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Registration of 3D objects using linear algebra

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Abstract. A method for estimating the orientation of 3D objects without point correspondence information is described. It is based on the decomposition of the object onto a basis of spherical harmonics. Tensors are obtained, and their normalization provides the orientation.

1 Introduction

Methods for estimating the orientation of 3D objects have largely focused on polyhedral models [5], and numerous methods need point correspondence information [6]. Another kind of approach is based on the minimization of a distance between the objects to register, with respect to a set of parameters modelizing the 3D transformation. Such approaches avoid the need of correspondence information, and may modelize non-rigid transformations, but they are computationally intensive. The use of genetic algorithms has been proposed recently to speed up the algorithm [3].
In this paper, a method which is not restricted to polyhedral objects, and which does not need point correspondence information is proposed. The method is fast because the 3D transformation is computed directly, without iterative search. The basic idea is to take profit of linear algebra theory. The 3D object is decomposed onto a basis of spherical harmonics, wherefrom tensors are obtained. The normalization of these tensors determines the orientation of the object with respect to a standard position. The input of the method is a 3D representation of the surface of the object. For instance, in the medical domain, such information can be easily derived from scanner data.
The paper is organized as follows. In the next section, the principle and the interest of the representation of a 3D object in the basis of spherical harmonics are presented. Then, the determination of the 3D transformation is explained. Finally, experimental results on a problem of registration of vertebrae are shown.

2 Decomposition onto the basis of spherical harmonics

Let us note $\mathcal{F}_\mathcal{S}$ the space of differentiable functions from $[0,\pi]$x$[0,2\pi]$ to $\mathcal{C}$, with finite energy. To each 3D object, we associate a function $|\Psi\rangle$ such that $\Psi(\theta,\phi)$ is the distance between the center of gravity of the object and the farthest point still belonging to the object in the direction given by the spherical coordinates (θ,ϕ). This kind of representation is usual for 3D medical data [1] [2].
The spherical harmonics are functions of $\mathcal{F}_\mathcal{S}$ which can be computed using Legendre polynomials [4]. In the medical domain, they have been used to represent

cranial surfaces [1]. The set of spherical harmonics $\{|Y_{lm}\rangle;\ l = 0,...,\infty;\ m = -l,...,l\}$ is an orthonormal basis of $\mathcal{F}_{\mathcal{S}}$. Hence, any function $\Psi(\theta,\phi)$ can be described by its coordinates in this basis:

$$c_l^m = \langle Y_{lm}|\Psi\rangle = \int_0^{2\pi} d\phi \int_0^{\pi} sin\theta\ d\theta\ Y_{lm}^*(\theta,\phi)\ \Psi(\theta,\phi) \tag{1}$$

The effect of a rotation of the object on these coordinates is given by:

$$\begin{pmatrix} \tilde{c}_0^0 \\ \tilde{c}_1^{-1} \\ \tilde{c}_1^0 \\ \tilde{c}_1^1 \\ \tilde{c}_2^{-2} \\ \tilde{c}_2^{-1} \\ \tilde{c}_2^0 \\ \tilde{c}_2^1 \\ \tilde{c}_2^2 \\ . \\ . \end{pmatrix} = \begin{pmatrix} D_0 & & & \\ & \begin{pmatrix} & & \\ & D_1 & \\ & & \end{pmatrix} & & \\ & & \begin{pmatrix} & & \\ & D_2 & \\ & & \end{pmatrix} & \\ & & & . \\ & & & & . \end{pmatrix} \begin{pmatrix} c_0^0 \\ c_1^{-1} \\ c_1^0 \\ c_1^1 \\ c_2^{-2} \\ c_2^{-1} \\ c_2^0 \\ c_2^1 \\ c_2^2 \\ . \\ . \end{pmatrix} \tag{2}$$

This equation shows the interest of reasoning in the basis of spherical harmonics: $\mathcal{F}_{\mathcal{S}}$ is decomposed into a direct sum of orthogonal subspaces globally invariant by rotation, such as $\mathcal{E}_2$ whose basis is $\{|Y_{2,-2}\rangle, |Y_{2,-1}\rangle, |Y_{20}\rangle, |Y_{21}\rangle, |Y_{22}\rangle\}$. Using group theory [7], one can prove that is is impossible to find a basis in which the rotation operator takes a simpler form.

Let us define a rotation by Euler angles. A rotation of the coordinates system (x,y,z) is decomposed into 3 elementary rotations: a rotation α around z, which transforms y into u, followed by a rotation β $(0 \leq \beta < \pi)$ around u, which transforms z into Z, and finally a rotation γ around Z. The effect of the rotation on c_l^m is [7]:

$$c_l^m(\alpha,\beta,\gamma) = \sum_n D_{nm}^l(\alpha,\beta,\gamma) c_l^n \tag{3}$$

$$D_{nm}^l(\alpha,\beta,\gamma) = e^{-i\alpha n}.d_{nm}^l(\beta).e^{-i\gamma m} \tag{4}$$

Let us note $c = \cos\beta$ and $s = \sin\beta$. In $\mathcal{E}_2$ we have:

$$d^2(\beta) = \begin{pmatrix} (\frac{1+c}{2})^2 & -\frac{(1+c)}{2}s & \frac{\sqrt{6}}{4}s^2 & -\frac{(1-c)}{2}s & (\frac{1-c}{2})^2 \\ \frac{(1+c)}{2}s & \frac{(1+c)}{2}(2c-1) & -\sqrt{\frac{3}{2}}sc & \frac{(1-c)}{2}(2c+1) & -\frac{(1-c)}{2}s \\ \frac{\sqrt{6}}{4}s^2 & \sqrt{\frac{3}{2}}sc & \frac{3}{2}c^2 - \frac{1}{2} & -\sqrt{\frac{3}{2}}sc & \frac{\sqrt{6}}{4}s^2 \\ \frac{(1-c)}{2}s & \frac{(1-c)}{2}(2c+1) & \sqrt{\frac{3}{2}}sc & \frac{(1+c)}{2}(2c-1) & -\frac{(1+c)}{2}s \\ (\frac{1-c}{2})^2 & \frac{(1-c)}{2}s & \frac{\sqrt{6}}{4}s^2 & \frac{(1+c)}{2}s & (\frac{1+c}{2})^2 \end{pmatrix} \tag{5}$$

3 Determination of the orientation

3.1 Principle of the method

The method determines the Euler angles that rotate the object to a standard orientation characterized by constraints on the tensor c_l^m. Using basic properties of the spherical harmonics, one can prove that $c_l^{-m} = (-1)^m (c_l^m)^*$. Hence, we consider only coefficients with $m \leq 0$. Since we have 3 degrees of freedom, we can, for instance, cancel one complex coefficient, plus one imaginary part. We will try to determine the rotation which yields to:

$$\begin{cases} c_2^{-1}(\alpha,\beta,\gamma) = 0 \\ c_2^{-2}(\alpha,\beta,\gamma) \text{ real, positive and maximal} \\ Re\{c_1^{-1}(\alpha,\beta,\gamma)\} \geq 0 \text{ and } Im\{c_1^{-1}(\alpha,\beta,\gamma)\} \geq 0 \end{cases} \tag{6}$$

We have $c_1^{-1}(\alpha,\beta) = \cos\beta\, Re\left\{c_1^{-1}(\alpha)\right\} + i\, Im\left\{c_1^{-1}(\alpha)\right\} - \frac{\sin\beta}{\sqrt{2}} c_1^0(\alpha)$. The interest of positivity and maximality constraints is to avoid residual ambiguities.

3.2 Determination of α and β

In $\mathcal{E}_2$, the effect of a rotation α is given by $c_2^m(\alpha) = e^{-im\alpha} c_2^m$. Let us note:

$$c_2^0(\alpha) = a_0 \qquad c_2^{-1}(\alpha) = -a_1 + ib_1 \qquad c_2^{-2}(\alpha) = a_2 - ib_2 \tag{7}$$

Then, according to equation (5) the effect of a rotation β is given by:

$$c_2^{-1}(\alpha,\beta) = A\, sin(2\beta) - a_1 cos(2\beta) + i(b_1 cos\beta - b_2 sin\beta) \tag{8}$$

where $A = (\frac{a_2}{2} - \frac{1}{2}\sqrt{\frac{3}{2}} a_0)$. To cancel $c_2^{-1}(\alpha,\beta)$, we must have:

$$\frac{2tan\beta}{1 - tan^2\beta} = \frac{a_1}{A} \qquad \text{and} \qquad tan(\beta) = \frac{b_1}{b_2} \tag{9}$$

By replacing the second equation in the first one, and assuming $Ab_2 \neq 0$ and $b_1^2 \neq b_2^2$, we get $\mathcal{F}(\alpha) = 0$, where:

$$\mathcal{F}(\alpha) = a_1(b_2^2 - b_1^2) - b_1 b_2 (a_2 - \sqrt{\frac{3}{2}} a_0) \tag{10}$$

Then, α must be a solution of $\mathcal{F}(\alpha) = 0$. One can prove that the number of solutions in the interval $[0,\pi[$ is always comprised between 1 and 3. These solutions can be found by any zero-finding method. Once α is determined, β is given by the second equation of (9). Finally, (α,β) which produces the largest value of $|c_2^{-2}(\alpha,\beta)|$ is kept.

3.3 Determination of γ

A rotation $\gamma \in [0,\pi[$ produces: $c_2^{-2}(\alpha,\beta,\gamma) = e^{2i\gamma} c_2^{-2}(\alpha,\beta)$. We obtain $c_2^{-2}(\alpha,\beta,\gamma)$ real and positive if:

$$\gamma = -\frac{1}{2} Arg(c_2^{-2}(\alpha,\beta)) \tag{11}$$

Until now, we restricted α and γ to $[0,\pi[$. One can prove that, when this restriction is cancelled, we get 3 new candidates. Hence, the possible solutions are $\{(\alpha,\beta,\gamma), (\alpha,\beta,\gamma+\pi), (\alpha+\pi,\pi-\beta,\gamma), (\alpha+\pi,\pi-\beta,\gamma+\pi)\}$. The constraint on the sign of the real and imaginary parts of $c_1^{-1}(\alpha,\beta,\gamma)$ determines the solution to keep. In fact, the sign of the real and imaginary parts of any $c_l^{-1}(\alpha,\beta,\gamma)$ with $l \neq 2$ could be used for this determination.

4 Experimental results: registration of vertebrae

Let us note R_{std1}, R_{std2} the rotations which bring objects 1 and 2 to their standard position. Then the rotation between the two objects is given by: $R_{12} = R_{std1}^{-1}.R_{std2}$.
Figure 1 illustrates the result obtained for a medical imaging application. We have two 3D images of a vertebra provided by a scanner. The acquisitions took place at different times. Using the method described above, the second acquisition has been registered with respect to the first one. The residual angular errors are usually less than a degree. This registration helps the specialist to compare the 3D images.

The method does not need the determination of specific points for correspon-

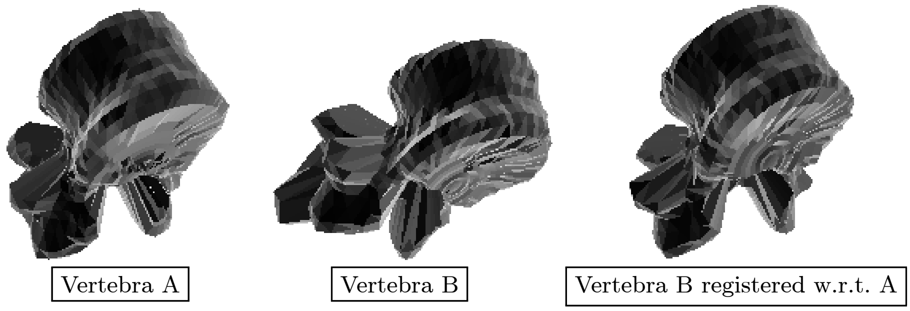

Fig. 1. *Medical imaging application: registration of a vertebra*

dence. Such points could be hard (and computationally expensive) to reliably determine on this kind of shape. Furthermore, because the vertebrae above have similar variances on two principal axes, methods based on the moments of inertia as in [2] do not apply.
The vertebra to register is originally represented by a 3D voxel matrix of size 100x120x150. Figure 2 shows the error on the estimation of angle β with respect to the error on the estimation of the barycenter position. An error on the estimation of the barycenter could be due to incomplete scanning, for instance. The effect on the estimation of β becomes noticeable when the translation estimation error is about 15 pixels (that is more than 20% of the including sphere radius).

The experimentations above have been done with a discretization step of 5^o for the computation of the c_l^m. Since the spherical harmonics can be precomputed, equation (1) shows that the number of multiplications is $(360/5)(180/5) = 2592$ for each of c_2^0, c_1^0, and $2\text{x}2592 = 5184$ for each of $c_2^{-2}, c_2^{-1}, c_1^{-1}$ (because in that case the spherical harmonic is complex). Hence, the total number of multiplications is 20736. On a standard PC machine realizing more than 1 multiplication per μs, this represents a computation time of $20ms$ only.

5 Conclusion

A method for determining the orientation of 3D objects has been proposed. It is not restricted to polyhedral objects, it does not need point matching, and it

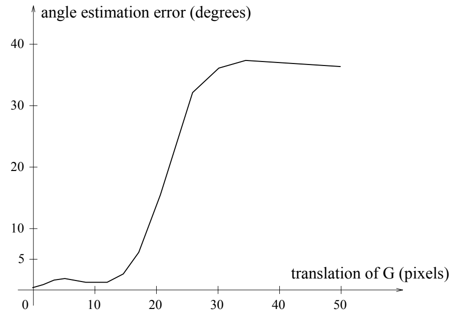

Fig. 2. *Evaluation of the robustness of the method*

is fast because it is not iterative. Since it needs 3D information on input, it can be applied to any domain in which such an information is available, but it is not appropriate for domains in which only 2D information is available, unless 3D reconstruction by computer tomography can be performed. On the theoretical point of view, this work opens new directions of investigation: approaches based on linear algebra and tensor theory instead of structural methods.
Determination of the orientation of 3D objects is a problem of practical interest in medical applications. It allows the registration of 3D data taken at different times or in different conditions. It might also be useful in future medical robotics applications. Since the method[1] is fast and simple it does not require expensive hardware or software.

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