to the surface. The elements of the inertia matrix are the second order moments, hence we will try to find a vector field whose divergence is equal to the moments. A radial vector field is chosen, because it produces the simplest expression of the surface integral. For a radial vector field, the divergence is given by:

$$div\vec{V} = \frac{2}{r}V + \frac{\partial V}{\partial r} \tag{30}$$

where V is the norm of vector $\vec{V}$. The first term of (29) will be a moment if $div\vec{V} = x^\alpha y^\beta z^\gamma$. Hence, we have to solve

$$\begin{aligned} \frac{2}{r}V + \frac{\partial V}{\partial r} &= x^\alpha y^\beta z^\gamma \\ &= r^{\alpha+\beta+\gamma} F_{\alpha,\beta,\gamma}(\theta,\phi) \end{aligned} \tag{31}$$

where $F_{\alpha,\beta,\gamma}(\theta,\phi)$ is easily obtained from equations (1). A solution to the differential equation is:

$$\begin{aligned} V &= \frac{1}{k+3} r^{k+1} F_{\alpha,\beta,\gamma}(\theta,\phi) \\ &= \frac{1}{k+3}\, r\, x^\alpha y^\beta z^\gamma \end{aligned} \tag{32}$$

where $k = \alpha + \beta + \gamma$. Finally, since the elementary surface is:

$$dS = r^2 \sin\theta \, d\theta \, d\phi \tag{33}$$

we get:

$$\int\int\int_{vol} x^\alpha y^\beta z^\gamma dxdydz = \frac{1}{k+3} \int\int_{surf} x^\alpha y^\beta z^\gamma \, r^3 \sin\theta \, d\theta \, d\phi \tag{34}$$

While our purpose is here to show the relationships between two approaches, it should be pointed out that this result may be useful for people who use the inertia-based approach, because it considerably reduces the computation time required to obtain the inertia matrix.

4.3 Link between the spherical harmonics method and the inertia method

In appendix 2, we show that $r^k \, Y_{lm}(\theta, \phi)$ is a homogeneous polynomial of degree k in (x, y, z), when $k - l$ is positive and even. Hence, we can write:

$$r^k \, Y_{lm}(\theta, \phi) = \sum_{\substack{u,v,w \\ u+v+w=k}} a_{u,v,w} x^u y^v z^w \tag{35}$$

Let us represent an object by the function below:

$$\Psi(\theta, \phi) = \frac{r^{k+3}}{k+3} \tag{36}$$

where r is the distance between the center of gravity and the farthest point of the object in direction (θ, ϕ). In order to avoid confusions, the tensor c_l^m corresponding to this representation will be denoted b_l^m. Then, according to equation (2), b_l^m is given by:

$$\begin{aligned}
b_l^m &= \int_0^\pi \int_0^{2\pi} Y_{lm}^*(\theta, \phi) \; \Psi(\theta, \phi) \; \sin\theta \; d\theta d\phi & (37)\\
&= \frac{1}{k+3} \int_0^\pi \int_0^{2\pi} r^k Y_{lm}^*(\theta, \phi) \; r^3 \sin\theta \; d\theta d\phi & (38)\\
&= \frac{1}{k+3} \int_0^\pi \int_0^{2\pi} \left(\sum_{u,v,w} a_{u,v,w}^* x^u y^v z^w \right) r^3 \sin\theta \; d\theta d\phi & (39)\\
&= \frac{1}{k+3} \sum_{\substack{u,v,w \\ u+v+w=k}} a_{u,v,w}^* \int\int\int x^u y^v z^w dxdydz & (40)
\end{aligned}$$

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Hence, b_l^m is a linear sum of moments of order k. As an example, let us show the link that exists between the second order moments and the coefficients b_2^m. We take $k = l = 2$, and, according to (36), the object is represented by $r^5/5$. Let us define:

$$Q_{lm}(x, y, z) = r^l \, Y_{lm}(\theta, \phi) \tag{41}$$

Using the explicit expressions of the spherical harmonics given in appendix 1, one can easily check that:

$$Q_{2,2}(x, y, z) = \sqrt{\frac{3}{2}} h(x + iy)^2 \tag{42}$$

$$Q_{2,1}(x, y, z) = -\sqrt{6} hz(x + iy) \tag{43}$$

$$Q_{2,0}(x, y, z) = h(2z^2 - x^2 - y^2) \tag{44}$$

where $h = \sqrt{\frac{5}{16\pi}}$. Then, using (38), (34) and (41), we can write:

$$b_l^m = \frac{1}{l+3} \int\int Q^*_{lm}(r, \theta, \phi) r^3 \sin\theta d\theta d\phi \tag{45}$$

$$= \int\int\int Q^*_{lm}(x, y, z) dx dy dz \tag{46}$$

we have:

$$b_2^2 = \sqrt{\frac{3}{2}} h(M_{x^2} - M_{y^2} - 2iM_{xy}) \tag{47}$$

$$b_2^1 = \sqrt{6} h(-M_{xz} + iM_{yz}) \tag{48}$$

$$b_2^0 = h(2M_{z^2} - M_{x^2} - M_{y^2}) \tag{49}$$

where $M_{x^2}, M_{y^2}, \ldots$ are the second order moments (i.e. the elements of the inertia matrix).

Conversely, the 6 second order moments can be expressed as functions of b_2^m (i.e. five independent real values). This is because the trace of the inertia matrix is preserved when the object is rotated. Hence, we have an additional equation:

$$M_{x^2} + M_{y^2} + M_{z^2} = constant \tag{50}$$

These results show that the inertia-based approach can be seen as a variant of the spherical harmonics approach, restricted to subspace $\mathcal{E}_2$, and using a highly distorted representation of the object (i.e. $r^5/5$ instead of r). These theoretical results also explain why the inertia-based method can be highly corrupted by noise: it is well known that use of high powers of the coordinates increases the impact of noise. This

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will be confirmed by the experimentations.

Finally, the equations above also help to understand the fundamental indetermination which remains when the inertia-based method is used: the reason is that this method is restricted to $\mathcal{E}_2$. The 4 possibilities mentioned in section 4.1 do not remain undetermined with the spherical harmonics approach because we use an additional constraint on the sign of the real and imaginary parts of c_1^1.

5 Experimental results

Figure 3 illustrates the result obtained for a medical imaging application. We have two 3D images of a vertebra provided by a scanner. They come from the same individual, but with a long delay between both acquisitions. Using the method described above, the second acquisition has been registered with respect to the first one. The angular estimation errors are usually less than half a degree. This registration helps the specialist to compare the 3D images.

The method does not need the determination of specific points for correspondence. Such points could be hard (and computationally expensive) to reliably determine on this kind of shape. Furthermore, because the vertebrae above have similar variances on two principal axes, methods based on the moments of inertia, as in [2], do not apply.

The vertebra to register is originally represented by a 3D voxel matrix of size 100x120x150. Figure 4 shows the error on the estimation of angle β with respect to the error on the estimation of the barycenter position. An error on the estimation of the barycenter could be due to incomplete scanning, for instance. The effect on the estimation of β becomes noticeable when the translation estimation error is about 15 pixels (that is more than 20% of the including sphere radius).

Figure 5 shows the accuracy of angles estimation with respect to the standard deviation of a gaussian noise. Here, a vertebra with different variances along the inertia axes has been used, to allow comparison with the estimation provided by the inertia matrix. The noise is added to the function $\Psi(\theta, \phi)$ representing the vertebra, and it is registered with respect to the uncorrupted version. Hence, the true rotation angles are $\beta = 0$ and $\alpha + \gamma = 0$ (because, when $\beta = 0$, α and γ cannot be separated). Denoting $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ the estimated angles, the graphs show $\sqrt{E\left\{\hat{\beta}^2\right\}}$

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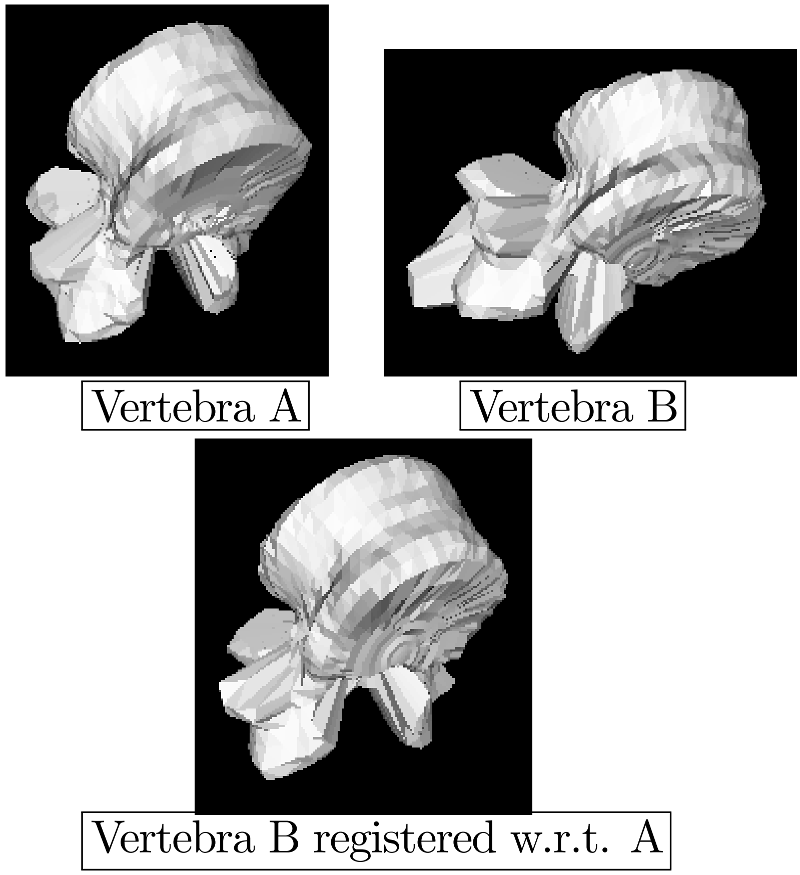

Figure 3: Medical imaging application: registration of a vertebra

and $\sqrt{E\left\{(\hat{\alpha}+\hat{\gamma})^2\right\}}$ with respect to the standard deviation of the noise (measured in pixels). These results show that cancelling a coefficient of $\mathcal{E}_1$ does not provide very accurate estimation, while the cancellation of an $\mathcal{E}_2$ coefficient clearly outperforms the estimation provided by the inertia matrix. The accuracy is even better if fine tuning in $\mathcal{E}_4$ is further performed.

The experimentations above have been done with a discretization step of 5^o for the computation of the c_l^m. Since the spherical harmonics can be precomputed, equation (2) shows that the number of multiplications is $(360/5)(180/5) = 2592$ for each of c_2^0, c_1^0, and 2x2592 $= 5184$ for each of c_2^2, c_2^1, c_1^1 (because in that case the spherical harmonic is complex). Hence, to compute the coefficients required by the method described in section 3.3, the total number of multiplications is 20736. On a standard PC machine realizing more than 1 multiplication per μs, this represents a computation time of $20ms$ only.

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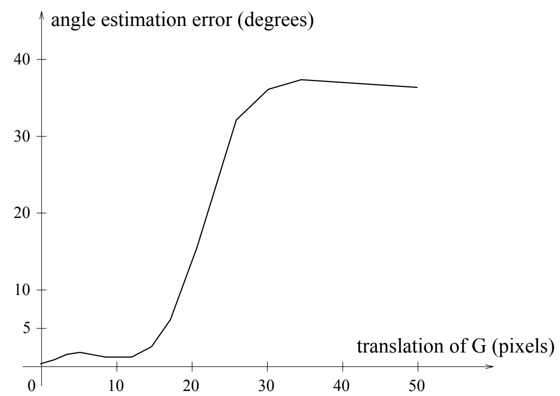

Figure 4: Evaluation of the robustness of the method

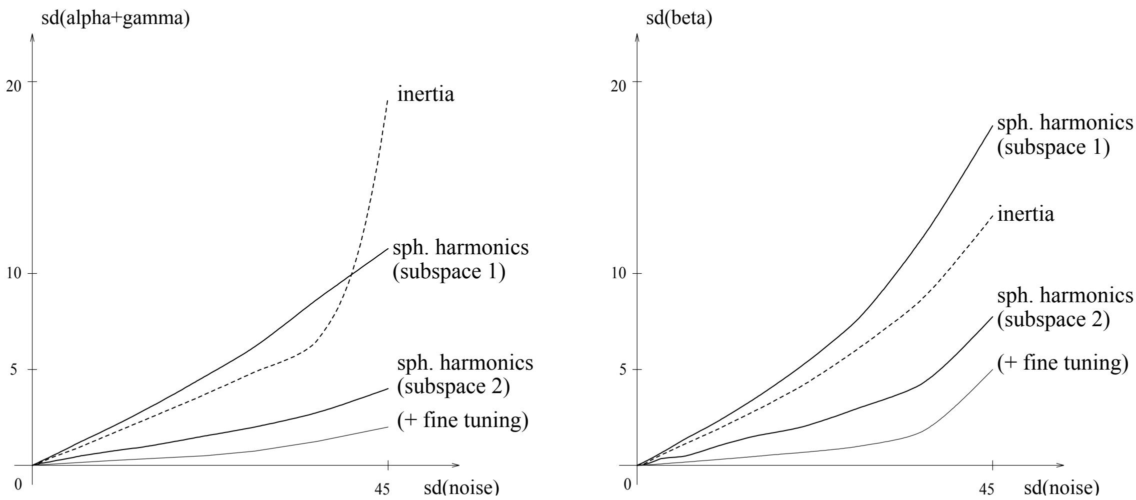

Figure 5: Accuracy with respect to noise

6 Conclusion

A method for determining the orientation of 3D objects has been proposed. It is not restricted to polyhedral objects, it does not need point matching, and it is fast because it is not iterative. The method outperforms the classical approach based on the inertia matrix, and avoids residual ambiguities. Since it needs 3D information on input, it can be applied to any domain in which such an information is available, but it is not appropriate for domains in which only 2D information is available, unless 3D reconstruction by computer tomography can be performed.

Determination of the orientation of 3D objects is a problem of practical interest in medical applications. It allows the registration of 3D data taken at different times or in different conditions. It might also be useful in future medical robotics applications. Since the method is fast and simple it does not require expensive hardware or software.

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Appendix 1: Definition of spherical harmonics

The spherical harmonics have been widely used and studied in quantum mechanics. They are defined as the the eigenvectors of the angular momentum operator:

$$\begin{aligned} \mathbf{L}^2|Y_{lm}\rangle &= l(l+1)|Y_{lm}\rangle \\ L_Z|Y_{lm}\rangle &= m|Y_{lm}\rangle \end{aligned}$$

where $\mathbf{L}^2$ and L_Z are respectively the operators "square of angular momentum" and "projection of angular momentum onto the Z axis". These operators are defined below ($\mathbf{L}^2$ is a Laplacian):

$$\begin{aligned} \mathbf{L}^2 &= -\left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{tg\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \\ L_Z &= \frac{1}{i}\frac{\partial}{\partial\phi} \end{aligned}$$

The spherical harmonics can be computed using recurrence equations [8]:

$$\begin{aligned} Y_{lm}(\theta,\phi) &= Z_{lm}(\theta)e^{im\phi} \text{ where } Z_{lm}(\theta) \text{ is real} \\ Z_{l+1,m}(\theta) &= \sqrt{\frac{(2l+1)(2l+3)}{(l+m+1)(l-m+1)}}\left\{\cos\theta Z_{lm}(\theta) - \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}}Z_{l-1,m}(\theta)\right\} \\ Z_{ll}(\theta) &= d_l(\sin\theta)^l \text{ with } d_l = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}} \\ Z_{l,-m}(\theta) &= (-1)^m Z_{lm}(\theta) \end{aligned}$$

A useful property is $Y_{l,-m}(\theta,\phi) = (-1)^m Y^*_{lm}(\theta,\phi)$

The explicit formulas for the first spherical harmonics are given below:

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$$Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$$

$$Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{2,\pm2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi}$$

$$Y_{2,\pm1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi}$$

$$Y_{2,0}(\theta,\phi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

Appendix 2: Proof that the spherical harmonics are homogeneous polynomials in (x, y, z)

The demonstration uses Legendre polynomials. It can be proved [8] that there exists a link between the spherical harmonics and Legendre Polynomials:

$$Y_{lm}(\theta,\phi) = (-1)^m\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}P_{lm}(\cos\theta)e^{im\phi}$$

This relation is true for $m \geq 0$ (for $m < 0$, the property given in appendix 1 can be used).

$$P_{lm}(u) = \sqrt{(1-u^2)^m}\frac{d^m}{du^m}P_l(u)$$

and $P_l(u)$ is the Legendre Polynomial of order l:

$$P_l(u) = \frac{(-1)^l}{2^l l!}\frac{d^l}{du^l}(1-u^2)^l$$

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P_l is a linear sum of $u^l, u^{l-2}, u^{l-4}, ...$ Hence, $P_{lm}(\cos\theta)$ is the product of $\sin^m\theta$ with a polynomial of degree $l-m$ in $\cos\theta$ (which is a linear sum of $\cos^{l-m}\theta, \cos^{l-m-2}\theta, \cos^{l-m-4}\theta, ...$). Therefore, since $\cos(m\phi)$ and $\sin(m\phi)$ are homogeneous polynomials of degree m in $(\cos\phi, \sin\phi)$, the spherical harmonic Y_{lm} is a linear sum of terms in $\sin^m\theta\,\cos^a\theta\,\sin^{m-b}\phi\,\cos^b\phi$, where $0 \leq a \leq l-m$ (with $l-m-a$ even) and $0 \leq b \leq m$.

Due to the relation between (x, y, z) and (r, θ, ϕ), given in section 2, we obtain:

$$\sin^m\theta\,\cos^a\theta\,\sin^{m-b}\phi\,\cos^b\phi = x^b y^{m-b} z^a / r^{m+a}$$

Now, let us consider $r^k Y_{lm}(\theta, \phi)$, where $k = l + 2n$. This function is a linear sum of terms in $r^{k-m-a}x^b y^{m-b} z^a$. Since $a = l-m-2q$ and $k = l+2n$, $r^{k-m-a} = r^{2(q+n)}$ can be replaced by $(x^2+y^2+z^2)^{q+n}$. Hence, $r^k Y_{lm}(\theta, \phi)$ is a homogeneous polynomial of degree k in (x, y, z) (because the sum of the exponents is [2q + 2n] + b + [m-b] + [l-m-2q] = l + 2n = k).