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# Pertinent parameters for Kautz approximation

R. Morvan, N. Tanguy, P.Vilbé, and L.C. Calvez.<sup>1</sup>

## Abstract

A procedure for determining two parameters to be used in Kautz approximation is presented. It is based on minimisation of an upper bound of the error energy.

*Index terms:* Orthonormal approximation, Signal representation, Modelling, Mathematical techniques.

## Introduction:

Poorly damped systems are difficult to approximate with a reasonable number of Laguerre functions, so the so-called two-parameter Kautz functions which can approximate more efficiently signals with strong oscillatory behavior, have received much attention in the recent mathematical modelling and identification literature (see, e.g., [[1]] and the references therein). These functions can be defined by their Laplace transforms

$$\begin{aligned}\widehat{\varphi}_{2k}(s) &= \frac{\sqrt{2bc}}{s^2 + bs + c} \left( \frac{s^2 - bs + c}{s^2 + bs + c} \right)^k \\ \widehat{\varphi}_{2k+1}(s) &= \frac{s\sqrt{2b}}{s^2 + bs + c} \left( \frac{s^2 - bs + c}{s^2 + bs + c} \right)^k \\ b &> 0, c > 0, k = 0, 1, 2, \dots\end{aligned}$$

where the numbering of the functions as defined in [[1]] has been slightly modified for suitability. The time functions are written  $\varphi_n(t)$  or as  $\varphi_n(t, b, c)$  whenever it is desirable to exhibit the parameters. The orthonormal set  $\{\varphi_n\}$  is complete in  $L^2[0, \infty[$ , thus any finite energy real causal signal  $f(t)$  can be approximated within any prescribed accuracy by truncating its infinite expansion  $f(t) = \sum_{n=0}^{\infty} a_n \varphi_n(t)$  where  $a_n = \langle f, \varphi_n \rangle$  is the  $n + 1$  th Fourier coefficient. The  $N$ -term truncated expansion yields the best approximation to  $f(t)$  of the form  $\tilde{f}(t) = \sum_{n=0}^{N-1} a_n \varphi_n(t)$  in the sense of minimising the integrated squared error (ISE)

$$Q = \int_0^{\infty} [f(t) - \tilde{f}(t)]^2 dt = \|f\|^2 - \sum_{n=0}^{N-1} a_n^2 = \sum_{n=N}^{\infty} a_n^2. \quad (1)$$

Usually, since the  $a_n$  depend on  $b$  and  $c$ ,  $Q$  can be reduced further by a proper choice of these parameters. Nice optimality conditions for Kautz approximation, generalizing that of the Laguerre case [[2], [3]], have been derived by Oliveira e Silva [[4]] and den Brinker [[5]]. However, these conditions of great theoretical interest can result in complicated computations in practical cases. For Laguerre functions [[6], [7]] and other classical functions [[8], [9]] an alternative easy-to-use and efficient approach, based on minimisation of an upper bound of the error energy, has been proposed. It is the purpose of this Letter to derive a somewhat similar procedure for the specific set of non-classical two-parameter Kautz functions.

## Key relationship:

Recently [[1]], it has been shown that the coefficients  $a_n$  can be found from power series calculations in the following manner. Denoting by  $\widehat{f}(s)$  the Laplace transform of  $f(t)$ , assumed to be analytic outside an appropriate region in the  $s$ -plane, let  $F_i$ ,  $i = 1, 2$ , be defined by

$$F_1(s) = \left[ s^2 \widehat{f}(s) - c \widehat{f}(c/s) \right] / (s^2 - c) \quad (2)$$

$$F_2(s) = \left[ \widehat{f}(c/s) - \widehat{f}(s) \right] s\sqrt{c} / (s^2 - c). \quad (3)$$

Since  $F_i(c/s) = F_i(s)$ , both these functions are symmetric functions of  $c/s$  and  $s$  and so they can be represented as functions of  $(c/s) + s$  and  $(c/s) s = c$ , whence

$$F_i(s) = \widehat{f}_i(s + c/s, c) \quad , \quad i = 1, 2. \quad (4)$$

<sup>1</sup>Laboratoire d'Electronique et Systèmes de Télécommunications (LEST),  
UMR CNRS n° 6616, Université de Bretagne Occidentale (UBO),  
29285 BREST cedex, FRANCE {Riwal.Morvan, Noel.Tanguy, Pierre.Vilbe, Leon-Claude.Calvez }@univ-brest.fr

Notice that the transformation  $s \rightarrow s + c/s$  is familiar in filter design where it is used to design a band-pass filter from a low-pass filter. The trick to relate Kautz coefficients and power series is to observe the remarkable relationship

$$\widehat{f}_i(s, c) = \sum_{n=0}^{\infty} a_{2n+2-i} \widehat{l}_n(s, b) \quad , \quad i = 1, 2 \quad (5)$$

where  $\widehat{l}_n(s, b) = \sqrt{2b}(s-b)^n / (s+b)^{n+1}$  denotes the Laplace transform of the normalised Laguerre function  $l_n(bt)$ . Thus, the Fourier coefficients associated with the expansion of  $f(t)$  with respect to the orthonormal set  $\{\varphi_n\}$  can be obtained via Laguerre expansions.

### ***Proposed procedure for pertinent parameters:***

Denoting by  $f_i(t, c)$  the inverse Laplace transform of  $\widehat{f}_i(s, c)$ , let us define moments  $M_j$  by

$$M_0 = \int_0^{\infty} [(f_1)^2 + (f_2)^2] dt \quad (6)$$

$$M_1(c) = \int_0^{\infty} t [(f_1)^2 + (f_2)^2] dt \quad (7)$$

$$M_2(c) = \int_0^{\infty} t \left[ \left( \frac{df_1}{dt} \right)^2 + \left( \frac{df_2}{dt} \right)^2 \right] dt \quad (8)$$

where we have used  $f_i$  as shorthand for  $f_i(t, c)$ . Since the Laguerre functions are orthonormal and the Kautz functions are orthonormal also, both  $M_0$  and  $\|f\|^2$  are equal to  $\sum_{n=0}^{\infty} a_n^2$ , hence  $M_0 = \|f\|^2$  is a constant. On the other hand,  $M_1$  and  $M_2$  depend on  $c$  but do not depend on  $b$ .

#### **Theorem 1:**

Let  $q = Q / \|f\|^2$ ,  $m_i(c) = M_i(c) / \|f\|^2$ ,  $i = 1, 2$ . Then, the normalised ISE associated with a  $2K$ -term Kautz approximation is bounded by

$$q \leq B = \frac{1}{2K} \left[ \frac{m_2(c)}{b} + bm_1(c) - 1 \right]. \quad (9)$$

This bound attains its minimum when  $b = \sqrt{m_2(c)/m_1(c)}$ . The minimum itself is  $B_{min} = (2\sqrt{m_1(c)m_2(c)} - 1) / (2K)$ .

#### **Proof:**

Let  $M_j = M_{j1} + M_{j2}$  where  $M_{ji}$  denotes the contribution of  $f_i$  ( $M_{01} = \int_0^{\infty} (f_1)^2 dt$ , ...). Then, the ISE  $Q_i = \sum_{n=K}^{\infty} a_{2n+2-i}^2$  associated with the  $K$ -term Laguerre approximation of  $f_i(t, c)$  (see eqn. 5) is known [[6]] to be bounded by  $(M_{2i}/b + M_{1i}b - M_{0i}) / (2K)$ , provided that  $(2K+1) \geq (M_{2i}/b + M_{1i}b) / M_{0i}$ ,  $i = 1, 2$ , a condition which is assumed to hold in the following ( $K$  is sufficiently large). In view of eqn. 1 the ISE associated with the ( $N = 2K$ )-term Kautz approximation of  $f(t)$  is  $Q = Q_1 + Q_2$  and can then be bounded as  $Q \leq (M_2/b + M_1b - M_0) / (2K)$ . Dividing throughout by  $\|f\|^2 = M_0$  achieves the proof of eqn. 9. Writing  $\partial B / \partial b = 0$ , the last part of the theorem follows readily.

For a fixed  $c > 0$ , let  $\mathbf{C} = \mathbf{C}(c; m_1, m_2)$  denote the class of signals  $f \in L^2[0, \infty[$  with given  $m_1(c) = m_1$  and  $m_2(c) = m_2$ . There exist signals  $f \in \mathbf{C}$  that achieve the bound in eqn. 9; as a simple example, consider  $\mathbf{C}(5; 0.4, 1.6)$ : it is a standard exercise to show that  $f(t) = 3\varphi_0(t, 2, 5) + \varphi_6(t, 2, 5)$  is in this class. Clearly, the 6-term Kautz approximation using  $\varphi_n(t, 2, 5)$  is  $\tilde{f}(t) = 3\varphi_0(t, 2, 5)$  with  $q = 0.1$  and  $B = (1.6/2 + 2 \times 0.4 - 1) / 6 = 0.1$ , whence  $q = B$ . Therefore, the bound in eqn. 9 is actually the maximum ISE for signals in  $\mathbf{C}$  and theorem 1 gives the best  $b$ , in the sense of minimising the maximum integrated squared error, that can be obtained with the knowledge of the signal limited to  $m_1(c)$  and  $m_2(c)$ .

Now, suppose that  $m_1(c)$  and  $m_2(c)$  are known for more than one value of  $c$ , say for  $c \in C$  where  $C$  represents a discrete or continuous set of positive numbers. Since the lowest  $m_1(c)m_2(c)$  will result in the lowest  $B_{min}$ , we have the following theorem.

#### **Theorem 2:**

Let  $c_0$  denote that value of  $c \in C$  at which the product  $m_1(c)m_2(c)$  is minimum and let  $b_0 = \sqrt{m_2(c_0)/m_1(c_0)}$ . Then, a pertinent choice for the pair of Kautz parameters is  $(b_0, c_0)$ , which yields  $(B_{min})_0 = (2\sqrt{m_1(c_0)m_2(c_0)} - 1) / (2K)$ .

#### **Remark:**

Notice that  $b_0$  and  $c_0$  do not depend on the number  $N = 2K$  of functions to be used. Thus  $b_0$  and  $c_0$  can be computed in a first time and  $N$  can be chosen afterwards: for instance, one can choose  $N$  such that the upper bound  $(B_{min})_0$  is small enough or such that the exact  $q = 1 - \sum_{n=0}^{N-1} a_n^2 / \|f\|^2$  is small enough.

## Illustrative example:

Consider the Laplace transform

$$\widehat{f}(s) = \frac{s^3 + 4s^2 + 8s + 1}{s^4 + 5s^3 + 13s^2 + 19s + 18}$$

with a view to deriving a second-order approximation ( $N = 2$  Kautz functions). Letting for example  $c = 4$ , eqns. 2-4 yield

$$\widehat{f}_1(s, 4) = \frac{9s^3 + 72s^2 + 182s + 127}{9s^4 + 83s^3 + 267s^2 + 349s + 164}$$

$$\widehat{f}_2(s, 4) = \frac{s^3 + 19s^2 + 81s + 87}{9s^4 + 83s^3 + 267s^2 + 349s + 164}$$

Using one of the available techniques (e.g. [[10]]), the required moments are computed as  $M_0 = \|f\|^2 = 0.5183$ ,  $M_1(4) = 0.2531$ ,  $M_2(4) = 0.3278$  and the error bound is minimised when  $b = \sqrt{m_2(4)/m_1(4)} = \sqrt{M_2(4)/M_1(4)} = 1.138$ . With  $b = 1.138$  and  $c = 4$ , the first and second coefficients of the Kautz expansion are  $a_0 = 0.2965$  and  $a_1 = 0.6365$  from which the exact normalised ISE is obtained as  $q = 4.878 \times 10^{-2}$ .

The normalised moments computed by repeating the procedure for  $c = 2$  and  $c = 3$  are shown in Table 1. For  $c \in \{2, 3, 4\}$ , the product  $m_1(c)m_2(c)$  is minimum if  $c = 3$ ; therefore, in agreement with theorem 2, we select  $b_o = \sqrt{0.4980/0.5138} = 0.9846$  and  $c_o = 3$ , improving the normalised ISE which becomes  $q = q_o = 2.505 \times 10^{-3}$ . It is worth noting that  $q_o$  obtained using limited knowledge of the signal (Table 1) is, for this example, very close to the best possible value  $q_{opt} = 2.486 \times 10^{-3}$  that can be achieved with complete knowledge of the signal.

$c$	2	3	4
$m_1(c)$	0.5329	0.5138	0.4883
$m_2(c)$	0.6747	0.4980	0.6325
$m_1(c)m_2(c)$	0.3596	0.2559	0.3088

**Table 1:** Normalised moments for  $c = 2, 3, 4$

## Conclusion:

A procedure for improving a Kautz approximation, in the case of a limited number of expansion terms, by a proper choice of a pair of free parameters, has been presented. It possesses desirable features and can be readily adapted to the discrete time case. This work is underway.

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