Parameter optimization of orthonormal basis functions for efficient rational approximations

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Abstract: In this paper, the authors present an efficient procedure for optimal placement of poles in rational approximations by Müntz-Laguerre functions. The technique is formulated as the minimization of a quadratic criterion and the linear equations involved are efficiently expressed using the orthonormal basis functions. The presented technique has direct application in rational approximation and model order reduction of large-degree or infinite-dimensional systems.

Keywords: Müntz-Laguerre functions, orthogonal projection, pole placement, model order reduction.

Highlights:

An efficient choice of parameters in orthogonal Müntz-Laguerre approximation

Model order reduction of large-degree or infinite-dimensional systems

The choice of Müntz-Laguerre parameters is based on a least squares optimization

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1. Introduction

Rational orthogonal basis functions (OBF) are useful tools in the identification and modeling of linear dynamical systems and found numerous applications in control and signal processing [1]. In approximation problems using OBF, one of the major difficulties is the choice of the poles defining the functions. Due to their simplicity, Laguerre basis functions are often used. They have a real multiple-order pole whose choice is of great importance for computing low-order and good quality models. Much work has been done on the subject and optimal methods [2-5] or sub-optimal methods [6-8] have been proposed in literature. However Laguerre functions are poorly suited to compact modeling of systems possessing several time constants or resonant characteristics. Two-parameter Kautz functions are more adequate for modeling such systems but their efficiency is also limited. Techniques for an optimal or a suboptimal choice of the two-parameter Kautz poles are respectively presented in [9] and [10].

Finally for an effective approximation with a limited number of functions, the use of a more general orthogonal basis is preferable. Among them, Müntz-Laguerre basis functions that result from orthogonalization of a set of complex exponentials, have interesting properties [11]. Nevertheless few methods exist for properly choosing the poles in generalized OBF approximations. The conditions that the optimal poles of OBF models must satisfy have been investigated in [1, 12] (in the discrete time case). These conditions are of great theoretical interest but generally cannot be solved in practical cases. On the other hand, asymptotically optimal pole locations aim to increase the convergence rate of the norm of the approximation error [1,13]. In this method, the minimization problem involves several independent variables and is faced with local minima and a cost function that is usually not differentiable. A simplistic approach consists in choosing the poles in accordance with the underlying dynamics of the system [14]. Such heuristics are rarely satisfactory in practice.

In a modeling context, the optimal choice of Müntz-Laguerre parameters (poles) is closely related to the model order reduction (MOR) problem whose main objective is the computation of a low order denominator (or, in a state space representation, a low order state matrix) that can be used to efficiently approximate the original system. Most MOR methods require that the original model be in a rational form. The major interest in the use of Müntz-Laguerre functions is that one can deal with original systems described by rational or irrational transfer functions and also by physical measurements in both time and frequency domains. The present paper does not focus on the actual computation of the Müntz-Laguerre expansion but it is important to note that effective technics exist such as the ones described in [1,15].

In the following sections an original method of pole selection for the Müntz-Laguerre functions is presented. It is based on the construction of a family of functions related to the original transfer function and on the minimization of a modified quadratic error criterion. An efficient method for computing the required gramian is also proposed. Gramians in general, have several useful properties [16] that have often attracted the interest of researchers working in various fields of system theory.

The article is organized as follows: Section II introduces the Müntz-Laguerre functions and describes the proposed procedure for parameters optimization. Section III details some properties of the procedure. Section IV illustrates the performance of the method with a variety of numerical examples and comparisons with existing methods. Some demonstrations are available in the appendices.

2. Proposed procedure

2.1. Background

Let the Hardy space $\mathcal{H}_2$ consisting of all analytic and square-integrable functions in the open right half-plane with scalar product

$$\langle F,G\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(i\omega)\overline{G(i\omega)}d\omega, \quad (1)$$

and the norm $\|F\| = \sqrt{\langle F,F\rangle}$. In $\mathcal{H}_2$, Müntz-Laguerre bases are built by orthogonalization of complex exponential functions. They are defined in the Laplace domain by

$$\Phi_n(s) = \frac{\sqrt{2\mathcal{R}\{\alpha_n\}}}{s+\alpha_n} \prod_{l=1}^{n-1} \left(\frac{s-\overline{\alpha}_l}{s+\alpha_l} \right) \quad (2)$$

for $n = 1,2,...,$ with $\mathcal{R}\{\alpha_l\} > 0 \quad \forall l$ where $\overline{\alpha}_l$ denotes the complex conjugate of α_l. Note that taking $\alpha_l = \alpha \in \mathcal{R} \quad \forall l$ yields the 'single parameter' Laguerre functions. Moreover when α_l parameters are grouped by pairs of real or complex conjugate values, Müntz-Laguerre functions can be directly linked to Kautz functions [1].

Müntz-Laguerre functions are orthonormal, $\langle \Phi_n, \Phi_m \rangle = \delta_{n,m}$, and form a complete set in $\mathcal{H}_2$ under an easily satisfied condition [17], [18]. It follows that any strictly proper transfer function $F(s)$ in the Hardy space $\mathcal{H}_2$ could be exactly represented with an infinite Müntz-Laguerre expansion as

$$F_\infty(s) = \sum_{n=1}^{\infty} d_n \Phi_n(s) = F(s), \quad (3)$$

where the expansion coefficients d_n are given by the inner products

$$d_n = \langle F, \Phi_n \rangle. \quad (4)$$

In practice series (3) is truncated

$$F_k(s) = \sum_{n=1}^{k} d_n \Phi_n(s) \quad (5)$$

where $\mathbf{d} = [d_1, d_2, ..., d_k]$ satisfy the optimality condition

$$\min_{\mathbf{d}} \| F - F_k \|. \quad (6)$$

The truncated series (5) defines a k-order rational approximation for $F(s)$ where the first Müntz-Laguerre parameters α_l ($l = 1,2,...,k$) have a great impact on the quality of the model. Nevertheless their optimal choice, minimizing the quadratic error norm, is a nonlinear problem that usually cannot be solved in a simple way.

2.2. Model definition and identification

Due to the nonlinearity of the problem, the major difficulty in rational approximation is to take the 'best' choice for the poles of the model. The original idea developed in this paper is to consider the following expression

$$G(s)=\frac{P(s)}{Q(s)}=\frac{\sum_{k=1}^{R} q_k Z_k^{-1}(s) F_k(s)}{\sum_{k=0}^{R} q_k Z_k^{-1}(s)}, \quad (7)$$

where $q_R = 1$. $Z_k(s)$ are the all-pass filters appearing in the expression of the Müntz-Laguerre functions (2),

$$Z_k(s)=\prod_{l=1}^{k} \frac{s-\overline{\alpha}_l}{s+\alpha_l} \quad (8)$$

and $F_k(s)$ ($k = 1,2,\ldots,R$) are the Müntz-Laguerre approximations of $F(s)$ computed for a predefined set of parameters $\mathbf{a} = \lfloor \alpha_1, \alpha_2, \ldots, \alpha_R \rfloor$. It will be noticed that expression (7) can be simplified by cancelling out the common poles α_l ($l = 1,2,\ldots,R$). It then follows that the R poles of the transfer function $G(s)$ are actually the R zeros of the denominator $Q(s)$.

An interesting property of $G(s)$ is that the numerator $P(s)$ in (7) is defined so that $G(\overline{\alpha}_\lambda) = F(\overline{\alpha}_\lambda)$ for $\lambda = 1,\ldots,R$ (see demonstration in appendix A). Therefore the first R expansion coefficients are identical for $F(s)$ and its rational approximation $G(s)$. It follows that

$$G(s)=\sum_{n=1}^{R} d_n \Phi_n(s) + \sum_{n=R+1}^{\infty} \tilde{d}_n \Phi_n(s). \quad (9)$$

To obtain the best rational R-order model for $F(s)$ we have to solve $J = \min_q \|F - G\|$ for $\mathbf{q} = \left[q_0, q_1, \cdots, q_{R-1}\right]^T$. Nevertheless this is also a nonlinear problem that cannot be easily solved. The difficulty is often circumvented by linearizing the problem as follows

$$\begin{aligned} \breve{J} &= \min_{\mathbf{q}} \|F.Q - P\| \\ &= \min_{\mathbf{q}} \left\| F.\sum_{k=0}^{R} q_k Z_k^{-1} - \sum_{k=1}^{R} q_k Z_k^{-1} F_k \right\| \end{aligned}. \quad (10)$$

At this point, let us define

$$E_k(s) = Z_k^{-1}(s)\left(F(s) - F_k(s)\right), \; k = 0,\ldots,R. \quad (11)$$

For $k = 0$ ones admits by convention that $Z_0(s) = 1$, $F_0(s) = 0$ and thus $E_0(s) = F(s)$. Given (5), (8) and the definition (2) it is straightforward to verify that functions $E_k(s)$ ($k = 0,...,R$) have the energy

$$\|E_k\|^2 = \|F - F_k\|^2 = \sum_{n=k+1}^{\infty} |d_n|^2 \quad (12)$$

and consequently belong to $\mathcal{H}_2$. Using (11) the minimization problem (10) can therefore be recast as

$$\breve{\mathcal{E}} = \min_{\mathbf{q}} \left\| \sum_{k=0}^{R} q_k E_k \right\| \; (q_R = 1). \quad (13)$$

The solution of (13), if it exist, is obtained by solving the linear algebra problem

$$\tilde{\mathbf{u}}\mathbf{q} = -\mathbf{b} \quad (14)$$

where $\tilde{\mathbf{u}}$ is a Gram matrix with elements $\gamma_{l,k} = \langle E_l, E_k \rangle$, $\mathbf{q} = [q_0, q_1, \cdots, q_{R-1}]^T$ and vector $\mathbf{b}$ contains the inner products $b_l = \langle E_l, E_R \rangle = \gamma_{l,R}$. The $\mathbf{q}$ vector solution of (14) then completely defines a rational model (7) for $F(s)$. The poles of $G(s)$ can then be used to define an improved set of parameters for the original Müntz-Laguerre basis.

3. Discussion

3.1. Existence of a unique solution

If $F(s)$ is a strictly proper transfer function in $\mathcal{H}_2$ of degree $R_0 \geq R$ then the solution of (13) is unique. To show that, one can use a proof by contradiction. Suppose that the subset of functions $\Omega_{R-1} = \{E_0, E_1, \cdots, E_{R-1}\}$ are linearly dependent and allow more than one solutions in (13). It follows from the hypothesis that a non-null vector $\mathbf{q} = [q_0, q_1, \cdots, q_{R-1}]^T$ exists such that $\sum_{k=0}^{R-1} q_k E_k(s) = 0$.

Substituting $E_k(s)$ with the definition (11) and rearranging this equation yields an *R-1* order rational expression for $F(s)$, i.e.

$$F(s) = \frac{\sum_{k=1}^{R-1} q_k Z_k^{-1}(s) F_k(s)}{\sum_{k=0}^{R-1} q_k Z_k^{-1}(s)},$$

which contradicts the hypothesis that $F(s)$ is a transfer function of degree $R_0 \geq R$ ✓

3.2. Computational aspect

One of the main advantages of the technique is that it was mathematically designed to exploit the properties of the Müntz-Laguerre basis making the required computations numerically efficient. From (5) and the definition of the Müntz-Laguerre functions (2) one can write

$$\begin{aligned} F_k(s) &= F_{k-1}(s) + d_k \Phi_k(s) \\ &= F_{k-1}(s) + d_k \frac{\sqrt{2\mathcal{R}\{\alpha_k\}}}{s+\alpha_k} Z_{k-1}(s) \end{aligned}. \quad (15)$$

Taking into account that

$$Z_k(s) = \frac{s - \overline{\alpha}_k}{s + \alpha_k} Z_{k-1}(s), \quad (16)$$

it follows that the definition (11) allows to derive

$$E_k(s) = \frac{s+\alpha_k}{s-\overline{\alpha}_k} \left(E_{k-1}(s) - d_k \frac{\sqrt{2\mathcal{R}\{\alpha_k\}}}{s+\alpha_k} \right) \quad (17)$$

for $k = 1,\ldots,R$. Equation (17) clearly shows that $E_k(s)$ functions can be computed recursively from $E_0(s) = F(s)$.

Note that relation (17) is a generalization of the technique already presented by some of the authors in [19] in a context of model order reduction of 'single parameter' Laguerre expansions. The same relation (17) is also exploited in [20] to derive rational models using a different approach.

With a view to compute the inner products $\langle E_k, E_l \rangle$ required in (14), it is useful to consider the Müntz-Laguerre expansions

$$E_k(s) = \sum_{n=1}^{\infty} e_{k,n} \Phi_n(s). \quad (18)$$

Starting with $e_{0,n} = d_n$ for all *n*, the Müntz-Laguerre spectrum of $E_0(s)$, one can be easily and recursively compute those of $E_1(s)$, $E_2(s)$, etc. (see Appendix B for the relationships). Therefore the inner products required in (14) can be practically computed by $\langle E_k, E_l \rangle = \sum_{n=1}^{\infty} e_{k,n} \overline{e_{l,n}}$. Also note the relation already given by (12) allowing the computation of the functions' energies using the expansion coefficients.

3.3. Only the poles of $F(s)$ appear in the minimization problem (13)

Suppose that $E_{k-1}(s)=\dfrac{N_{k-1}(s)}{D(s)}$. Taking $s=\overline{\alpha}_k$ in (11), with (2) in mind, allows to derive

$E_{k-1}(\overline{\alpha}_k)=\dfrac{N_{k-1}(\overline{\alpha}_k)}{D(\overline{\alpha}_k)}=\dfrac{d_k}{\sqrt{2\mathcal{R}\{\alpha_k\}}}$. From (17) it follows that

$$\begin{aligned} E_k(s) &= \frac{(s+\alpha_k)N_{k-1}(s)-d_k\sqrt{2\mathcal{R}\{\alpha_k\}}D(s)}{(s-\overline{\alpha}_k)D(s)} \\ &= \frac{D(\overline{\alpha}_k)(s+\alpha_k)N_{k-1}(s)-2\mathcal{R}\{\alpha_k\}N_{k-1}(\overline{\alpha}_k)D(s)}{D(\overline{\alpha}_k)(s-\overline{\alpha}_k)D(s)}. \qquad (19) \\ &= \frac{N_k(s)}{D(s)} \end{aligned}$$

Hence, one can conclude that recursive relation (17) preserves the poles or natural frequencies of the original function $E_0(s)=F(s)$.

Moreover, from approximation theory and the Bessel inequality it follows that the minimum value for the criterion (10), or (13), is less than the *R*-order optimal Müntz-Laguerre model computed for the initial set of parameters $\mathbf{a}$ i.e.

$$\breve{\mathcal{F}} = \min_{\mathbf{q}} \left\| \sum_{k=0}^{R} q_k E_k \right\| \le \|E_R\| = \|F-F_R\|.$$

3.4. Stability

The stability of the resulting rational transfer function $G(s)$ has not been analytically proved yet. A large number of numerical tests consistently yielding stable models tends to show this property exists. Despite investigations by the authors, the subject is still open. Note that for the special case of single parameter Laguerre functions, i.e. $\alpha_l=\alpha\in\mathcal{R}$, the stability of $G(s)$ was already proven in [19].

The use of Müntz-Laguerre functions instead of Laguerre functions allows a greater flexibility in the set of initial parameters $\mathbf{a}$ but, more important the procedure can be iterated to slightly improve the rational approximation (9).

3.5. Iterative algorithm

An iterative procedure could be established to refine the rational model $G(s)$ and further improve the

Müntz-Laguerre parameters. The algorithm follows.

1: Initialize $\mathbf{a}^{(0)}$

2: While ñnot convergeò do

3: Compute (4) the Müntz-Laguerre spectrum of $F(s)$ relative to $\mathbf{a}^{(t)}$

4: Construct the Gram matrix using (B.2) to (B.4) then solve (14) and construct $G^{(t)}(s)$ with (7)

5: Set $\mathbf{a}^{(t+1)}$ as the opposite of the poles of $G^{(t)}(s)$

6: End while

7: Compute the R-order Müntz-Laguerre model relative to $\mathbf{a}^{(t+1)}$

Remarks: The initial choice $\mathbf{a}^{(0)}$ is not crucial and has mainly an impact on the rate of convergence of the algorithm. In practice (3) will be truncated at an order N and $N >> R$ is a good choice. The algorithm yields R poles. A larger set of Müntz-Laguerre parameters may be obtained by a periodic repetition of the latter with $\alpha_{\lambda R+k} = \alpha_k$ ($k = 1,\ldots,R$ and $\lambda = 1,2,\ldots$). These points are illustrated by examples in section IV.

3.6. Rational model with real impulse response

In numerous applications the rational model must have a real impulse response. To guarantee this property one must start with a vector of Müntz-Laguerre parameters $\mathbf{a}$ composed of real or pairs of complex conjugate values. Once the minimization problem (13) is solved for $\mathbf{q}$, only the real parts of the error term $\sum_{k=0}^{R} q_k E_k$ in (13) or equivalently in (10) is then considered to derive $G(s)$ using (7). Note that in this particular case one may consider using the Kautz basis to implement the algorithm with minimal modifications.

4. Numerical examples

Several examples available in literature were selected in order to illustrate the algorithm. They were already used as a benchmark in two previous papers providing an overview of model order reduction [21], [22]. Table I collects the results of this analysis and allows a comparison in terms of relative quadratic errors $\|F - G\| / \|F\|$. The modeled systems are

System 1: the fourth-order transfer function in [23, ex. 6.1]

System 2: the seventh-order transfer function from [24]

System 3: the fourth-order transfer function in [25, ex. 1]

System 4: the second-order transfer function in [25, ex. 2]

The techniques used to derive the different lower orders are

MLM: the Müntz-Laguerre Model obtained via the algorithm presented above.

IRKA: the iterative rational Krylov algorithm [22]

GFM: the gradient flow method [21]

OPM: the orthogonal projection method [23]

BTM: Balanced truncation method

LPMV: the algorithm proposed in [24]

SMM: the algorithm proposed in [25]

GRSM: the algorithm based on the impulseï response Gramian of the reciprocal system [26]

We note that in terms of relative quadratic error, the proposed method yields results either identical or very close to some of the best MOR techniques already available. Furthermore the initialization is trivial. In all cases identical starting parameters were used, i.e. $\alpha_l^{(0)} = \alpha = 1 \quad \forall l$. The number of iterations required to achieve convergence varies from 2 to 5 for any of the considered systems and the total number of Müntz-Laguerre functions used in the procedure was N = 20 in all cases. Moreover the procedure only requires elementary arithmetic operations and a low order matrix inversion. It follows that the computational cost required to evaluate the inner products in (B.4) and of the method in general tends to be low.

Few methods exist for a proper choice of Müntz-Laguerre parameters. One of the more elaborated is asymptotical optimal pole locations method that aims to decrease the exponential convergence factor of the norm of the approximation error [1,13]. We have applied this method to choose the poles of a Müntz-Laguerre model for the system 1, the relative quadratic errors for model orders R ranging from 3 down to 1 are respectively given by 9.84e-03, 4.23e-01 and 9.17e-01. These results show the inaccuracy of this technics when very low order models are sought-for. Moreover, the knowledge of the poles of the original function are required and, due to the non-differentiability in some search regions of the cost function considered by this method, its minimization must be done numerically.

System	Lower order R	MLM	IRKA	GFM	OPM	BTM	LPMV	SMM	GRSM
1	3	1.3048e-3	1.3047e-3	1.3107e-3	1.3047e-3	1.3107e-3			6.55e-3
	2	3.9449e-2	3.9290e-2	3.9299e-2	3.9290e-2	3.9378e-2			4.51e-2
	1	4.4527e-1	4.2683e-1	4.2709e-1	4.2683e-1	4.3212e-1			5.05e-1
2	6	5.817e-5	5.817e-5	5.817e-5	5.817e-5	5.822e-5	2.864e-4		2.23e-3
	5	2.132e-3	2.132e-3	2.132e-3	Divergent	2.452e-3	2.132e-3		1.55e-2
	4	8.202e-3	8.199e-3	8.199e-3	8.199e-3	8.266e-3	8.199e-3		2.45e-2
	3	1.175e-1	1.171e-1	1.171e-1	Divergent	2.384e-1	1.171e-1		3.37e-2
3	3	5.74e-2	5.74e-2	5.98e-2	5.74e-2	5.99e-2		5.74e-2	4.05e-1
	2	2.527e-1	2.443e-1	2.443e-1	Divergent	3.332e-1		2.443e-1	3.11e-1
	1	5.259e-1	4.818e-1	4.818e-1	4.818e-1	4.848e-1		4.818e-1	5.58e-1
4	1	9.85e-2	9.85e-2	9.85e-2	9.85e-2	9.949e-1		9.85e-2	>1

Table 1: Comparison of relative errors

The second example deals with an infinite dimensional function. This is a far more complex issue. Classical MOR methods that only apply to rational original models are not suitable in this case. Müntz-Laguerre functions constitute an efficient tool in computing a rational approximation. With the algorithm described in the present paper, both the issue of parameter selection and the issue of subsequently deriving a low order model are solved. The considered system is a distributed RC-circuit whose irrational transfer function is [27], [19]

$$F(s)=\frac{1}{1+\frac{\sinh\left(\sqrt{RCs}\right)}{\sqrt{RCs}}}, \text{ with } RC = 1.$$

An $R=4$ order model was purchase for this transfer function. Figure 1 shows that the choice of the initial parameters has little impact on the convergence which is in any case fast. A relative quadratic error of $\|F-G\|/\|F\|=1.25\text{e-}02$ is achieved in all cases regardless of the initial guess. Figure 2 shows that the optimization technique works well for various values of N and furthermore the method converges to the same result. One also noticed that even with a mediocre initial approximation ($N=2R$) the resulting model is accurate and therefore the optimization precision turns out to be effective.

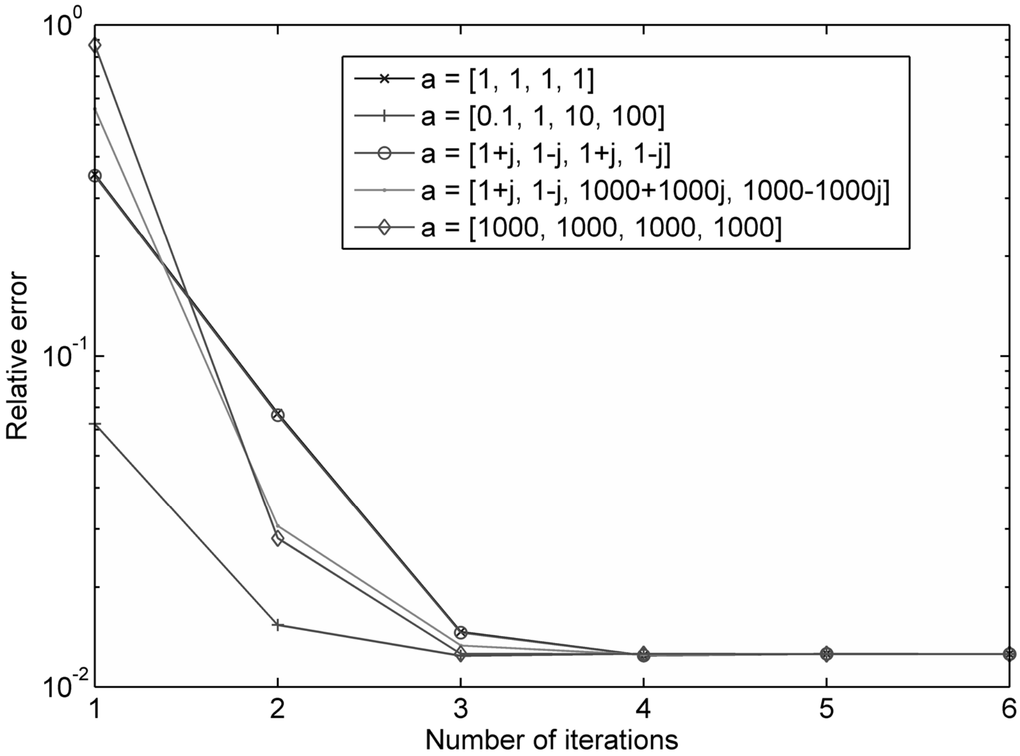

Fig. 1: Evolution of the relative errors for different sets of initial parameters (N = 48)

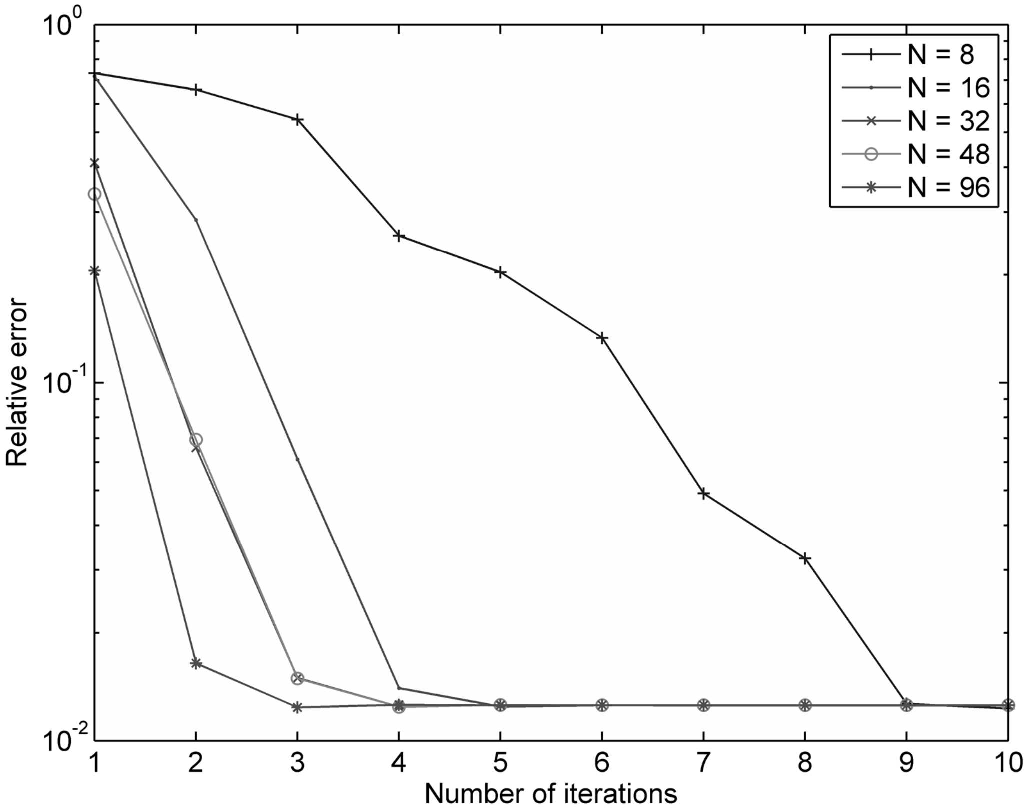

Fig. 2: Evolution of the relative errors for values of N ($\alpha_l^{(0)} = 1 \; \forall l$).

5. Conclusion

An efficient procedure for an optimization of the Müntz-Laguerre functions in rational approximations has been presented. The Müntz-Laguerre parameters are obtained by the minimization of a quadratic error criterion based on the linearization of the original optimization problem. The properties of the basis functions are carefully exploited in order to make the overall algorithm computationally efficient. Illustrative examples have been shown in order to demonstrate the procedure. A particularly interesting field of application is the modeling of infinite-dimensional systems, but the results presented in this paper may prove useful in any application using rational orthogonal bases.

Appendix A

$F(s)$ and $G(s)$ possess the same moments around the expansion points $\overline{\alpha}_\lambda$ ($1 \le \lambda \le R$).

To prove that, normalize $G(s)$ as given in (7) by multiplying the denominator and the numerator by $Z_R(s)$, then substitute $F_k(s)$ and $Z_k(s)$ with their definition (5), (8) and reorganize the summation at the numerator, it follows

$$G(s) = \frac{\sum_{n=1}^{R} d_n \Phi_n(s) \sum_{k=n}^{R} q_k \prod_{l=k+1}^{R} \frac{s-\overline{\alpha}_l}{s+\alpha_l}}{\sum_{k=0}^{R} q_k \prod_{l=k+1}^{R} \frac{s-\overline{\alpha}_l}{s+\alpha_l}}.$$

Now, consider $s = \overline{\alpha}_\lambda$ for $1 \le \lambda \le R$ in $G(s)$. Taking into account that the Müntz-Laguerre functions (2) verify $\Phi_n(\overline{\alpha}_\lambda) = 0$ if $n > \lambda$, and that $\prod_{l=k+1}^{R} \frac{\overline{\alpha}_\lambda - \overline{\alpha}_l}{\overline{\alpha}_\lambda + \alpha_l} = 0$ if $k < \lambda$, simplifying by the null terms yields

$$\begin{aligned} G(\overline{\alpha}_\lambda) &= \frac{\sum_{n=1}^{\lambda} d_n \Phi_n(\overline{\alpha}_\lambda) \sum_{k=\lambda}^{R} q_k \prod_{l=k+1}^{R} \frac{\overline{\alpha}_\lambda - \overline{\alpha}_l}{\overline{\alpha}_\lambda + \alpha_l}}{\sum_{k=\lambda}^{R} q_k \prod_{l=k+1}^{R} \frac{\overline{\alpha}_\lambda - \overline{\alpha}_l}{\overline{\alpha}_\lambda + \alpha_l}} \\ &= \sum_{n=1}^{\lambda} d_n \Phi_n(\overline{\alpha}_\lambda) \\ &= F(\overline{\alpha}_\lambda) \end{aligned}$$

This equality implies that the proposed method retains the moments of $F(s)$ computed around the expansion points $s = \overline{\alpha}_\lambda$ ($1 \le \lambda \le R$) and consequently the *R*-first Müntz-Laguerre coefficients spectrum of $F(s)$ and $G(s)$ computed for $\mathbf{a} = \lfloor \alpha_1, \alpha_2, ..., \alpha_R \rfloor$ are identical.

Appendix B

The computation of the Müntz-Laguerre spectrum of the $E_k(s)$ functions for $k = 1,...,R$ is discussed.

One notes

$$E_k(s) = \sum_{m=1}^{\infty} e_{k,m} \Phi_m(s).$$

From the orthonormal property of Müntz-Laguerre functions and the recursive property of functions $E_k(s)$ (17), it follows

$$\begin{aligned} e_{k,m} &= \langle E_k, \Phi_m \rangle \\ &= \langle z_k^{-1} E_{k-1}, \Phi_m \rangle \qquad , \text{(B.1)} \\ &= \sum_{n=1}^{\infty} e_{k-1,n} \langle z_k^{-1} \Phi_n, \Phi_m \rangle \end{aligned}$$

where $z_k(s) = \dfrac{s - \overline{\alpha}_k}{s + \alpha_k}$. The second part of (17) has no pole in the complex left half-plane and therefore has a null contribution in (B.1). The scalar product in (B.1) is given by

$$\langle z_k^{-1} \Phi_n, \Phi_m \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{i\omega + \alpha_k}{i\omega - \overline{\alpha}_k} \Phi_n(i\omega) \overline{\Phi_m(i\omega)} d\omega,$$

and could be evaluated using Cauchy's residue theorem. It yields

$$\langle z_k^{-1} \Phi_n, \Phi_m \rangle = \begin{cases} 0 & \text{for } n < m \\ \dfrac{\alpha_n - \alpha_k}{\alpha_n + \overline{\alpha}_k} & \text{for } n = m \\ \dfrac{(\alpha_k + \overline{\alpha}_k)\sqrt{\alpha_n + \overline{\alpha}_n}\sqrt{\alpha_m + \overline{\alpha}_m}}{(\overline{\alpha}_k + \alpha_n)(\overline{\alpha}_k + \alpha_m)} \prod_{l=m+1}^{n-1} \dfrac{\overline{\alpha}_k - \overline{\alpha}_l}{\overline{\alpha}_k + \alpha_l} & \text{for } n > m \end{cases}. \text{ (B.2)}$$

In practice, a periodic set of Müntz-Laguerre parameters satisfying $\alpha_{\lambda R+k} = \alpha_k$ ($k = 1,...,R$ and $\lambda = 1,.2,...$) may be used and a useful procedure [15] based on the FFT algorithm can then be employed to compute the Müntz-Laguerre spectrum of $E_0(s) = F(s)$. Moreover the periodicity $\alpha_{\lambda R+k} = \alpha_k$ ($k = 1,...,R$) leads to $\langle z_k^{-1} \Phi_n, \Phi_m \rangle = 0$ for $n > m + R$ and the sum (B.1) is computed over a finite interval, i.e

$$e_{k,m} = \sum_{n=m}^{m+R} e_{k-1,n} \left\langle z_k^{-1} \Phi_n , \Phi_m \right\rangle \text{ (B.3)}$$

with $k = 1,\ldots,R$ and $m = 1,2,\ldots$

Once the Müntz-Laguerre spectrum of each function $E_k(s)$ is computed, the inner products required to solve the minimization problem (13) are evaluated by

$$\left\langle E_k , E_l \right\rangle = \sum_{n=1}^{\infty} e_{k,n} \overline{e_{l,n}} \text{ . (B.4)}$$

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