Pertinent parameters for Kautz approximation

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Pertinent parameters for Kautz approximation


Abstract

A procedure for determining two parameters to be used in Kautz approximation is presented. It is based on minimisation of an upper bound of the error energy.

Index terms: Orthonormal approximation, Signal representation, Modelling, Mathematical techniques.

Introduction:

Poorly damped systems are difficult to approximate with a reasonable number of Laguerre functions, so the so-called two-parameter Kautz functions which can approximate more efficiently signals with strong oscillatory behavior, have received much attention in the recent mathematical modelling and identification literature (see, e.g., [1] and the references therein). These functions can be defined by their Laplace transforms

\[
\hat{\varphi}_{2k} (s) = \frac{\sqrt{2bc}}{s^2 + bs + c} \left( \frac{s^2 - bs + c}{s^2 + bs + c} \right)^k
\]

\[
\hat{\varphi}_{2k+1} (s) = \frac{s\sqrt{2b}}{s^2 + bs + c} \left( \frac{s^2 - bs + c}{s^2 + bs + c} \right)^k
\]

where the numbering of the functions as defined in [1] has been slightly modified for suitability. The time functions are written \( \varphi_n(t) \) or as \( \varphi(t, b, c) \) whenever it is desirable to exhibit the parameters. The orthonormal set \( \{\varphi_n\} \) is complete in \( L^2[0, \infty] \), thus any finite energy real causal signal \( f(t) \) can be approximated within any prescribed accuracy by truncating its infinite expansion \( f(t) = \sum_{n=0}^{\infty} a_n \varphi_n(t) \) where \( a_n = \langle f, \varphi_n \rangle \) is the \( n+1 \) th Fourier coefficient. The \( N \)-term truncated expansion yields the best approximation to \( f(t) \) of the form \( \tilde{f}(t) = \sum_{n=0}^{N-1} a_n \varphi_n(t) \) in the sense of minimising the integrated squared error (ISE)

\[
Q = \int_0^{\infty} [f(t) - \tilde{f}(t)]^2 dt \leq \|f\|_*^2 - \sum_{n=0}^{N-1} a_n^2 = \sum_{n=N}^{\infty} a_n^2
\]

Usually, since the \( a_n \) depend on \( b \) and \( c \), \( Q \) can be reduced further by a proper choice of these parameters. Nice optimality conditions for Kautz approximation, generalizing that of the Laguerre case [2, 3], have been derived by Oliveira e Silva [4] and den Brinker [5]. However, these conditions of great theoretical interest can result in complicated computations in practical cases. For Laguerre functions [6, 7] and other classical functions [8, 9] an alternative easy-to-use and efficient approach, based on minimisation of an upper bound of the error energy, has been proposed. It is the purpose of this Letter to derive a somewhat similar procedure for the specific set of non-classical two-parameter Kautz functions.

Key relationship:

Recently [1], it has been shown that the coefficients \( a_n \) can be found from power series calculations in the following manner. Denoting by \( \tilde{f}(s) \) the Laplace transform of \( f(t) \), assumed to be analytic outside an appropriate region in the \( s \)-plane, let \( F_i, i = 1, 2 \), be defined by

\[
F_1 (s) = \left[ s^2 \tilde{f}(s) - c \tilde{f}(c/s) \right] / (s^2 - c) \]

\[
F_2 (s) = \left[ \tilde{f}(c/s) - \tilde{f}(s) \right] s \sqrt{c} / (s^2 - c)
\]

Since \( F_i (c/s) = F_i (s) \), both these functions are symmetric functions of \( c/s \) and \( s \) and so they can be represented as functions of \( (c/s) + s \) and \( (c/s) s = c \), whence

\[
F_i (s) = \tilde{f}_i (s + c/s, c), \quad i = 1, 2
\]

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Notice that the transformation $s \rightarrow s + c/s$ is familiar in filter design where it is used to design a band-pass filter. The trick to relate Kautz coefficients and power series is to observe the remarkable relationship

$$\hat{f}_i(s, c) = \sum_{n=0}^{\infty} a_{2n+2-i} \hat{\lambda}_n(s, b) \quad , \quad i = 1, 2$$

(5)

where $\hat{\lambda}_n(s, b) = \sqrt{2b} (s - b)^n / (s + b)^{n+1}$ denotes the Laplace transform of the normalised Laguerre function $\lambda_n(bt)$. Thus, the Fourier coefficients associated with the expansion of $f(t)$ with respect to the orthonormal set $\{\varphi_n\}$ can be obtained via Laguerre expansions.

**Proposed procedure for pertinent parameters:**

Denoting by $f_i(t, c)$ the inverse Laplace transform of $\hat{f}_i(s, c)$, let us define moments $M_j$ by

$$M_0 = \int_0^\infty [(f_1)^2 + (f_2)^2] \, dt$$

(6)

$$M_1(c) = \int_0^\infty t [(f_1)^2 + (f_2)^2] \, dt$$

(7)

$$M_2(c) = \int_0^\infty t \left[ \left( \frac{df_1}{dt} \right)^2 + \left( \frac{df_2}{dt} \right)^2 \right] \, dt$$

(8)

where we have used $f_i$ as shorthand for $f_i(t, c)$. Since the Laguerre functions are orthonormal and the Kautz functions are orthonormal also, both $M_0$ and $\|f\|^2$ are equal to $\sum_{n=0}^{\infty} a_n^2$, hence $M_0 = \|f\|^2$ is a constant. On the other hand, $M_1$ and $M_2$ depend on $c$ but do not depend on $b$.

**Theorem 1:**

Let $q = Q/\|f\|^2$, $m_i(c) = M_i(c)/\|f\|^2$, $i = 1, 2$. Then, the normalised ISE associated with a $2K$-term Kautz approximation is bounded by

$$q \leq B = \frac{1}{2K} \left[ \frac{m_2(c)}{b} + bm_1(c) - 1 \right].$$

(9)

This bound attains its minimum when $b = \sqrt{m_2(c)/m_1(c)}$. The minimum itself is $B_{\min} = \left( 2\sqrt{m_1(c)} m_2(c) - 1 \right) / (2K)$.

**Proof:**

Let $M_j = M_{j1} + M_{j2}$ where $M_{ji}$ denotes the contribution of $f_i$ ($M_{01} = \int_0^\infty (f_1)^2 \, dt$, ...). Then, the ISE $Q_i = \sum_{n=K}^{\infty} a_{2n+2-i}$ associated with the $K$-term Laguerre approximation of $f_i(t, c)$ (see eqn. 5) is known [6] to be bounded by $\left( M_{j1} / b + M_{j2} / M_0 \right) / (2K)$, provided that $(2K + 1) \geq (M_{j1} / b + M_{j2} / M_0)$, $i = 1, 2$, a condition which is assumed to hold in the following ($K$ is sufficiently large). In view of eqn. 1 the ISE associated with the $(N = 2K)$-term Kautz approximation of $f(t)$ is $Q = Q_1 + Q_2$ and can then be bounded as $Q \leq \left( M_{j1} / b + M_{j2} / M_0 \right) / (2K)$. Dividing throughout by $\|f\|^2 = M_0$ achieves the proof of eqn. 9. Writing $\partial B / \partial b = 0$, the last part of the theorem follows readily.

For a fixed $c > 0$, let $C = C(c; m_1, m_2)$ denote the class of signals $f \in L^2 [0, \infty]$ with given $m_1(c) = m_1$ and $m_2(c) = m_2$. There exist signals $f \in C$ that achieve the bound in eqn. 9; as a simple example, consider $C(5; 0.4, 1.6)$: it is a standard exercise to show that $f(t) = \varphi_0(t, 2, 5) + \varphi_0(t, 2, 5)$ is in this class. Clearly, the 6-term Kautz approximation using $\varphi_n(t, 2, 5)$ is $\hat{f}(t) = 3\varphi_0(t, 2, 5)$ with $q = 0.1$ and $B = (1.6 / 2 + 2 \times 0.4 - 1) / 6 = 0.1$, whence $q = B$. Therefore, the bound in eqn. 9 is actually the maximum ISE for signals in $C$ and theorem 1 gives the best $b$, in the sense of minimising the maximum integrated squared error, that can be obtained with the knowledge of the signal limited to $m_1(c)$ and $m_2(c)$.

Now, suppose that $m_1(c)$ and $m_2(c)$ are known for more than one value of $c$, say for $c \in C$ where $C$ represents a discrete or continuous set of positive numbers. Since the lowest $m_1(c), m_2(c)$ will result in the lowest $B_{\min}$, we have the following theorem.

**Theorem 2:**

Let $c_0$ denote that value of $c \in C$ at which the product $m_1(c) m_2(c)$ is minimum and let $b_0 = \sqrt{m_2(c_0)/m_1(c_0)}$. Then, a pertinent choice for the pair of Kautz parameters is $(b_0, c_0)$, which yields $(B_{\min})_0 = \left( 2\sqrt{m_1(c_0)} m_2(c_0) - 1 \right) / (2K)$.

**Remark:**

Notice that $b_0$ and $c_0$ do not depend on the number $N = 2K$ of functions to be used. Thus $b_0$ and $c_0$ can be computed in a first time and $N$ can be chosen afterwards: for instance, one can choose $N$ such that the upper bound $(B_{\min})_0$ is small enough or such that the exact $q = 1 - \sum_{n=0}^{N-1} a_n^2 / \|f\|^2$ is small enough.
Illustrative example:
Consider the Laplace transform
\[ \tilde{f}(s) = \frac{s^3 + 4s^2 + 8s + 1}{s^4 + 5s^3 + 13s^2 + 19s + 18} \]
with a view to deriving a second-order approximation \((N = 2\) Kautz functions). Letting for example \(c = 4\), eqns. 2-4 yield
\[ \tilde{f}_1(s, 4) = \frac{9s^3 + 72s^2 + 182s + 127}{9s^4 + 83s^3 + 267s^2 + 349s + 164} \]
\[ \tilde{f}_2(s, 4) = \frac{s^3 + 19s^2 + 81s + 87}{9s^4 + 83s^3 + 267s^2 + 349s + 164} \]
Using one of the available techniques (e.g. \([10]\)), the required moments are computed as
\[ M_0 = \| \tilde{f} \|^2 = 0.5183, \quad M_1(4) = 0.2531, \quad M_2(4) = 0.3278 \]
and the error bound is minimised when \( b = \sqrt{m_2(4)/m_1(4)} = \sqrt{M_2(4)/M_1(4)} = 1.138 \).
With \( b = 1.138 \) and \( c = 4 \), the first and second coefficients of the Kautz expansion are \( a_0 = 0.2965 \) and \( a_1 = 0.6365 \) from which the exact normalised ISE is obtained as \( q = 4.878 \times 10^{-2} \).

The normalised moments computed by repeating the procedure for \( c = 2 \) and \( c = 3 \) are shown in Table 1. For \( c \in \{2, 3, 4\} \), the product \( m_1(c) m_2(c) \) is minimum if \( c = 3 \); therefore, in agreement with theorem 2, we select \( b_0 = \sqrt{0.4980/0.5138} = 0.9846 \) and \( c_0 = 3 \), improving the normalised ISE which becomes \( q = q_0 = 2.505 \times 10^{-3} \). It is worth noting that \( q_0 \) obtained using limited knowledge of the signal (Table 1) is, for this example, very close to the best possible value \( q_{\text{opt}} = 2.486 \times 10^{-3} \) that can be achieved with complete knowledge of the signal.

<table>
<thead>
<tr>
<th>( c )</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<td>( m_1(c) )</td>
<td>0.5329</td>
<td>0.5138</td>
<td>0.4883</td>
</tr>
<tr>
<td>( m_2(c) )</td>
<td>0.6747</td>
<td>0.4980</td>
<td>0.6325</td>
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<td>( m_1(c) m_2(c) )</td>
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<td>0.2559</td>
<td>0.3088</td>
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</tbody>
</table>

Table 1: Normalised moments for \( c = 2, 3, 4 \)

Conclusion:
A procedure for improving a Kautz approximation, in the case of a limited number of expansion terms, by a proper choice of a pair of free parameters, has been presented. It possesses desirable features and can be readily adapted to the discrete time case. This work is underway.

References