Pertinent parameters for Kautz approximation

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Abstract

A procedure for determining two parameters to be used in Kautz approximation is presented. It is based on minimisation of an upper bound of the error energy.

Index terms: Orthonormal approximation, Signal representation, Modelling, Mathematical techniques.

Introduction:

Poorly damped systems are difficult to approximate with a reasonable number of Laguerre functions, so the so-called two-parameter Kautz functions which can approximate more efficiently signals with strong oscillatory behavior, have received much attention in the recent mathematical modelling and identification literature (see, e.g., [[1]] and the references therein). These functions can be defined by their Laplace transforms

$$\begin{aligned}
\widehat{\varphi}_{2k}(s) &= \frac{\sqrt{2bc}}{s^2+bs+c}\left(\frac{s^2-bs+c}{s^2+bs+c}\right)^k \\
\widehat{\varphi}_{2k+1}(s) &= \frac{s\sqrt{2b}}{s^2+bs+c}\left(\frac{s^2-bs+c}{s^2+bs+c}\right)^k \\
b &> 0, c>0, k=0,1,2,...
\end{aligned}$$

where the numbering of the functions as defined in [[1]] has been slightly modified for suitability. The time functions are written $\varphi_n(t)$ or as $\varphi_n(t,b,c)$ whenever it is desirable to exhibit the parameters. The orthonormal set $\{\varphi_n\}$ is complete in $L^2[0,\infty[$, thus any finite energy real causal signal $f(t)$ can be approximated within any prescribed accuracy by truncating its infinite expansion $f(t)=\sum_{n=0}^{\infty} a_n\varphi_n(t)$ where $a_n=\langle f,\varphi_n\rangle$ is the $n+1$ th Fourier coefficient. The N-term truncated expansion yields the best approximation to $f(t)$ of the form $\widetilde{f}(t)=\sum_{n=0}^{N-1} a_n\varphi_n(t)$ in the sense of minimising the integrated squared error (ISE)

$$Q=\int_0^{\infty}\left[f(t)-\widetilde{f}(t)\right]^2 dt = \|f\|^2-\sum_{n=0}^{N-1} a_n^2=\sum_{n=N}^{\infty} a_n^2\ . \tag{1}$$

Usually, since the a_n depend on b and c, Q can be reduced further by a proper choice of these parameters. Nice optimality conditions for Kautz approximation, generalizing that of the Laguerre case [[2], [3]], have been derived by Oliveira e Silva [[4]] and den Brinker [[5]]. However, these conditions of great theoretical interest can result in complicated computations in practical cases. For Laguerre functions [[6], [7]] and other classical functions [[8], [9]] an alternative easy-to-use and efficient approach, based on minimisation of an upper bound of the error energy, has been proposed. It is the purpose of this Letter to derive a somewhat similar procedure for the specific set of non-classical two-parameter Kautz functions.

Key relationship:

Recently [[1]], it has been shown that the coefficients a_n can be found from power series calculations in the following manner. Denoting by $\widehat{f}(s)$ the Laplace transform of $f(t)$, assumed to be analytic outside an appropriate region in the s-plane, let F_i, $i=1,2$, be defined by

$$F_1(s)=\left[s^2\widehat{f}(s)-c\widehat{f}(c/s)\right]/\left(s^2-c\right) \tag{2}$$

$$F_2(s)=\left[\widehat{f}(c/s)-\widehat{f}(s)\right]s\sqrt{c}/\left(s^2-c\right)\ . \tag{3}$$

Since $F_i(c/s)=F_i(s)$, both these functions are symmetric functions of c/s and s and so they can be represented as functions of $(c/s)+s$ and $(c/s)s=c$, whence

$$F_i(s)=\widehat{f}_i(s+c/s\,,c)\quad,\quad i=1,2\ . \tag{4}$$

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Notice that the transformation $s \to s + c/s$ is familiar in filter design where it is used to design a band-pass filter from a low-pass filter. The trick to relate Kautz coefficients and power series is to observe the remarkable relationship

$$\widehat{f_i}(s,c) = \sum_{n=0}^{\infty} a_{2n+2-i} \widehat{l_n}(s,b) \quad , \quad i = 1,2 \tag{5}$$

where $\widehat{l_n}(s,b) = \sqrt{2b}\,(s-b)^n \Big/ (s+b)^{n+1}$ denotes the Laplace transform of the normalised Laguerre function $l_n(bt)$. Thus, the Fourier coefficients associated with the expansion of $f(t)$ with respect to the orthonormal set $\{\varphi_n\}$ can be obtained via Laguerre expansions.

Proposed procedure for pertinent parameters:

Denoting by $f_i(t,c)$ the inverse Laplace transform of $\widehat{f_i}(s,c)$, let us define moments M_j by

$$M_0 = \int_0^{\infty} \left[(f_1)^2 + (f_2)^2 \right] dt \tag{6}$$

$$M_1(c) = \int_0^{\infty} t \left[(f_1)^2 + (f_2)^2 \right] dt \tag{7}$$

$$M_2(c) = \int_0^{\infty} t \left[\left(\frac{df_1}{dt} \right)^2 + \left(\frac{df_2}{dt} \right)^2 \right] dt \tag{8}$$

where we have used f_i as shorthand for $f_i(t,c)$. Since the Laguerre functions are orthonormal and the Kautz functions are orthonormal also, both M_0 and $\|f\|^2$ are equal to $\sum_{n=0}^{\infty} a_n^2$, hence $M_0 = \|f\|^2$ is a constant. On the other hand, M_1 and M_2 depend on c but do not depend on b.

Theorem 1:

Let $q = Q \Big/ \|f\|^2$, $m_i(c) = M_i(c) \Big/ \|f\|^2$, $i = 1,2$. Then, the normalised ISE associated with a $2K$-term Kautz approximation is bounded by

$$q \le B = \frac{1}{2K} \left[\frac{m_2(c)}{b} + b m_1(c) - 1 \right]. \tag{9}$$

This bound attains its minimum when $b = \sqrt{m_2(c)/m_1(c)}$. The minimum itself is $B_{min} = \left(2\sqrt{m_1(c)\, m_2(c)} - 1 \right)/(2K)$.

Proof:

Let $M_j = M_{j1} + M_{j2}$ where M_{ji} denotes the contribution of f_i ($M_{01} = \int_0^{\infty} (f_1)^2 dt$, ...). Then, the ISE $Q_i = \sum_{n=K}^{\infty} a_{2n+2-i}^2$ associated with the K-term Laguerre approximation of $f_i(t,c)$ (see eqn. 5) is known [[6]] to be bounded by $(M_{2i}/b + M_{1i}b - M_{0i})/(2K)$, provided that $(2K+1) \ge (M_{2i}/b + M_{1i}b)/M_{0i}$, $i = 1,2$, a condition which is assumed to hold in the following (K is sufficiently large). In view of eqn. 1 the ISE associated with the ($N = 2K$)-term Kautz approximation of $f(t)$ is $Q = Q_1 + Q_2$ and can then be bounded as $Q \le (M_2/b + M_1 b - M_0)/(2K)$. Dividing throughout by $\|f\|^2 = M_0$ achieves the proof of eqn. 9. Writing $\partial B/\partial b = 0$, the last part of the theorem follows readily.

For a fixed $c > 0$, let $\mathbf{C} = \mathbf{C}(c; m_1, m_2)$ denote the class of signals $f \in L^2[0,\infty[$ with given $m_1(c) = m_1$ and $m_2(c) = m_2$. There exist signals $f \in \mathbf{C}$ that achieve the bound in eqn. 9; as a simple example, consider $\mathbf{C}(5; 0.4, 1.6)$: it is a standard exercise to show that $f(t) = 3\varphi_0(t,2,5) + \varphi_6(t,2,5)$ is in this class. Clearly, the 6-term Kautz approximation using $\varphi_n(t,2,5)$ is $\widetilde{f}(t) = 3\varphi_0(t,2,5)$ with $q = 0.1$ and $B = (1.6/2 + 2 \times 0.4 - 1)/6 = 0.1$, whence $q = B$. Therefore, the bound in eqn. 9 is actually the maximum ISE for signals in $\mathbf{C}$ and theorem 1 gives the best b, in the sense of minimising the maximum integrated squared error, that can be obtained with the knowledge of the signal limited to $m_1(c)$ and $m_2(c)$.

Now, suppose that $m_1(c)$ and $m_2(c)$ are known for more than one value of c, say for $c \in C$ where C represents a discrete or continuous set of positive numbers. Since the lowest $m_1(c)\, m_2(c)$ will result in the lowest B_{min}, we have the following theorem.

Theorem 2:

Let c_0 denote that value of $c \in C$ at which the product $m_1(c)\, m_2(c)$ is minimum and let $b_0 = \sqrt{m_2(c_0)/m_1(c_0)}$. Then, a pertinent choice for the pair of Kautz parameters is (b_0, c_0), which yields $(B_{min})_0 = \left(2\sqrt{m_1(c_0)\, m_2(c_0)} - 1 \right)/(2K)$.

Remark:

Notice that b_0 and c_0 do not depend on the number $N = 2K$ of functions to be used. Thus b_0 and c_0 can be computed in a first time and N can be chosen afterwards: for instance, one can choose N such that the upper bound $(B_{min})_0$ is small enough or such that the exact $q = 1 - \sum_{n=0}^{N-1} a_n^2 \Big/ \|f\|^2$ is small enough.

Illustrative example:

Consider the Laplace transform

$$\widehat{f}(s) = \frac{s^3 + 4s^2 + 8s + 1}{s^4 + 5s^3 + 13s^2 + 19s + 18}$$

with a view to deriving a second-order approximation ($N = 2$ Kautz functions). Letting for example $c = 4$, eqns. 2-4 yield

$$\widehat{f_1}(s, 4) = \frac{9s^3 + 72s^2 + 182s + 127}{9s^4 + 83s^3 + 267s^2 + 349s + 164}$$

$$\widehat{f_2}(s, 4) = \frac{s^3 + 19s^2 + 81s + 87}{9s^4 + 83s^3 + 267s^2 + 349s + 164}$$

Using one of the available techniques (e.g. [[10]]), the required moments are computed as $M_0 = \|f\|^2 = 0.5183$, $M_1(4) = 0.2531$, $M_2(4) = 0.3278$ and the error bound is minimised when $b = \sqrt{m_2(4)/m_1(4)} = \sqrt{M_2(4)/M_1(4)} = 1.138$. With $b = 1.138$ and $c = 4$, the first and second coefficients of the Kautz expansion are $a_0 = 0.2965$ and $a_1 = 0.6365$ from which the exact normalised ISE is obtained as $q = 4.878 \times 10^{-2}$.

The normalised moments computed by repeating the procedure for $c = 2$ and $c = 3$ are shown in Table 1. For $c \in \{2, 3, 4\}$, the product $m_1(c)\, m_2(c)$ is minimum if $c = 3$; therefore, in agreement with theorem 2, we select $b_o = \sqrt{0.4980/0.5138} = 0.9846$ and $c_o = 3$, improving the normalised ISE which becomes $q = q_o = 2.505 \times 10^{-3}$. It is worth noting that q_o obtained using limited knowledge of the signal (Table 1) is, for this example, very close to the best possible value $q_{opt} = 2.486 \times 10^{-3}$ that can be achieved with complete knowledge of the signal.

c	2	3	4
$m_1(c)$	0.5329	0.5138	0.4883
$m_2(c)$	0.6747	0.4980	0.6325
$m_1(c)\, m_2(c)$	0.3596	0.2559	0.3088

Table 1: Normalised moments for $c = 2, 3, 4$

Conclusion:

A procedure for improving a Kautz approximation, in the case of a limited number of expansion terms, by a proper choice of a pair of free parameters, has been presented. It possesses desirable features and can be readily adapted to the discrete time case. This work is underway.

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