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The purpose of this paper is to establish a geometric scattering result for a conformally invariant nonlinear wave equation on an asymptotically simple space-time. The scattering operator is obtained via trace operators at null infinities. The proof is achieved in three steps. A priori linear estimates are obtained via an adaptation of the Morawetz vector field in the Schwarzschild space-time and a method used by Hörmander for the Goursat problem consisting in writing conformal null infinity as a graph. A well-posedness result for the characteristic Cauchy problem on a light cone at infinity for small data is then obtained. This requires uniform Sobolev estimates on a timelike foliation coming from a control of the norms of extension operators and Sobolev embeddings. Finally, the trace operators on conformal infinities are built and used to define the conformal scattering operator.

Introduction

Scattering theory is one of the most precise tools to analyze and describe the asymptotic behavior of solutions of evolution equations. As a consequence, it has a great importance in relativity to understand the influence of the geometry on the propagation of waves. Scattering in relativity was developed by many authors: Dimock ([17]), Dimock and Kay ([20, 18, 19]) and more recently, Bachelot ([1, 4, 2, 3]), Bachelot and Bachelot-Motet ([5]), Häfner ([25, 26]), Häfner-Nicolas ([28]), Melnyk ([34]) and Daudé [15, 16].

The scattering method used by these authors relies on spectral theory: this requires the metric to be independent of a certain time function. It has consequently been necessary to develop a scattering theory which is not time dependent. As remarked by Friedlander in [22], it is possible to use a conformal rescaling to study an asymptotic behavior. This method was used for the first time by Baez-Segal-Zhou in [6] for the wave equation on the Minkowsky space-time. Their method consisted in embedding conformally the Minkowski space-time in a bigger compact manifold.

The conformal compactification of a space-time was first introduced by Penrose in the sixties in [37] to describe the asymptotic behavior of solutions of the Dirac equations. A boundary, which represents in some way the infinity for causal curves, is added to the manifold. This boundary is divided into two connex components $\mathscr{I}^+$ and $\mathscr{I}^-$. When the spacetime satisfies the Einstein equations (with no cosmological constant), $\mathscr{I}^+$ and $\mathscr{I}^-$ are light cones from two singularities i^+ and i^-. The asymptotic behavior can then be obtained by considering the traces on these

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hypersurfaces of a solution of the conformal wave equation (more precisely the scattering operator is obtained from the trace operators on $\mathscr{I}^+$ and $\mathscr{I}^-$). Asymptotically simple curved spacetimes, that is to say spacetimes admitting a conformal compactification, with specifiable regularity at i^+ and i^-, were constructed by Chrusciel-Delay ([11, 12]), Corvino ([13]) and Corvino-Schoen ([14]). Mason and Nicolas successfully adapted the method of Baez-Segal-Zhou in the linear case for the Dirac and Dirac-Maxwell equation by in [32] on this curved background. They also obtained a compete peeling result for the wave equation on a Schwarzschild background in [33].

This paper presents the construction of a conformal scattering operator for the conformally invariant defocusing cubic wave equation:

$$\nabla_a \nabla^a \Phi + b\Phi^3 = 0$$

on the asymptotically simple space-time obtained by Chrusciel-Delay and Corvino-Schoen. Our construction relies on vector fields methods which were previously used to obtain the well-posedness of the Cauchy problem equation (see the result of Cagnac–Choquet-Bruhat in [9]) to obtain energy estimates. The choice which is made here is for the vector field is the same as the one made in [33] and the techniques are essentially the same as in [32].

A specific method is used to handle the singularity in i^+: the characteristic hypersurface is described as the graph of a function. This method was introduced by Hörmander in [30] and generalized in [36, 35] by Nicolas to establish the well-posedness of the characteristic Cauchy problem.

The main obstacle and difference with the linear case are the necessity to obtain uniform estimates of the non linearity. This requires to obtain uniform Sobolev embeddings from H^1 into L^6. This is achieved by considering results concerning the constant associated with the embeddings given by Stein ([39] for extension theorem) and Hébey ([29] for Sobolev embeddings).

The paper is organized as follows:

- the first section introduces the geometrical and analytical background: the space-times obtained by Corvino-Schoen and Chrusciel-Delay are precisely defined and the function space on the characteristic hypersurface at infinity are given.

- The a priori estimates are derived in section 2: these estimates are established in three specific subsets of M: a neighborhood of i^0 where the estimates come from the asymptotic behavior of the chosen vector field (the Morawetz vector field), a neighborhood of i^+ where the estimates are established by following the method developed by Hörmander, and finally in a neighborhood of a Cauchy hypersurface. The techniques consist essentially in the use of Gronwall lemma and Stokes theorem.

- Section 3 is devoted to the well-posedness for small data of the Cauchy problem. The proof is made as follows: estimates on the propagator of the cubic wave equation are established from uniform Sobolev estimates. Using a contraction result, a local existence theorem for the characteristic problem is then obtained for small data: the solution is constructed up to a uniformly spacelike hypersurface close enough to the conformal infinity. Finally, a Cauchy problem from this hypersurface gives a global solution.

- Finally, we prove in section 4 the existence of a Lipschitz conformal scattering operator obtained from two trace operators.

- Section 5 introduces another approach for the a priori estimates based on a weakly spacelike foliation. This part of the work remains unachieved because it requires a control of the Killing form associated with the foliation. The author has not yet been able to obtain it.

Conventions and notation

Let (M, g) be a 4 dimensional manifold of Chrusciel-Delay/ Corvino-Schoen type. Its compactification is denoted by $(\hat{M}, \hat{g})$. The associated connections are denoted ∇ and $\hat{\nabla}$.

Let us consider on M the following nonlinear wave equation:

$$\nabla_a \nabla^a \Phi + b\Phi^3 = 0$$

We assume that:

a. b is positive;

b. b admits a continuous extension to $\hat{M}$ such that b vanishes at $\mathscr{I}$;

c. b satisfies: there exists a constant c such that, uniformly on $\hat{M}$:

$$\exists c, |\hat{T}^a \nabla_a b| \leq cb.$$

Remark 0.1. *a. The positivity of b corresponds to the defocusing case.*

b. The vanishing of b on $\mathscr{I}$ implies that the non linearity vanishes at infinity. This hypothesis is made so that we do not have to deal with Sobolev embeddings on $\mathscr{I}$.

c. Since $\hat{M}$ is compact, the differential inequality satisfied by b does in fact not impose another specific asymptotic behaviour than the fact that $\hat{T}_a \nabla^a \phi$ decrease and vanishes at the same rate of b.

The following notations will be used:

- we will note:

$$\phi \lesssim \psi$$

where ϕ and ψ are two functions over U, a subset of M, whenever there exists a constant C, depending only on the geometry, the vector $\hat{T}^a$, the Killing form $\hat{\nabla}^{(a}\hat{T}^{b)}$ and the function b, such as:

$$\psi \leq C\phi \text{ on } U.$$

If the ψ and ϕ both satisfy:

$$\phi \lesssim \psi \text{ and } \psi \lesssim \phi,$$

we say that ϕ and ψ are equivalent and note:

$$\phi \approx \psi.$$

- The geometric notations are the following:
  - The quantities with $\hat{}$ are geometric quantities related to the unphysical metric.
  - $\mu[\hat{g}]$ is the volume form associated with the metric $\hat{g}$.
  - If ν is a form over $\hat{M}$, then $\star\nu$ is its Hodge dual. If ν is a 1-form and V the vector field associated to ν via the metric $\hat{g}$, then:

$$\star\nu = V \lrcorner \mu[\hat{g}] \text{ or } \star V_a = V^a \lrcorner \mu[\hat{g}]$$

    where $\lrcorner\mu[\hat{g}]$ is the contraction with the volume form $\mu[\hat{g}]$
  - i_Σ is the restriction to the submanifold Σ. The pull-forward of a form ν on $\hat{M}$ over the tangent space to Σ is denoted by $i^\star_\Sigma(\Sigma)$.

1 Functional and geometric preliminaries

We present in this section the geometric and analytic background to the present work. A specific care is brought to the structure at null infinity and the definition of function spaces on that structure.

1.1 Geometric framework

The geometric framework is based on the results of Corvino-Schoen ([14, 13]) and Chrusciel-Delay([11, 12]). They gave a construction of asymptotically simple spacetimes satisfying the Einstein equations with specifiable regularity at null and timelike infinities.

1.1.1 Asymptotic simplicity

The notion of asymptotically simple spacetimes was introduced by Penrose as a general model for asymptotically flat Einstein spacetimes and their conformal compactification (see [38] definition 9.6.11):

Definition 1.1. *A smooth Lorentzian manifold M satisfying the Einstein equations is said to be (C^k) asymptotically simple if there exists a smooth Lorentzian manifold $\hat{M}$ with boundary, a metric $\hat{g}$ and a conformal factor Ω such that:*

a. M is the interior of $\hat{M}$;

b. $\hat{g} = \Omega^2 g$ in M;

c. $\hat{g}$ and Ω are C^k on $\hat{M}$;

d. Ω is positive in M; Ω vanishes at the boundary $\mathscr{I}$ of $\hat{M}$ and $d\Omega$ does not vanish at $\mathscr{I}$;

e. every null geodesic in M acquires a past and future end-point in $\mathscr{I}$

We assume that the boundary $\mathscr{I}$ is C^2 (which is sufficient for this work). It is known that this boundary is a null hypersurface (that it is to say that the restriction of the metric to $\mathscr{I}$ is degenerate) provided that the cosmological constant is zero. Furthermore, $\mathscr{I}$ has two connected components $\mathscr{I}^+$ and $\mathscr{I}^-$ consisting of, respectively, the future and past endpoints of null geodesics. $\mathscr{I}^+$ and $\mathscr{I}^-$ are both diffeomorphic to $\mathbb{R} \times \mathbb{S}^2$.

The manifold (M, g) is usually referred to as the physical space-time and its compactification is referred to as the unphysical space-time. In order to remain consistent with this notation all along this paper, the quantities associated with the unphysical metric are denoted with a "$\hat{}$".

1.1.2 Global hyperbolicity

An important assumption in the context of the Cauchy problem for a wave equation is the possibility to write the equation as an evolution partial differential equation. This is usually achieved by requiring that the manifold M is globally hyperbolic:

Definition 1.2. *A Lorentzian manifold (M, g) is said to be globally hyperbolic if, and only if, there exists in M a global Cauchy hypersurface, i.e. a spacelike hypersurface such that any inextendible timelike curve intersects this surface exactly once.*

A useful consequence of this is the existence of a time function on M and the parallelizability of M, that is to say the existence of a global section of the principal bundle of orthonormal frames (see the work of Geroch in [23, 24] and Bernal-Sanchez in [7]).

In the case of an asymptotically simple manifold (M, g), this property extends of course to the manifold $(\hat{M}, \hat{g})$.

1.1.3 Corvino-Schoen/Chrusciel-Delay space-times

We can then introduce the spacetimes obtained by Corvino-Schoen and Chrusciel-Delay:

Definition 1.3. *A space-time M is of Chrusciel-Delay/Corvino-Schoen type if:*

a. M is asymptotically simple;

b. M is globally hyperbolic; let Σ_0 be a spacelike Cauchy hypersurface; M is then diffeomorphic to $\mathbb{R} \times \Sigma_0$

c. $\hat{M}$ can be completed into a compact manifold by adding three points i^0, i^+ and i^- such that $\mathscr{I}^+$ and $\mathscr{I}^-$ are respectively the past null and future null cones from i^+ and i^- and i^0 is the conformal infinity of the spacelike hypersurface Σ_0 for the metric $\hat{g}$;

d. there exists a compact set K in $\Sigma 0$ such that $(\mathbb{R}\times\Sigma_0 \backslash K, g)$ is isometric to $(\mathbb{R}\times]r_0, +\infty[\times\mathbb{S}^2, g_S)$, where g_S is the Schwarzschild metric with mass m and $r_0 > 2m$;

e. there exists a neighborhood of i^+ such that the metric $\hat{g}$ is obtained in this neighborhood as the restriction of a smooth (C^2) Lorentzian metric of an extension of $\hat{M}$ in the given neighborhood of i^+. The same property holds in i^-.

Remark 1.4. *a. The result of Corvino-Schoen/Chrusciel-Delay states that the metric is isometric to the Kerr metric outside a compact set; we restrict ourself to a Schwarzschild metric.*

b. The extension of the manifold $\hat{M}$ in the neighborhood of i^+ was used by Mason-Nicolas (see [32, 33]). The point i^+ remains singular in $\hat{M}$ but it is nonetheless possible, because of the existence of this extension, to consider geometric data in i^+ (metric, exponential map, connection, curvature) as being the one obtained from the Lorentzian manifold extending $(\hat{M}, \hat{g})$ in the a neighborhood of this point.

c. The point i^0 is a "real" singularity of $\hat{M}$; nonetheless, the geometry is completely known in its neighborhood. This singularity is the main problem we have to deal with in this paper, being an obstacle to global estimates and Sobolev embeddings for instance.

The last part of this section is devoted to the geometric description around the point i^0, that is to say in the Schwarzschild part of the manifold.

We consider here a neighborhood O in $\hat{M}$ of i^0 where the metric g is the Schwarzschild metric. This metric, in the Schwarzschild coordinates $(t, r, \omega_{\mathbb{S}^2})$, can be written:

$$g = F(r)\mathrm{d}t^2 - \frac{1}{F(r)}\mathrm{d}r^2 - r^2\mathrm{d}^2\omega_{\mathbb{S}^2}$$

where:

$$F(r) = 1 - \frac{2m}{r}$$

with m a positive constant. Introducing the new coordinates

$$r^* = r + 2m\log(r - 2m), u = t - r^* \text{ and } R = \frac{1}{r}.$$

The metric can then be expressed:

$$g = (1 - 2mR)\mathrm{d}^2u - \frac{2}{R^2}\mathrm{d}u\mathrm{d}R - \frac{1}{R^2}\mathrm{d}^2\omega_{\mathbb{S}^2}.$$

where $\mathrm{d}^2\omega_{\mathbb{S}^2}$ stands for the standard volume form on the 2-sphere, which can be written in polar coordinates (θ, ϕ):

$$\mathrm{d}^2\omega_{\mathbb{S}^2} = \sin\theta\mathrm{d}\theta \wedge \mathrm{d}\phi.$$

Its inverse is:

$$g^{-1} = \frac{-2}{R^2}\partial_u\partial_R - (1-2mR)\partial_R^2 - \frac{1}{R^2}\eth\eth'.$$

The part of the Cauchy hypersurface Σ_0 is then given by the equation $\{t = 0\}$ in these coordinates.

In the neighborhood O, the conformal factor is chosen to be:

$$\Omega = R$$

and is extended smoothly in $M \backslash O$. The conformal metric is then:

$$\hat{g} = R^2(1-2mR)\mathrm{d}^2u - 2\mathrm{d}u\mathrm{d}R - \mathrm{d}^2\omega_{\mathbb{S}^2}.$$

and its inverse:

$$\hat{g}^{-1} = -2\partial_u\partial_R - R^2(1-2mR)\partial_R^2 - \eth\eth'.$$

The point i^0 is given in these coordinates by $u = -\infty$, $R = 0$.

The description of the geometry around i^0 is completed by the following lemma (lemma A.1 in [33]):

Lemma 1.5. *Let $\epsilon > 0$. There exists $u_0 < 0$, $|u_0|$ large enough, such that the following decay estimates in the coordinate (u, r, θ, ψ) hold:*

$$r < r^* < (1+\epsilon)r, 1 < Rr^* < 1+\epsilon, 0 < R|u| < 1+\epsilon,$$

$$1-\epsilon < 1-2mR < 1, 0 < s = \frac{|u|}{r^*} < 1$$

Proof. The proof of this lemma is straightforward: it only consists in writing simultaneously the asymptotic behavior or the continuity over $\hat{M}$ of each of the coordinates involved in the lemma. ⊛

Remark 1.6. a. *As mentioned in the introduction, we choose to work in the neighborhood of i^0 with the Morawetz vector $\hat{T}^a$ defined by:*

$$\hat{T}^a = u^2\partial_u - 2(1+uR)\partial_R.$$

The squared norm of this vector is:

$$\hat{g}_{ab}\hat{T}^a\hat{T}^b = u^2(4(1+uR) + u^2R^2(1-2mR)).$$

This polynomial in uR vanishes at:

$$2\frac{1\pm\sqrt{2mR}}{1-2mR} = \frac{-2}{1\mp\sqrt{2mR}},$$

so that, if R is chosen small enough, these roots are arbitrarily close to -2. Let ϵ be a positive number chosen such that the inequalities in lemma 1.5 hold for a given u_0. The larger zero of this polynomial satisfies:

$$\frac{-2}{1+\sqrt{2mR}} \leq \frac{-2}{1+\sqrt{\epsilon}}.$$

Choosing ϵ such that

$$\frac{-2}{1+\sqrt{\epsilon}} \leq -1-\epsilon,$$

the norm of $\hat{T}^a$ is then uniformly controlled by:

$$\hat{g}_{ab}\hat{T}^a\hat{T}^b = u^2(4(1+uR) + u^2R^2(1-2mR)) \geq 4u_0^2\epsilon$$

on the domain $\Omega^+_{u_0} = \{(u, R, \omega_{\mathbb{S}^2})|u < u_0\} \cap J^+(\Sigma_0)$, $J^+(\Sigma_0)$ being the future of Σ_0.

b. *Another criterion, given in proposition 2.5, will be required to define ϵ.*

1.2 Analytical requirements

We introduce in this section the technical and analytical tools which are required to the description of the solution for the wave equation.

1.2.1 Conformal wave equation

We recall here how the problem on the physical space time is transformed into a problem on the unphysical space-time. This is based on the classic transformation of the wave d'Alembertian operator:

Lemma 1.7. *Let M a Lorentzian manifold endowed with the conformal metrics g and $\hat{g} = \Omega^2 g$ where Ω is a conformal factor in $C^2(M)$. The connections associated with g and $\hat{g}$ are denoted by ∇ and $\hat{\nabla}$ respectively.*
Then, for any smooth function ϕ on M, the following equality holds:

$$\nabla_a \nabla^a \phi + \frac{1}{6} Scal_g \phi = \Omega^{-3} \left(\hat{\nabla}_a \hat{\nabla}^a \left(\Omega^{-1}\phi\right) + \frac{1}{6} Scal_{\hat{g}} \left(\Omega^{-1}\phi\right) \right)$$

where $Scal_g$ and $Scal_{\hat{g}}$ are the scalar curvatures associated with g and $\hat{g}$ respectively.

Assuming that we are working on a vacuum space-time, for which the scalar curvature vanishes, the equation becomes:

$$\nabla_a \nabla^a \phi = \Omega^{-3} \left(\hat{\nabla}_a \hat{\nabla}^a \left(\Omega^{-1}\phi\right) + \frac{1}{6} \mathrm{Scal}_{\hat{g}} \left(\Omega^{-1}\phi\right) \right) \tag{1.1}$$

We obtain in particular the useful formula:

$$\Omega^3 \nabla_a \nabla^a \Omega = \frac{1}{6} \mathrm{Scal}_{\hat{g}}. \tag{1.2}$$

Let us now consider the Cauchy problem on the physical spacetime M:

$$\left\{ \begin{array}{l} \Box\phi + b\phi^3 = 0 \\ \phi\big|_{\Sigma_0} = \theta \in C_0^\infty(\Sigma_0) \\ \hat{T}^a \nabla_a \phi\big|_{\Sigma_0} = \xi \in C_0^\infty(\Sigma_0) \end{array} \right. . \tag{1.3}$$

Using this transformation, this Cauchy problem is transformed into a Cauchy problem on the unphysical spacetime $\hat{M}$ as followed:

Lemma 1.8. *The function ϕ is a solution of problem 1.3 if, and only if, the function*

$$\psi = \Omega^{-1}\phi$$

is solution of the problem on Σ_0:

$$\left\{ \begin{array}{l} \hat{\Box}\psi + \frac{1}{6} Scal_{\hat{g}} \psi + b\psi^3 = 0 \\ \psi\big|_{\Sigma_0} = \Omega^{-1}\theta \in C_0^\infty(\Sigma_0) \\ \hat{T}^a \hat{\nabla}_a \psi\big|_{\Sigma_0} = \frac{1}{\Omega}\left(\xi - (\hat{T}^a \hat{\nabla}_a \Omega)\frac{\theta}{\Omega}\right) \in C_0^\infty(\Sigma_0) \end{array} \right. .$$

Remark 1.9. *a. Because of the finite speed propagation, since the data on the physical space-time are smooth with compact support on Σ_0, the data on the unphysical space-time are smooth with compact support in Σ_0.*

b. Another consequence of the finite speed propagation is that, because the data remain with compact support in Σ_0, we do not have to deal with the singularity in i^0.

c. *Conversely, it is possible to start with a Cauchy problem on Σ_0 in the unphysical space-time and obtain a Cauchy problem on the physical space-time: starting with the Cauchy problem on $\hat{M}$:*

$$\begin{cases} \hat{\Box}\psi + \frac{1}{6}Scal_{\hat{g}}\psi + b\psi^3 = 0 \\ \psi\big|_{\Sigma_0} = \hat{\theta} \in C_0^\infty(\Sigma_0) \\ \hat{T}^a\hat{\nabla}_a\psi\big|_{\Sigma_0} = \hat{\xi} \in C_0^\infty(\Sigma_0) \end{cases},$$

then $\phi = \Omega\psi$ satisfies the Cauchy problem:

$$\begin{cases} \hat{\Box}\phi + b\psi^3 = 0 \\ \psi\big|_{\Sigma_0} = \Omega\hat{\theta} \in C_0^\infty(\Sigma_0) \\ \hat{T}^a\nabla_a\psi\big|_{\Sigma_0} = \Omega\hat{\xi} + (\hat{T}^a\nabla_a\Omega)\hat{\theta} \in C_0^\infty(\Sigma_0) \end{cases}.$$

1.2.2 Function spaces

The purpose of this section is to give a precise description of the Sobolev spaces which are used in the present paper. Two problems are encountered in this section: the first one consists in adapting the definition of Sobolev spaces on a null hypersurface and the second is the difficulty coming from the singularity at i^0. This difficulty has two aspects: the necessity to adapt the definition of the Sobolev space to the singularity: this is solved using weighted Sobolev spaces on $\mathscr{I}$. Another aspect of this singularity is encountered in section 3.1.3.

We recall the definition of a Sobolev space on a uniformly spacelike hypersurface Σ: for a smooth function u on Σ, consider the norm, when the integral exists:

$$||u||_p^2 = \int_\Sigma \sum_{k=0}^{p} ||D^k u||_h^2 \mathrm{d}\mu[h],$$

where h is the restriction of the metric $\hat{g}$ and D is the restriction of the connection $\hat{\nabla}$ to Σ.

Definition 1.10. *The completion of the space:*

$$\Big\{u \in C^\infty(\Sigma)\Big|\, ||u||_p < +\infty\Big\}$$

in the norm $||\star||_p$is denoted by $H^p(\Sigma)$.
When Σ is is a compact spacelike hypersurface with boundary, the completion of the space of smooth functions with compact support in the interior of Σ in the norm $||\star||_p$is denoted by $H_0^p(\Sigma)$.

Remark 1.11. *It is known that, when working on a Riemannian closed manifold, the Sobolev spaces are independent of the choice of the metric. Nonetheless, this fact is not true any more when working with a weakly spacelike hypersurface, as we are about to see. Arbitrary choices are made for their definitions.*

Because of the degeneracy of the metric, it is not possible to define on $\mathscr{I}^+$ geometric quantities that only depend on the metric $\hat{g}$. Two solutions can be provided:

- lifting the metric from Σ_0 to $\mathscr{I}^+$;
- adding geometric information on $\mathscr{I}^+$ by using the uniformly timelike vector field $\hat{T}^a$.

Following [27, 31] and using Geroch-Held-Penrose formalism, $\mathscr{I}^+$ is endowed with a basis (l^a, n^a, e_3^a, e_4^a) such that:

- l^a and n^a are two future directed null vectors; n^a is tangent to $\mathscr{I}^+$; they satisfy:

$$l^a + n^a = \hat{T}^a;$$

in the neighborhood of i^0, they are chosen to be:

$$l^a = -2\partial_R \text{ and } n^a = u^2\partial_u$$

- the set $\{l^a, n^a\}$ is completed by two vectors (e_3^a, e_4^a) orthogonal to $\{l^a, n^a\}$, orthogonal to each other and normalized.

$\mathscr{I}^+$ is then endowed with the volume form $i^\star_{\mathscr{I}^+}(l^a \lrcorner \mu[\hat{g}])$. The following norm is defined on $\mathscr{I}^+$, for u a smooth function with compact support which does not contain i^0 or i^+:

$$||u||^2_{H^1(\mathscr{I}^+)} = \int_{\mathscr{I}^+} \left(\frac{\left(n^a\hat{\nabla}u\right)^2}{\hat{g}_{cd}\hat{T}^c\hat{T}^d} + \left|\hat{\nabla}_{\mathbb{S}^2}u\right|^2 + u^2 \right) i^\star_{\mathscr{I}^+}(l^a \lrcorner \mu[\hat{g}])$$

where $\left|\hat{\nabla}_{\mathbb{S}^2}u\right|$ stands for the derivatives with respect to (e_3^a, e_4^a).

Following chapter 5.4.3 of Friedlander's book ([21]), the Sobolev space H^1 on $\mathscr{I}^+$ is finally defined:

Definition 1.12. *Let $\tilde{M}$ be an extension of $\hat{M}$ behind i^+ and consider the function space $\mathcal{D}(\mathscr{I}^+)$ on $\mathscr{I}^+$ obtained as the trace of smooth functions with compact support in $\tilde{M}$ which does not contain i^0. The weighted Sobolev space $H^1(\mathscr{I}^+)$ is defined as the completion of the space $\mathcal{D}(\mathscr{I}^+)$ in the norm $||\star||_{H^1(\mathscr{I}^+)}$.*

Since the volume form is written on the Schwarzschildean part of $\hat{M}$ as, using polar coordinates:

$$\mu[\hat{g}] = \sin(\theta)\mathrm{d}u \wedge \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\psi,$$

this norm is written in the Schwarzschild part of the manifold:

$$||\phi||^2_{H^1(\mathscr{I}^+)} = 2\int_{\mathscr{I}^+} \left(\frac{1}{4}u^2(\partial_u\phi)^2 + |\hat{\nabla}_{\mathbb{S}^2}\phi|^2 + \phi^2\right) \mathrm{d}u\mathrm{d}\omega_{\mathbb{S}^2}$$

since, on $\mathscr{I}^+$,

$$n^a = u^2\partial_u \text{ and } \hat{g}_{cd}\hat{T}^c\hat{T}^d = 4u^2$$

The metric at i^+ is obtained as the restriction of a smooth metric of an extension of $\hat{M}$ beyond $\mathscr{I}^+$. As a consequence, a trace theorem could give another way, more intrinsic, to define $H^1(\mathscr{I}^+ \cap O)$ where O is a bounded open set around i^+. Another way to obtain the fact that the point i^+ does not matter in the definition of the Sobolev space over $\mathscr{I}^+$ is to notice the following property:

Proposition 1.13. *The set of smooth functions with compact support on $\mathscr{I}^+$, whose support does not contain i^+ is dense in $H^1(\mathscr{I}^+)$.*

Proof.The method of the proof relies on the construction of an identification between $H_0^1(\Sigma_0)$ and $H^1(\mathscr{I}^+)$. This identification is brought by Hörmander in [30] and is obtained as follows.

Let t be a smooth time function in the future of Σ_0 in $\hat{M}$. This time function gives rise to a local coordinate system, where t is the time coordinate. We denote by ∂_t the vector field associated with this coordinate. The flow associated with this vector field is denoted Ψ_t.

For x in Σ_0, let $\phi(x)$ be the time at which the curve $\Psi_t(x)$ hits $\mathscr{I}^+$ and consider the application defined by:

$$\xi \in C_0^\infty(\Sigma_0) \longmapsto \left(y \in \mathscr{I}^+ \mapsto \xi\left(\Psi_{-\phi(y)}(y)\right)\right)$$

This application has value in $C_0^\infty(\mathscr{I}^+)$ since the future of a compact set in Σ_0 has compact support in $\mathscr{I}^+$. Furthermore, this application can easily be inverted:

$$\xi \in C_0^\infty(\mathscr{I}^+) \longmapsto \left(x \in \mathscr{I}^+ \mapsto \xi\left(\Psi_{-\phi(x)}(x)\right)\right) \in C_0^\infty(\Sigma_0)$$

and can consequently be used to define on $\mathscr{I}^+$ a Sobolev space by pushing forward the H^1-norm on Σ_0. The Sobolev spaces which are obtained are then equivalent on $H^1(\mathscr{I}^+)$ since the norm are equivalent on any compact set of $\mathscr{I}^+$.

To prove the theorem, it is then sufficient to prove that the smooth functions with compact support in Σ_0 wich does not contain the preimage of i^+ by the flow associated wit the time function t. Since Σ_0 has no topology, it is sufficient to establish the following lemma:

Lemma 1.14. *Let us consider the set of smooth functions defined in $\overline{B(0,1)} \subset \mathbb{R}^3$ with support which does not contain* 0*. Then this set is dense in $H^1(B(0,1))$.*

Proof. It is sufficient to prove that the constant function 1 can be approximated by a sequence of smooth function whose compact support does not contain 0.

Let f be the function defined on $\mathbb{R}^+$ by:

- f is a smooth function on $\mathbb{R}^+$ with value in $[0,1]$;
- $f = 1$ in $[\frac{1}{2}, +\infty)$;
- f vanishes in $[0, \frac{1}{3}]$.

Let us consider the sequence of smooth spherically symmetric functions defined by:

$$\forall n \in \mathbb{N}, \forall x \in B(0,1), \psi_n(x) = f(n||x||).$$

They satisfies, for all n in $\mathbb{N}$:

- $\psi_n = 1$ in $B(0,1)\backslash B(0, \frac{1}{2n})$;
- ψ_n vanishes in $B(0, \frac{1}{3n})$;
- ψ_n is a smooth function on $B(0,1)$ with value in $[0,1]$ since it vanishes in a neighborhood of zero.

Finally, the difference $(1-\psi_n)_n$ converges towards 0 in H^1-norm:

$$\begin{aligned} ||1-\psi_n||_{H^1}^2 &= \int_{B(0,1)} \left((1-f(nr))^2 + n^2(f'(nr))^2\right) r^2 \mathrm{d}r\mathrm{d}\omega_{\mathbb{S}^2} \\ &= \frac{4}{3}\pi \left(\int_0^1 (1-f(nr))^2 \mathrm{d}r + n^2 \int_{\frac{1}{3n}}^{\frac{1}{2n}} (f'(nr))^2 r^2 \mathrm{d}r\right) \\ &\leq \frac{4}{3}\pi \left(\int_0^1 (1-f(nr))^2 \mathrm{d}r + \frac{\sup_{\mathbb{R}}((f')^2)}{n}\right). \end{aligned}$$

The remaining integral converges towards 0 by Lebesgue theorem. As a consequence, the sequence $(\psi_n)_n$ converges towards the constant 1 in $H^1(B(0,1))$.

Finally, let f be a function $H^1(B(0,1))$ (or in $H_0^1(B(0,1))$). Cauchy-Schwarz inequality gives, for all n in $\mathbb{N}$:

$$\begin{aligned} ||f(1-\psi_n)||_{H^1}^2 &= ||f(1-\psi_n)||_{L^2}^2 + \left|\left|f|\nabla(1-\psi_n)|\right|\right|_{L^2}^2 + \left|\left||\nabla f|(1-\psi_n)\right|\right|_{L^2}^2 \\ &\leq 2||f||_{H^1}^2 ||1-\psi_n||_{H^1}^2. \end{aligned}$$

$(f\psi_n)_n$ is then a sequence of functions in $H^1(B(0,1))$ whose support does not contain 0 which converges in H^1 towards f. ⊛

Using this lemma in the neighborhood of the preimage of i^+ immediatly gives the result. ⊛

1.3 Cauchy problem

A well-posedness theorem in our framework is now stated. It is based on a result of Choquet-Bruhat and Cagnac in [9] (and see also [10], appendix III for a survey on the wave equation, and appendix III chapter 5 for our problem).

The geometric framework for this well-posedness theorem is the following (definition 11.8 in [10]):

Definition 1.15. *A space-time (M, g) is said to be regularly sliced if there exists a smooth 3-manifold Σ endowed with coordinates (x^i) and an interval I of $\mathbb{R}$ such that M is $I \times \Sigma$ and the metric g can be written:*

$$g = N^2 dt^2 - g_{ij}(dx^i + \beta^i dt).$$

and its coefficients satisfy:

a. the lapse function N is bounded above and below by two positives constants:

$$\exists (c, C), 0 < c \leq N \leq C;$$

b. for t in I, the 3-dimensional Riemannian manifolds $(\{t\} \times M, g_{t,ij} = g_{ij}\big|_{\{t\}\times M})$ are complete and the metrics g_t are bounded below by a metric h i.e.:

$$\forall V \in T\Sigma, h_{ij} V^i V^j \leq g_{t,ij} V^i V^j;$$

c. and, finally, the norm for the metric g_t of the vector β is uniformly bounded on M.

Remark 1.16. *a. This hypothesis implies that the space-time is globally hyperbolic (theorem 11.10 in [10]).*

b. The asymptotically simple space-time and its compacification which we are working with do not satisfy this property.

The following theorem, obtained by Choquet-Bruhat and Cagnac in [9] , gives existence and uniqueness of solutions to the Cauchy problem for a cubic wave equation:

Theorem 1.17 (Cauchy problem for a nonlinear wave equation)**.** *Let us consider the Cauchy problem on the regularly sliced manifold $(M = \mathbb{R} \times \Sigma, g)$:*

$$\begin{cases} \hat{\Box}\phi + \frac{1}{6} Scal_{\hat{g}}\phi + b\phi^3 = 0 \\ \phi\big|_{\{0\}\times\Sigma} = \theta \in H^1(\Sigma) \\ \partial_t\phi\big|_{\{0\}\times\Sigma} = \tilde{\theta} \in L^2(\Sigma) \end{cases}$$

Then this problem admits a unique global solution on M in $C^0(\mathbb{R}, H^1(\Sigma)) \cap C^1(\mathbb{R}, L^2(\Sigma))$.

As already noted (see remark 1.16), this theorem cannot of course be applied directly to the compactification of M because of the geometry in the Schwarzschild part. This problem can be solved using an extension of $\hat{M}$ constructed as follows:

a. Let θ and ξ be two function in $H^1(\Sigma_0)$ and $L^2(\Sigma_0)$ with compact support in the interior of Σ_0. Let K be a compact subset of Σ_0 containing the support of θ and ξ.

b. Let $\hat{t}$ be a time function on $\hat{M}$ such that Σ_0 is given by $\{\hat{t} = 0\}$; the associated foliation is denoted by $(\Sigma_{\hat{t}})$ for $\hat{t} \in [0, \hat{T}]$; we assume that the gradient of this time function is uniformly timelike for the metric $\hat{g}$.

c. The manifold $(J^+(K), g)$ is a 4-dimensional Lorentzian manifold with boundary. This boundary is constituted of K, the light cone from K, $C^+(K)$, and the part of $\mathscr{I} \cup \{i^+\}$ in the future of K. Since $(\hat{M}, g)$ is extendible smoothly in the neighborhood of i^+, there exists a smooth extension of $(J^+(K), g)$ into a 4-dimensional Lorentzian manifold $(\tilde{M}, \tilde{g})$, depending on the support of K such that the manifold $\tilde{M}$ is diffeomorphic to $[0, \hat{T}] \times \tilde{\Sigma}$, where $\tilde{\Sigma}$ is topologically equivalent to $\mathbb{S}^3$ and such that the foliation $(\Sigma_{\hat{t}})$ is extended into the uniformly spacelike foliation $(\{t\} \times \tilde{\Sigma})$.

Since $\tilde{M}$ is compact, conditions 1 ,2 and 3 of definition 1.15 are clearly satisfied. As a consequence, using theorem 1.17, we obtain the well-posedness in $C^0(\mathbb{R}, H^1(\Sigma_0))$ of the Cauchy problem for

$$\begin{cases} \hat{\Box}\phi + \frac{1}{6}\mathrm{Scal}_{\hat{g}}\phi + b\phi^3 = 0 \\ \phi\big|_{\{0\}\times\Sigma} = \theta \in H^1(\Sigma) \text{ with compact support in } K \\ \partial_t\phi\big|_{\{0\}\times\Sigma} = \tilde{\theta} \in L^2(\Sigma) \text{ with compact support in } K \end{cases}$$

As a consequence, we obtain by restriction to $J^+(K)$ and on $\hat{M}$ well posedness of the Cauchy problem for data in $H^1_0(\Sigma_0) \times L^2(\Sigma_0)$.

Remark 1.18. *a. The extension of the solution of the wave equation to a cylinder was also used by Mason-Nicolas in [33] (proposition 6.1) to obtain energy estimates.*

b. The space $C^0(\mathbb{R}, H^1(\Sigma_0)) \cap C^1(\mathbb{R}, L^2(\Sigma_0)) \subset L^\infty(\mathbb{R}, L^2(\Sigma_0))$ is the space in which the Cauchy problem is well-posed. $\hat{M}$ is nonetheless not diffeomorphic to a product $\mathbb{R} \times \Sigma$. If $\hat{M}$ is extended in the same way, it is then possible to set a well-posedness theorem in this space.

2 A priori estimates

The purpose of this section is to establish a priori estimates for solutions of the wave equation, in the sense that it is possible to control the energy on $\mathscr{I}^+$ by the energy on Σ_0 and reciprocally. These a priori estimates will be used in the next section to establish the continuity of the conformal wave operator, its domain of definition and the existence of trace operators.

Let us consider in this section a smooth solution with compactly supported data of the problem:

$$\hat{\Box}\phi + \frac{1}{6}\mathrm{Scal}_{\hat{g}}\phi + b\phi^3 \quad = \quad 0 \tag{2.1}$$

and the associated stress-energy tensor:

$$T_{ab} = \hat{\nabla}_a\phi\hat{\nabla}_b\phi + \hat{g}_{ab}\left(-\frac{1}{2}\hat{\nabla}_c\phi\hat{\nabla}^c\phi + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right).$$

The contraction of this tensor with $\hat{T}^a$, $\hat{T}^a T_{ab}$, is called "energy 3-form"; it satisfies an "approximate conservation law":

Lemma 2.1. *The derivative of the energy 3-form satisfies:*

$$\hat{\nabla}^a\left(\hat{T}^b T_{ab}\right) \quad = \quad \left(\hat{\nabla}^{(a}\hat{T}^{b)}\right)T_{ab} + \left(1 - \frac{1}{6}Scal_{\hat{g}}\right)\phi\hat{T}^a\hat{\nabla}_a\phi + \hat{T}^a\hat{\nabla}_a b\frac{\phi^4}{4}.$$

This derivative will be designated as "the error term" since it arises in the volume term when applying Stokes theorem.

A quantity which is equivalent to the integral:

$$\int_S i^\star(\star T^a T_{ab})$$

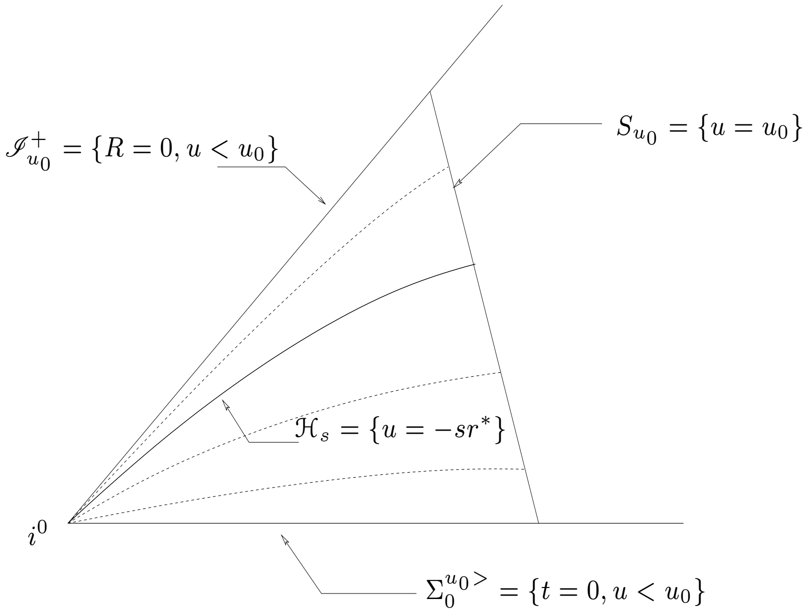


Figure 1: Neighborhood of i^0

on the hypersurface S is called an energy and is denoted by $E(S)$. A global $E(S)$ is piecewise defined in the future of Σ_0 in propositions 2.5, 2.16 and 2.21. The purpose of this multiple definitions is to simplify the comparison with other quantities.

2.1 Estimates in the neighborhood of i^0

The purpose of this section is to obtain a priori estimates for the energy associated with the energy 3-form on $\mathscr{I}^+$ and Σ_0, using the fact that the geometry is known almost completely.

The estimates which are obtained in this section are close to the ones obtained for the linear wave equation by Mason-Nicolas in [33]. These estimates are based on two main tools:

- an explicit control of the decay of the physical metric in the neighborhood of i^0
- and the use of Gronwall lemma.

We define, in $\Omega^+_{u_0} = \{t > 0, u < u_0\}$, the following hypersurfaces, for u_0 given in $\mathbb{R}$:

- $S_{u_0} = \{u = u_0\}$, a null hypersurface transverse to $\mathscr{I}^+$;
- $\Sigma_0^{u_0>} = \Sigma_0 \cap \{u_0 > u\}$, the part of the initial data surface Σ_0 beyond S_{u_0};
- $\mathscr{I}^+_{u_0} = \Omega^+_{u_0} \cap \mathscr{I}^+$, the part of $\mathscr{I}^+$ beyond S_{u_0};
- $\mathcal{H}_s = \Omega^+_{u_0} \cap \{u = -sr^*\}$, for s in $[0, 1]$, a foliation of $\Omega^+_{u_0}$ by spacelike hypersurfaces accumulating on $\mathscr{I}$.

The volume form associated with $\hat{g}$ in the coordinates $(R, u, \omega_{\mathbb{S}^2})$ is then:

$$\mu[\hat{g}] = \mathrm{d}u \wedge \mathrm{d}R \wedge \mathrm{d}^2\omega_{\mathbb{S}^2}. \tag{2.2}$$

Finally, we consider the approximate conformal Killing vector field $\hat{T}^a$:

$$\hat{T}^a = u^2\partial_u - 2(1 + uR)\partial_R.$$

Remark 2.2. *a. This vector field is timelike for the unphysical metric and, as such, is transverse to $\mathscr{I}$. More precisely, it is uniformly timelike in a neighborhood of i^0 (see remark 1.6 for the choice of the neighborhood).*

b. Its expression is derived from the so-called Morawetz vector field in the Minkowski space and was previously used to obtain pointwise estimates in the flat case.

The strategy of the proof in this section is the following:

a. writing an explicit description of the hypersurfaces S_{u_0} and $\mathcal{H}_s^{u_0}$;

b. proving energy equalities in both ways for $\int \star \hat{T}^a T_{ab}$ using the Stokes theorem between the hypersurfaces $\Sigma^+_{u_0}$ and $\mathcal{H}_s^{u_0}$;

c. determining an energy $E(\mathcal{H}_s^{u_0})$ equivalent to $\int \star \hat{T}^a T_{ab}$ from the decay of the metric g;

d. obtaining an integral inequality for $E(\mathcal{H}_s^{u_0})$ to apply the Gronwall lemma;

e. starting from point b, doing the same work from a Stokes theorem applied between $\mathscr{I}^+_{u_0}$ and $\mathcal{H}_s^{u_0}$.

2.1.1 Geometric description

This section is devoted to the description of the energy associated with the nonlinear wave equation in the neighborhood of i^0.

Proposition 2.3. *The energy 3-form, written in the coordinates (R, u, θ, ψ), is given by:*

$$\begin{aligned}
\star \hat{T}^a T_{ab} = &\Big[u^2(\partial_u\phi)^2 + R^2(1-2mR)\left(u^2\partial_R\phi\partial_u\phi - (1+uR)(\partial_R\phi)^2\right) \\
&+\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi|^2 + \frac{\phi^2}{4} + b\frac{\phi^4}{4}\right)\Big]\sin(\theta)\,du\wedge d\theta\wedge d\psi \\
&+\left[\frac{1}{2}\left((2+uR)^2 - 2mR^3u^2\right)\left(\partial_R\phi^2\right) + u^2\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi|^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\right]\sin(\theta)\,dR\wedge d\theta\wedge d\psi \\
&+\sin(\theta)\left[u^2\partial_u\phi - 2(1+uR)\partial_R\phi\right]\left(-\partial_\theta\phi\, du\wedge dR\wedge d\psi + \partial_\psi\phi\, du\wedge dR\wedge d\theta\right)
\end{aligned}$$

The restriction of the energy 3-form can be written:

- *to $\mathcal{H}_s$:*

$$\begin{aligned}
i^\star_{\mathcal{H}_s}(\star\hat{T}^a T_{ab}) = &\big(u^2(\partial_u\phi)^2 + R^2(1-2mR)u^2\partial_R\phi\partial_u\phi \\
&+R^2(1-2mR)\left(\frac{(2+uR)^2}{2s} - \frac{mu^2R^3}{s} - (1+uR)\right)(\partial_R\phi^2) \\
&+\left(\frac{u^2R^2(1-2mR)}{s} + 2(1+uR)\right)\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi| + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\Big)\sin(\theta)\,du\wedge d\theta\wedge d\psi;
\end{aligned}$$

- *to S_u:*

$$\begin{aligned}
i^\star_{S_u}(\star\hat{T}^a T_{ab}) = &\Big(\frac{1}{2}\left((2+uR)^2 - 2mR^3u^2\right)\left(\partial_R\phi^2\right) \\
&+u^2\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi|^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\Big)\sin(\theta)\,dR\wedge d\theta\wedge d\psi;
\end{aligned}$$

- *to* $\mathscr{I}^+_{u_0}$

$$i^\star_{\mathscr{I}^+}(\star\hat{T}^aT_{ab}) = \left(u^2(\partial_u\phi)^2 + \|\nabla_{\mathbb{S}^2}\phi| + \phi^2 + b\frac{\phi^4}{2}\right)\sin(\theta)du\wedge d\theta\wedge d\psi.$$

Proof. For the calculation to come, let us denote by A, the quantity:

$$A = -\frac{1}{2}\hat{\nabla}_c\phi\hat{\nabla}^c\phi + \frac{\phi^2}{2} + b\frac{\phi^4}{4}$$

We calculate the general form of $\star T^aT_{ab}$:

$$\begin{aligned}\star\hat{T}^aT_{ab} &= \star(u^2\partial_u^a - 2(1+uR)\partial_R^a)T_{ab}\\ &= (u^2\partial_u\phi - 2(1+uR)\partial_R\phi)\star\hat{\nabla}_b\phi + u^2A\partial_u^a\lrcorner\mu[\hat{g}] - 2A(1+uR)\partial_R\lrcorner\mu[\hat{g}].\end{aligned}$$

Using the expression of the volume form in the coordinate system (equation (2.2)), we obtain:

$$\begin{array}{rclrcl}\partial_u\lrcorner\mu[\hat{g}] &=& \mathrm{d}R\wedge\mathrm{d}^2\omega_{\mathbb{S}^2}, & \partial_\theta\lrcorner\mu[\hat{g}] &=& \sin(\theta)\mathrm{d}R\wedge\mathrm{d}u\wedge\mathrm{d}\theta,\\ \partial_R\lrcorner\mu[\hat{g}] &=& -\mathrm{d}u\wedge\mathrm{d}^2\omega_{\mathbb{S}^2}, & \partial_\phi\lrcorner\mu[\hat{g}] &=& -\sin(\theta)\mathrm{d}R\wedge\mathrm{d}u\wedge\mathrm{d}\phi.\end{array}$$

$\hat{\nabla}\phi$ is written in the coordinates (R, u, ω_{S^2}) as follows:

$$\hat{\nabla}\phi = -\partial_R\phi\partial_u - (\partial_u\phi + R^2(1-2mR)\partial_R\phi)\partial_R - \hat{\nabla}_{\mathbb{S}^2}\phi;$$

its norm is then:

$$\begin{aligned}\hat{\nabla}_c\phi\hat{\nabla}^c\phi &= (-\partial_R\phi)^2g(\partial_u,\partial_u) + 2\partial_R\phi(\partial_u\phi + R^2(1-2mR)\partial_R\phi))g(\partial_u,\partial_R) - |\hat{\nabla}_{\mathbb{S}^2}\phi|^2\\ &= -R^2(1-2mR)(\partial_R\phi)^2 - 2\partial_R\phi\partial_u\phi - |\hat{\nabla}_{\mathbb{S}^2}\phi|^2.\end{aligned}$$

The Hodge dual of $\nabla_a\phi$ is calculated by splitting $\hat{\nabla}_a u$ over $(\mathrm{d}u, \mathrm{d}R, \mathrm{d}\omega_{S^2})$:

$$\begin{array}{rcl}\star\hat{\nabla}_b\phi &=& \star\left(\partial_u\phi\mathrm{d}u + \partial_R\phi\mathrm{d}R + \partial_\theta\phi\mathrm{d}\theta + \partial_\psi\phi\mathrm{d}\psi\right)\\ &=& \partial_u\phi\mathrm{d}R\wedge\mathrm{d}\omega_{\mathbb{S}^2} - \partial_R\phi\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}\end{array}$$

The gradient of each coordinates is calculated:

$$\begin{array}{rclrcl}\hat{\nabla}^b u &=& \hat{g}^{ab}\hat{\nabla}_a u = -\partial_R & \hat{\nabla}^b\theta &=& -\sin(\theta)\partial_\theta\\ \hat{\nabla}^b R &=& -\partial_u - R^2(1-2mR)\partial_R & \hat{\nabla}^b\psi &=& -\partial_\psi,\end{array}$$

and, as a consequence,

$$\begin{array}{rclrcl}\star\mathrm{d}u &=& \mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2} & \star\mathrm{d}\theta &=& -\sin\theta\mathrm{d}u\wedge\mathrm{d}R\wedge\mathrm{d}\psi\\ \star\mathrm{d}R &=& -\mathrm{d}R\wedge\mathrm{d}\omega_{\mathbb{S}^2} + R^2(1-2mR)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2} & \star\mathrm{d}\psi &=& \mathrm{d}u\wedge\mathrm{d}R\wedge\mathrm{d}\theta\end{array},$$

so that $\star\nabla_a\phi$ is:

$$\begin{array}{rcl}\star\nabla_a\phi &=& \partial_u\phi\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2} + \partial_R\phi\left(-\mathrm{d}R\wedge\mathrm{d}\omega_{\mathbb{S}^2} + R^2(1-2mR)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}\right)\\ && -\partial_\theta\phi\sin\theta\mathrm{d}u\wedge\mathrm{d}R\wedge\mathrm{d}\psi + \partial_\psi\mathrm{d}u\wedge\mathrm{d}R\wedge\mathrm{d}\theta\\ &=& \left(\partial_u\phi + R^2(1-2mR)\partial_R\phi\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2} - \partial_R\phi\mathrm{d}R\wedge\mathrm{d}\omega_{\mathbb{S}^2}\\ && -\partial_\theta\phi\sin\theta\mathrm{d}u\wedge\mathrm{d}R\wedge\mathrm{d}\psi + \partial_\psi\mathrm{d}u\wedge\mathrm{d}R\wedge\mathrm{d}\theta\end{array}$$

The energy 3-form is then:

$$\begin{aligned}\star\hat{T}^aT_{ab} = &\Big[u^2(\partial_u\phi)^2 + R^2(1-2mR)\left(u^2\partial_R\phi\partial_u\phi - (1+uR)(\partial_R\phi)^2\right.\\ &+2(1+uR)\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi|^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\Big]\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}\\ &+\left[\frac{1}{2}\left((2+uR)^2 - 2mR^3u^2\right)\left(\partial_R\phi^2\right) + u^2\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi|^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\right]\mathrm{d}R\wedge\mathrm{d}\omega_{\mathbb{S}^2}\\ &+\sin(\theta)\left[u^2\partial_u\phi - 2(1+uR)\partial_R\phi\right]\left(-\partial_\theta\phi\mathrm{d}u\wedge\mathrm{d}R\wedge\mathrm{d}\psi + \partial_\psi\phi\mathrm{d}u\wedge\mathrm{d}R\wedge\mathrm{d}\theta\right)\end{aligned}$$

Remark 2.4. *To calculate the restrictions of $\star\hat{T}^a T_{ab}$ to the hypersurfaces $\mathcal{H}_s$, S_u and $\mathscr{I}^+$, it is necessary to give the restrictions of each of the differentials of the coordinates. Nonetheless, since ∂_θ and ∂_ϕ are tangent to $\mathcal{H}_s$, S_u and $\mathscr{I}^+$, the only remaining 3-forms to consider when restricting to these hypersurfaces are $du \wedge d\omega_{\mathbb{S}^2}$ and $dR \wedge d\omega_{\mathbb{S}^2}$. This means that only du and dR should be taken care of.*

Noticing that

$$\frac{\mathrm{d}r^*}{\mathrm{d}R} = \frac{-1}{R^2(1-2mR)},$$

we get, on $\mathcal{H}_s$, defined in $\Omega^+_{u_0}$ by $u = -sr^*$:

$$\mathrm{d}R|_{\mathcal{H}_s} = \frac{R^2(1-2mR)}{s}\,\mathrm{d}u|_{\mathcal{H}_s} = \frac{r^* R^2(1-2mR)}{|u|}\,\mathrm{d}u|_{\mathcal{H}_s}.$$

The restriction to $\mathcal{H}_s$ of the energy 3-form is then:

$$\begin{aligned} i^\star_{\mathcal{H}_s}(\star\hat{T}^a T_{ab}) = \Big(\Big[&u^2(\partial_u\phi)^2 + R^2(1-2mR)\left(u^2\partial_R\phi\partial_u\phi - (1+uR)(\partial_R\phi\right)^2 \\ &+2(1+uR)\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi|^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\Big] \\ +\frac{R^2(1-2mR)}{s}&\left[\frac{1}{2}\left((2+uR)^2 - 2mR^3u^2\right)\left(\partial_R\phi^2\right) + u^2\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi|^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\right] \\ &\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2} \end{aligned}$$

$$\begin{aligned} i^\star_{\mathcal{H}_s}(\star\hat{T}^a T_{ab}) =& \left(u^2(\partial_u\phi)^2 + R^2(1-2mR)u^2\partial_R\phi\partial_u\phi\right. \\ &+R^2(1-2mR)\left(\frac{(2+uR)^2}{2s} - \frac{mu^2R^3}{s} - (1+uR)\right)(\partial_R\phi^2) \\ &+\left(\frac{u^2R^2(1-2mR)}{s} + 2(1+uR)\right)\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi| + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\Big)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}. \end{aligned}$$

$\mathscr{I}^+$ is defined by $R=0$; so the restriction to $\mathscr{I}^+$ of the energy 3-form is:

$$i^\star_{\mathscr{I}^+}(\star\hat{T}^a T_{ab}) = \left(u^2(\partial_u\phi)^2 + \left(|\nabla_{\mathbb{S}^2}\phi| + \phi^2 + b\frac{\phi^4}{2}\right)\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}.$$

Finally, for S_u which is defined by $\{u = \mathit{constant}\}$, we obtain:

$$i^\star_{S_u}(\star\hat{T}^a T_{ab}) = \left(\frac{1}{2}\left((2+uR)^2 - 2mR^3u^2\right)\left(\partial_R\phi^2\right) + u^2\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi|^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\right)\mathrm{d}R\wedge\mathrm{d}\omega_{\mathbb{S}^2}.\circledast$$

Proposition 2.5. *There exists u_0, such that the following energy estimates holds on $\mathcal{H}_s$ in $\Omega^+_{u_0}$:*

$$\int_{\mathcal{H}_s} i^\star_{\mathcal{H}_s}\left(\star\hat{T}^a T_{ab}\right) \approx \int_{\mathcal{H}_s}\left(u^2(\partial_u\phi)^2 + \frac{R}{|u|}(\partial_R\phi)^2 + |\nabla_{\mathbb{S}^2}\phi|^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right) du\wedge d\omega_{\mathbb{S}^2}$$

Proof. Let us consider the expression of $i^\star_{\mathcal{H}_s}(\star\hat{T}^a T_{ab})$:

$$\begin{aligned} i^\star_{\mathcal{H}_s}(\star\hat{T}^a T_{ab}) \;=\; &\Big(u^2(\partial_u\phi)^2 && (2.3)\\ &+R^2u^2(1-2mR)\partial_R\phi\partial_u\phi && (2.4)\\ &+R^2(1-2mR)\left(\frac{(2+uR)^2}{2s} - \frac{mu^2R^3}{s} - (1+uR)\right)(\partial_R\phi^2) && (2.5)\\ &+\left(\frac{u^2R^2(1-2mR)}{s} + 2(1+uR)\right)\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi| + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\Big)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}. && (2.6) \end{aligned}$$

Each of these terms is estimated separately using lemma 1.5 and the obtained estimates are summed. Let ϵ be a given positive number and let u_0 be the non positive constant associated to ϵ via lemma 1.5; ϵ will be chosen during the proof.

Nothing needs to be done for (2.3).

For (2.6), since u is non-positive and $s = -\frac{u}{r^*} = \frac{|u|}{r^*}$, we have, on one side:

$$\begin{aligned}\left(\frac{u^2R^2(1-2mR)}{s} + 2(1-|u|R)\right) &= (Rr^*)(R|u|)(1-2mR) + 2(1-|u|R)\\ &\leq (Rr^*)(R|u|)(1-2mR) + 2\\ &\leq (1-\epsilon)(1+\epsilon) + 2\end{aligned}$$

and, on the other side:

$$\begin{aligned}\left(\frac{u^2R^2(1-2mR)}{s} + 2(1-|u|R)\right) &= (Rr^*)(1-2mR)(R|u|) + 2(1-|u|R)\\ &\geq 1\cdot(1-\epsilon)\cdot(R|u|) + 2(1-|u|R)\\ &\geq 2-(1+\epsilon)(R|u|)\\ &\geq 2-(1+\epsilon)(1+\epsilon)\\ &\geq 1-2\epsilon-\epsilon^2. \end{aligned} \tag{2.7}$$

ϵ is chosen such as (2.7) is positive.

For (2.5), the proof slightly more complicated. We have, since u is non-positive and $s = -\frac{u}{r^*} = \frac{|u|}{r^*}$, on one hand:

$$R^2(1-2mR)\left(\frac{(2+uR)^2}{2s} - \frac{mu^2R^3}{s} - (1+uR)\right)(\partial_R\phi)^2$$

$$= R^2(1-2mR)\left(\frac{r^*(2-|u|R)^2}{2|u|} - (mR)(R|u|)(Rr^*) - (1+uR)\right)(\partial_R\phi)^2$$

$$= \left(\frac{R}{|u|}(\partial_R\phi)^2\right)(1-2mR)(Rr^*)\left(\frac{(2-|u|R)^2}{2} - (mR)(R|u|)^2 - \frac{|u|}{r^*} + \frac{|u|}{r^*}R|u|\right) \tag{2.8}$$

$$\leq \left(\frac{R}{|u|}(\partial_R\phi)^2\right)\cdot 1\cdot(1+\epsilon)\left(\frac{(3+\epsilon^2)^2}{2} + 1\cdot(1+\epsilon)\right).$$

Starting from equation (2.8), the lower bound for (2.5) is obtained as follows, setting $X = |u|R$:

$$\begin{aligned}&\left(\frac{R}{|u|}(\partial_R\phi)^2\right)(1-2mR)(Rr^*)\left(\frac{(2-|u|R)^2}{2} - (mR)(R|u|)^2 - \frac{|u|}{r^*} + \frac{|u|}{r^*}R|u|\right)\\ &\geq \left(\frac{R}{|u|}(\partial_R\phi)^2\right)(1-\epsilon)\cdot\left(\frac{(2-X)^2}{2}(Rr^*) - (mR)(Rr^*)X^2 - X + X^2\right)\\ &\geq \left(\frac{R}{|u|}(\partial_R\phi)^2\right)\frac{(1-\epsilon)}{2}\left(4-6X+3X^2-\epsilon(1+\epsilon)X^2\right)\end{aligned}$$

The polynomial $4-6X+3X^2$ reaches its minimum for $X=1$ and equals 1 at $X=1$ so that:

$$\begin{aligned}&\left(\frac{R}{|u|}(\partial_R\phi)^2\right)(1-2mR)(Rr^*)\left(\frac{(2-|u|R)^2}{2} - (mR)(R|u|)^2 - \frac{|u|}{r^*} + \frac{|u|}{r^*}R|u|\right)\\ &\geq \left(\frac{R}{|u|}(\partial_R\phi)^2\right)\frac{(1-\epsilon)}{2}\left(1-\epsilon(1+\epsilon)^3\right)\end{aligned}$$

To deal with (2.4), we write:

$$|R^2u^2(1-2mR)\partial_R\phi\partial_u\phi| = (1-2mR)\left(R^2|u|\sqrt{\frac{2}{3}}\partial_R\phi\right)\left(\sqrt{\frac{3}{2}}u\partial_u\phi\right),$$

so that:

$$
\begin{aligned}
|R^2u^2(1-2mR)\partial_R\phi\partial_u\phi| &\leq (1-2mR)\frac{1}{2}\left(\frac{3}{2}\left(u\partial_u\phi\right)^2+\frac{2}{3}(R^3|u|^3)\frac{R}{|u|}(\partial_R\phi)^2\right)\\
&\leq \frac{3}{4}\left(u\partial_u\phi\right)^2+\frac{1}{3}(1+\epsilon)^3\frac{R}{|u|}(\partial_R\phi)^2
\end{aligned}
$$

Finally, the following equivalence estimates hold:

$$
c_\epsilon\int_{\mathcal{H}_s}\left(u^2(\partial_u\phi)^2+\frac{R}{|u|}(\partial_R\phi)^2+|\nabla_{\mathbb{S}^2}\phi|^2+\frac{\phi^2}{2}+b\frac{\phi^4}{4}\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}\leq\int_{\mathcal{H}_s}i^\star_{\mathcal{H}_s}\left(\star\hat{T}^aT_{ab}\right)
$$

and

$$
\int_{\mathcal{H}_s}i^\star_{\mathcal{H}_s}\left(\star\hat{T}^aT_{ab}\right)\leq C_\epsilon\int_{\mathcal{H}_s}\left(u^2(\partial_u\phi)^2+\frac{R}{|u|}(\partial_R\phi)^2+|\nabla_{\mathbb{S}^2}\phi|^2+\frac{\phi^2}{2}+b\frac{\phi^4}{4}\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}
$$

where

$$
C_\epsilon=\max\left(1,\frac{1}{3}(1+\epsilon)^3,(1+\epsilon)\left(\frac{(3+\epsilon^2)^2}{2}+(1+\epsilon)\right),(1-\epsilon)(1+\epsilon)+2\right)
$$

and

$$
c_\epsilon=\min\left(\frac{1}{4},\frac{1}{6}-\epsilon P(\epsilon),1-2\epsilon-\epsilon^2\right),
$$

where $P(\epsilon)$ is a polynomial in ϵ. ϵ is chosen such that the constant c_ϵ is positive. Using lemma 1.5, there exists u_0, negative, $|u_0|$ large enough, such that the estimates of the coordinates hold in $\Omega^+_{u_0}$ and, consequently, the equivalence is true on this domain.⊛

u_0 is now fixed, being equal to the u_0 associated with the ϵ which ensures that the energy equivalence established in proposition 2.5 holds. The neighborhood of i^0 where the energy estimates are relevant is then $\Omega^+_{u_0}$.

2.1.2 Energy estimates near i^0

The energy estimates are established between $\Sigma_0^{u_0>}$, S_{u_0} and $\mathscr{I}^+_{u_0}$, by writing a Stokes theorem between $\mathcal{H}_s$, $S^s_{u_0}=\{(u,R,\omega_{\mathbb{S}^2})|u=u=u_0,u\leq -sr^*\}$ and $\mathscr{I}^+_{u_0}$.

The first step consists in evaluating the error term:

Lemma 2.6. *The error is given by:*

$$
\hat{\nabla}^a\left(\hat{T}^bT_{ab}\right)=4mR^2(3+uR)\left(\partial_R\phi\right)^2+(1-12mR)\,\phi\left(u^2\partial_u\phi-2(1+uR)\partial_R\phi\right)+\hat{T}^a\hat{\nabla}_ab\frac{\phi^4}{4}.
$$

Proof.The Killing form of the vector $\hat{T}^a$ is calculated via the Lie derivative of the metric:

$$
\begin{gathered}
\hat{\nabla}_{(a}\hat{T}_{b)}=\mathrm{L}_{\hat{T}}\hat{g}\\
=\mathrm{L}_{\hat{T}}(R^2(1-2mR))(\mathrm{d}u)^2+2R^2(1-2mR)\mathrm{L}_{\hat{T}}(\mathrm{d}u)\mathrm{d}u-2\mathrm{L}_{\hat{T}}(\mathrm{d}u)\mathrm{d}R-2\mathrm{L}_{\hat{T}}(\mathrm{d}R)\mathrm{d}u-2\mathrm{L}_{\hat{T}}(\mathrm{d}\omega_{\mathbb{S}^2})\\
=-4R(1+uR)(1-3mR)(\mathrm{d}u)^2+2R^2(1-2mR)\mathrm{d}\left(\mathrm{L}_{\hat{T}}(u)\right)\mathrm{d}u-2\mathrm{d}\left(\mathrm{L}_{\hat{T}}(u)\right)\mathrm{d}R-2\mathrm{d}\left(\mathrm{L}_{\hat{T}}(R)\right)\mathrm{d}u-0\\
=-4R(1+uR)(1-3mR)(\mathrm{d}u)^2+2R^2(1-2mR)\mathrm{d}(u^2)\mathrm{d}u-2\mathrm{d}\left(u^2\right)\mathrm{d}R-2\mathrm{d}\left(-2(1+uR)\right)\mathrm{d}u\\
=-4R(1+uR)(1-3mR)(\mathrm{d}u)^2+4uR^2(1-2mR)(\mathrm{d}u)^2-4u\mathrm{d}u\mathrm{d}R+4(R\mathrm{d}u+u\mathrm{d}R)\mathrm{d}u\\
=(-2R(1+uR)(1-3mR)+4uR^2(1-2mR)+4R)(\mathrm{d}u)^2\\
=(12mR^2+4muR^3)(\mathrm{d}u^2),
\end{gathered}
$$

or, in the vector form:

$$
\hat{\nabla}^{(a}\hat{T}^{b)}=4mR^2(3+uR)\partial_R\partial_R.
$$

A direct consequence of the above formula is that the Killling form is trace-free:

$$\begin{aligned}\nabla^a \hat{T}_a &= 4mR^2(3+uR)\hat{g}(\partial_R, \partial_R) \\ &= 0.\end{aligned}$$

The scalar curvature of the rescaled metric is given by equation (1.2); choosing $\Omega = R$ gives:

$$\begin{aligned}\frac{1}{6}\mathrm{Scal}_{\hat{g}} &= R^3\nabla_b\nabla^b R \\ &= 2mR.\end{aligned}$$

Finally, the error term is given by:

$$\begin{aligned}\nabla^{(a}\hat{T}^{b)}T_{ab} &= 4mR^2(3+uR)\left(\partial_R\phi\right)^2 + (1-12mR)\left(\hat{T}^a\nabla_a\phi\right)\phi + \hat{T}^a\hat{\nabla}_a b\frac{\phi^4}{4} \\ &= 4mR^2(3+uR)\left(\partial_R\phi\right)^2 + (1-12mR)\,\phi\left(u^2\partial_u\phi - 2(1+uR)\partial_R\phi\right) \\ &\quad + \hat{T}^a\hat{\nabla}_a b\frac{\phi^4}{4}. ⊛\end{aligned}$$

Remark 2.7. *As noticed in [33], one obstacle to the use of the parameter s for the foliation is the fact that this parametrization in s is not smooth in the sense that $(r^*)^{-1}$ is not a smooth function of R at $R=0$.*

In order to avoid this singularity, the speed of the identifying vector field is decreased by means of a change of variable: let τ be the function defined by:

$$\tau : \begin{array}{ccc} [0,1] & \longrightarrow & [0,2] \\ s & \longmapsto & -2(\sqrt{s}-1). \end{array} \tag{2.9}$$

$\mathscr{I}^+_{u_0}$ is then given by $s=0$ and $\tau=2$ and $\Sigma_0^{u_0>}$ is given by $s=1$ and $\tau=0$. The new identifying vector field V^a is then chosen such that:

$$\mathrm{d}\tau(V^a) = 1 \text{ so that } V^a = (r^*R)^{\frac{3}{2}}(1-2mR)\sqrt{\frac{R}{|u|}}\partial_R^a. \tag{2.10}$$

The foliation $\mathcal{H}_s$, when parametrized by τ, is denoted by Σ_τ

Finally, we can prove the following estimates:

Proposition 2.8. *The following equivalence holds:*

$$E(\mathscr{I}^+_{u_0}) + E(S_{u_0}) \approx E(\Sigma_0^{u_0<})$$

where

$$E(\mathscr{I}^+_{u_0}) = \int_{\mathscr{I}^+_{u_0}} i^\star_{\mathscr{I}^+}(\star\hat{T}^a T_{ab}) \text{ and } E(S_{u_0}) = \int_{S_{u_0}} i^\star_{S_{u_0}}(\star\hat{T}^a T_{ab}).$$

Proof.The proof of these estimates relies on Stokes theorem applied between the hypersufaces $S^s_{u_0} = \{(u, R, \omega_{\mathbb{S}^2}) | u = u_0, |u| \geq sr^*\}$, $\mathcal{H}_{s(\tau)} = \Sigma_\tau$ and $\Sigma_0^{u_0>}$. Stokes theorem can be used here since the data are compactly supported in Σ_0 and, as a consequence, the future of the support of the initial data does not contain the singularity i^0. Let be $M^s_{u_0}$ be the subset of $\Omega^+_{u_0}$ whose boundary consists of these hypersufaces. We have:

$$\int_{S^{s(\tau)}_{u_0}} i^\star_{S_u}(\star\hat{T}^a T_{ab}) + \int_{\Sigma_\tau} i^\star_{\Sigma_\tau}(\star\hat{T}^a T_{ab}) - \int_{\Sigma_0^{u_0>}} i^\star_{\Sigma_0}(\star\hat{T}^a T_{ab}) - = \int_{M^{s(\tau)}_{u_0}} \hat{\nabla}^a\left(\hat{T}^b T_{ab}\right)\mu[\hat{g}]$$

and, using the notations in the proposition, the foliation given by τ defined by equation (2.9) and lemma 2.6, this becomes:

$$E(S^{s(\tau)}_{u_0}) + \int_{\Sigma_\tau} i^\star_{\Sigma_\tau}(\star \hat{T}^a T_{ab}) - \int_{\Sigma_0^{u_0>}} i^\star_{\Sigma_0}(\star \hat{T}^a T_{ab})$$

$$= \int_0^\tau \left(\int_{\Sigma_\tau} \left\{ 4mR^2(3+uR)\left(\partial_R\phi\right)^2 + (1-12mR)\,\phi\left(u^2\partial_u\phi - 2(1+uR)\partial_R\phi\right) \right.\right.$$
$$\left.\left. + \hat{T}^a\hat{\nabla}_a b\frac{\phi^4}{4}\right\}(r^*R)^{\frac{3}{2}}(1-2mR)\sqrt{\frac{R}{|u|}}\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}\right)\mathrm{d}\tau$$

The error term is bounded above in absolute value; each term is evaluated separetly:

$$\begin{aligned}
&|(r^*R)^{\frac{3}{2}}(1-2mR)\sqrt{\frac{R}{|u|}}4mR^2(3+uR)\left(\partial_R\phi\right)^2| \\
&= (r^*R)^{\frac{3}{2}}(1-2mR)(3+|u|R)(R|u|)^{1/2}R\frac{R}{|u|}\left(\partial_R\phi\right)^2 \\
&\leq (1+\epsilon)^{\frac{3}{2}}\cdot 1\cdot(4+\epsilon)\cdot(1+\epsilon)\frac{\epsilon}{2m}\frac{R}{|u|}\left(\partial_R\phi\right)^2 \\
&\lesssim \frac{R}{|u|}\left(\partial_R\phi\right)^2.
\end{aligned}$$

and

$$\begin{aligned}
|\,(1-12mR)\,(r^*R)^{\frac{3}{2}}(1-2mR)\sqrt{\frac{R}{|u|}}u^2\phi\partial_u\phi| &= |\,(1-12mR)\,(r^*R)^{\frac{3}{2}}(1-2mR)\sqrt{R|u|}\phi(u\partial_u\phi|) \\
&\lesssim \frac{1}{2}(1+6\epsilon)(1+\epsilon)^{\frac{3}{2}}\cdot 1\cdot(1+\epsilon)\left(\phi^2+(u\partial_u\phi)^2\right) \\
&\lesssim \phi^2+(u\partial_u\phi)^2.
\end{aligned}$$

The remaining term is controlled by:

$$\begin{aligned}
|(1-12mR)(r^*R)^{\frac{3}{2}}(1-2mR)\sqrt{\frac{R}{|u|}}\phi\partial_R\phi &\leq (1+6\epsilon)(1+\epsilon)^{\frac{3}{2}}\phi\left(\sqrt{\frac{R}{|u|}}\partial_R\phi\right) \\
&\leq (1+6\epsilon)(1+\epsilon)^{\frac{3}{2}}\left(\phi^2+\frac{R}{|u|}(\partial_R\phi)^2\right).
\end{aligned}$$

Remark 2.9. *This term is the main obstacle in the use of the parameter s: if the foliation was parametrized by s, this term would be replaced by:*

$$\left|(1-12mR)\frac{(r^*R)^2(1-2mR)}{|u|}\phi\partial_R\phi\right| \leq (1+6\epsilon)(1+\epsilon)^2\left|\frac{\phi\partial_R\phi}{u}\right|$$

which cannot be compared to $\phi^2+\frac{R}{|u|}(\partial_R\phi)^2$.

Gathering these inequalities, it remains:

$$\left|\int_0^\tau \left(\int_{\Sigma_\tau} \left\{ 4mR^2(3+uR)\left(\partial_R\phi\right)^2 + (1-12mR)\,\phi\left(u^2\partial_u\phi - 2(1+uR)\partial_R\phi\right) \right.\right.\right.$$
$$\left.\left.\left. + \hat{T}^a\hat{\nabla}_a b\frac{\phi^4}{4}\right\}(r^*R)^{\frac{3}{2}}(1-2mR)\sqrt{\frac{R}{|u|}}\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}\right)\mathrm{d}\tau\right| \lesssim \int_0^\tau E(\Sigma_r)\mathrm{d}r.$$

Finally, the following inequalities hold:

$$\begin{array}{lll} E(\Sigma_\tau) + E(S_{u_0}^{s(\tau)}) & \lesssim & \int_0^\tau E(\Sigma_r)\mathrm{d}r + E(\Sigma_0^{u_0>}) \\ E(\Sigma_0^{u_0>}) & \lesssim & \int_0^\tau E(\Sigma_r)\mathrm{d}r + E(\Sigma_\tau) + E(S_{u_0}^{s(\tau)}) \end{array} \quad (2.11)$$

Since $E(S_{u_0}^{s(\tau)})$ is positive, the integral inequality holds:

$$E(\Sigma_\tau) \lesssim \int_0^\tau E(\Sigma_r)\mathrm{d}r + E(\Sigma_0^{u_0>}).$$

Using Gronwall's lemma, we obtain:

$$E(\Sigma_\tau) \lesssim E(\Sigma_0^{u_0>}). \quad (2.12)$$

Putting this inequality back into (2.11), we obtain the first part of the inequality, for $\tau = 2$:

$$E(\mathscr{I}_{u_0}^+) + E(S_{u_0}) \lesssim E(\Sigma_0^{u_0>}).$$

The other inequality is obtained by doing the same calculation between the hypersurfaces $S_{u_0,s} = \{(u, R, \omega_{\mathbb{S}^2}) | u = u_0, |u| \leq sr^*\}$, $\mathcal{H}_s$ and $\Sigma_0^{u_0>}$. Let be $M_s^{u_0}$ be the subset of $\Omega_{u_0}^+$ whose boundary consists of these hypersurfaces. Applying Stokes theorem, using the parametrization by τ and the previous estimates of the error term, we obtain:

$$E(\Sigma_\tau) \lesssim \int_0^\tau E(\Sigma_r)\mathrm{d}r + E(\mathscr{I}_{u_0}^+) + E(S^{u_0,s(\tau)})$$

As a consequence, since the integrand in $E(S^{u_0,s(\tau)})$ is positive, the following integral inequality holds:

$$E(\Sigma_\tau) \lesssim \int_0^\tau E(\Sigma_r)\mathrm{d}r + E(\mathscr{I}_{u_0}^+) + E(S^{u_0}).$$

The use of Gronwall lemma gives the second inequality:

$$E(\Sigma_0^{u_0<}) \lesssim E(\mathscr{I}_{u_0}^+) + E(S^{u_0}) \circledast$$

2.2 Energy estimates far from the spacelike infinity i^0

The estimates which are obtained in this section are widely inspired by the work of Hörmander [30] and generalized in [36] to establish the existence of solutions for the characteristic Cauchy problem for the wave equation on a curved background. The main tool consists in writing the characteristic, or weakly characteristic (that is to say is locally either spacelike or degenerate; this is also referred to as achronal) surface as the graph of a function and expressing all the relevant quantities in term of this graph. This method was also used by Mason-Nicolas in [32] to control how spacelike surfaces converge to null infinity.

$\hat{M} \backslash \Omega_{u_0}^+ \cap J^+(\Sigma_0)$ is divided in two parts as follows:

- let Σ be a spacelike hypersurface in $\hat{M}$ for the metric $\hat{g}$ such that $\Sigma \cap \mathscr{I}^+ = S_{u_0} \cap \mathscr{I}^+$ and Σ is orthogonal to $\hat{T}^a$.
- the part of $\hat{M} \backslash \Omega_{u_0}^+$ contained between Σ_0 and Σ, denoted by V;
- and finally the future of Σ, containing i^+, U. The subset of its boundary in $\mathscr{I}^+$ is denoted by $\mathscr{I}_T^+$.

This decomposition of the future of Σ_0 is represented in figure 2.2.

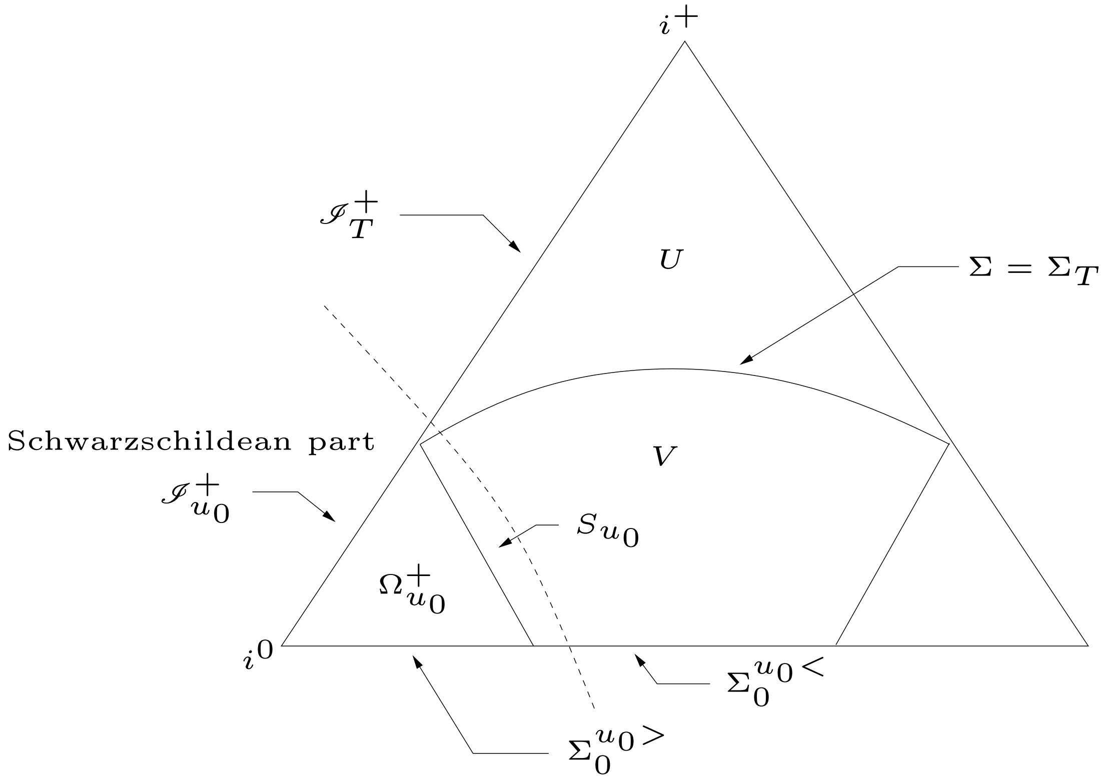


Figure 2: Future of Σ_0

Remark 2.10. *The fact that the timelike vector field $\hat{T}^a$ is orthogonal to the spacelike hypersurface is an assumption which is made about this timelike vector field. It is build outside the Schwarzshildean part of the manifold as follows:*

- *outside $\Omega^+_{u_0}$, $\hat{T}^a$ is smoothly extended such that it is orthogonal to a uniformly spacelike hypersurface Σ whose boundary is $\mathscr{I}^+ \cap S_{u_0}$.*
- *The intersection of S_{u_0} and $\mathscr{I}^+$ is the Schwarzschildean part and is given $\{u = u_0, R = 0\}$. The vector field $\hat{T}^a$ is then orthogonal to this 2-dimensional surface.*

This subsection deals with estimates on U. As previously said, we use here Hormander's techniques which consists in writing $\mathscr{I}$ as the graph of a function.

We consider on U the flow associated with $e_0^a = \frac{\hat{T}^a}{\hat{g}_{cd}\hat{T}^c\hat{T}^d}$. Let t be the time function induced by this flow. The set $\{e_0^a\}$ is completed in an orthonormal basis of $\hat{M}$ by choosing an orthonormal basis $\{e_i^a; i = 1, 2, 3\}$ of the spacelike foliation $\{\Sigma_t\}$ induced by t. For the sake of clarity, Σ is denoted Σ_T as corresponding to the slice $\{t = T\}$ (T is chosen to be non zero, in order not to introduce confusion with Σ_0).

2.2.1 Geometric description of $\mathscr{I}_T^+$

Using the flow Ψ_t associated with e_0^a, $\mathscr{I}_T^+$ can be identified with Σ_T:

$$\begin{array}{ccc} \Sigma_T & \longrightarrow & \mathscr{I}_T^+ \\ x & \longmapsto & \Psi_{\varphi(x)}(x) \end{array} \tag{2.13}$$

where $\varphi(x)$ is the time at which the curve $t \mapsto \Psi_t(x)$ hits $\mathscr{I}^+_T$. $\mathscr{I}^+_T$ can then be considered as defined by the graph of the function $\varphi : x \in \Sigma_T \longmapsto \varphi(x)$.

We denote by $\nabla_i\varphi$ the derivatives of φ with regard to the vector tangent to Σ_T $e^a_{\mathbf{i}}$ at time T.

Remark 2.11. *a. As noticed in the introduction, the spacetimes constructed by Chrusciel-Delay and Corvino-Schoen have the specificity that the regularity at $\mathscr{I}^+\backslash\{i^0, i^+\}$ can be specified arbitrarily. In order to insure that some geometric quantities are defined, we assumed that the manifold is C^2 differentiable at $\mathscr{I}^+\backslash\{i^0, i^+\}$. The use of the implicit functions theorem then insures that the function φ has the same regularity.*

b. The function φ is defined on a compact set and as such admits a maximum. This maximum is denoted by T_{max}.

The lapse function associated with this choice of time t is denoted by N. The metric can be decomposed as:

$$\hat{g} = N^2(\mathrm{d}t)^2 - h_{\Sigma_t}$$

where h is a Riemannian metric on Σ_t depending on the spacelike leaves of the foliation induced by the time function, and N is the (positive) lapse function. Since the time function is build from the vector e^a_0, the vector field ∂_t satisfies:

$$\partial_t = N e^a_0.$$

The following lemma describes the geometry of $\mathscr{I}^+_T$ in term of the parametrization:

Lemma 2.12. *The vector N^a defined by*

$$N^a = e^a_0 - \sum_{j\in\{1,2,3\}} N\nabla_j\varphi e^a_{\boldsymbol{j}}$$

is normal and tangent to the hypersurface $\mathscr{I}^+_T$.
The set of vectors defined by, for $i \in \{1, 2, 3\}$,

$$t^a_{\boldsymbol{i}} = N\nabla_i\varphi e^a_0 - e^a_{\boldsymbol{i}}$$

are normal to N^a and, as such, forms a basis of $T\mathscr{I}^+_T$.

Proof.The fact that N^a is null directly comes from the fact N^a is normal to $\mathscr{I}^+_T$, which is a null surface. The derivatives of φ then satisfies:

$$N^2 \sum_{i=1,2,3} (\nabla_i\varphi)^2 = 1$$

The tangent plane to $\mathscr{I}^+_T$ is given by the kernel of the differential of the application

$$x \longmapsto (\varphi(x), x)$$

which is given by

$$h^a \in \Sigma_T \longmapsto h^a + g_{ij}(\varphi(x), x)\nabla^i\varphi h^j \underbrace{Ne^a_0}_{\partial_t}.$$

It is then clear that the set of vectors

$$t^a_{\mathbf{i}} = N\nabla^i\varphi e^a_0 - e^a_{\mathbf{i}}$$

forms a basis of $T\mathscr{I}_T^+$. N^a is then a linear combination of them:

$$N^a = \sum_{i=1,2,3} N\nabla^i\varphi t^a_{\mathbf{i}}.\circledast$$

As a direct consequence of this lemma, the vector defined by

$$\tau^a = e_0^a + \sum_{j\in\{1,2,3\}} N\nabla_j\varphi e^a_{\mathbf{j}}$$

is null and transverse to $\mathscr{I}_T^+$.

In order to complete the geometric description of $\mathscr{I}_T^+$ and facilitate the calculation afterwards, we introduce the following objects:

- using the Geroch-Held-Penrose formalism, the set of two null vectors (τ^a, N^a) is completed by two normalized spacelike vectors (v_1^a, v_2^a) tangent to $\mathscr{I}_T^+$ to form a basis of $T\hat{M}$;
- $\mathscr{I}_T^+$ is endowed with the 3-form:

  $$\mu_{\mathscr{I}} = t_a^1 \wedge t_a^2 \wedge t_a^3$$

  which satisfies:

  $$\begin{aligned}\tau_a \wedge t_a^1 \wedge t_a^2 \wedge t_a^3 &= (1 + N^2 \sum_{i=1,2,3}(\nabla^i\varphi)^2)\mu[\hat{g}]\\ &= 2\mu[\hat{g}];\end{aligned}$$

  this 3-form will be used as the form of reference to calculate the energy on $\mathscr{I}_T^+$.

Remark 2.13. *The tangent vector n^a used in definition 1.12 for the H^1-norm on $\mathscr{I}^+$ is colinear to the vector N^a:*

$$N^a = \frac{n^a}{\hat{g}_{cd}\hat{T}^c\hat{T}^d}.$$

As a consequence, the norm used in definition 1.12 are equivalent.

Finally, in order to prepare the estimates, the expression of the 3-form $\star\hat{T}^aT_{ab}$ on the surfaces Σ_t and $\mathscr{I}_T^+$ is given:

Lemma 2.14. *The restrictions of the energy 3-form $\star\hat{T}^aT_{ab}$ to Σ_t, for t given in $[T, T_{max}]$, and $\mathscr{I}_T^+$ are given by, respectively:*

$$i^\star_{\Sigma_t}\left(\star\hat{T}^aT_{ab}\right) = ||\hat{T}^a||\left(\frac{1}{2}\sum_{i=0}^{4}(e^a_{\boldsymbol{i}}\nabla_a\phi)^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right) e_a^1\wedge e_a^2\wedge e_a^3$$

and

$$i^\star_{\mathscr{I}_T^+}\left(\star\hat{T}^aT_{ab}\right) = \frac{||\hat{T}^a||}{4}\left((N^a\hat{\nabla}_a\phi)^2 + (v_1^a\hat{\nabla}_a\phi)^2 + (v_2^a\hat{\nabla}_a\phi)^2) + \frac{\phi^2}{2}\right) t_a^1\wedge t_a^2\wedge t_a^3$$

Remark 2.15. *a. The expression which is given for the energy form on $\mathscr{I}$ is consistent with the one obtain by Hörmander, since it only depends on tangential derivatives to null infinity. It is nonetheless not identical: the result of Hörmander is similar to a calculation made with respect to the Riemannian metric obtained from $\hat{g}$ and the timelike vector field $\hat{T}^a$.*

b. *The part of the energy form on $\mathscr{I}$ given by $(v_1^a\hat{\nabla}_a\phi)^2+(v_2^a\hat{\nabla}_a\phi)^2$ is usually interpreted as the norm of the gradient on a 2 sphere, even though the distribution of 2-planes Span(v_1,v_2) is not integrable.*

Proof. Using the basis $(e_{\mathbf{i}}^a)_{i=0,\dots,4}$ which is adapted to the foliation, the energy 3-form over Σ_t can easily be calculated:

$$i^\star_{\Sigma_t}\left(\star\hat{T}^aT_{ab}\right) = \left(\hat{T}^a\hat{\nabla}_a\phi\right)i^\star_{\star\Sigma_t}(\hat{\nabla}_b\phi)+\left(-\frac{1}{2}\hat{g}_{cd}\hat{\nabla}^c\phi\hat{\nabla}^d\phi+\frac{\phi^2}{2}+b\frac{\phi^4}{4}\right)i^\star_{\Sigma_t}(\star T_b)$$

Since the vectors $\{e_{\mathbf{i}}^a\}_{i=1,2,3}$ are tangent to the hypersufaces Σ_t, we obtain:

- $i^\star_{\Sigma_t}(\star T_a)=||\hat{T}^a||e_0^a\lrcorner\mu[\hat{g}]=||\hat{T}^a||e_a^1\wedge e_a^2\wedge e_a^3$
- $i^\star_{\Sigma_t}(\hat{\nabla}_b\phi)=e_0^b\nabla_b\phi\,(e_0^a\lrcorner\mu[\hat{g}])=(e_0^b\nabla_b\phi)e_a^1\wedge e_a^2\wedge e_a^3$

and

$$i^\star_{\Sigma_t}\left(\star\hat{T}^aT_{ab}\right)=||\hat{T}^a||\left(\frac{1}{2}\sum_{i=0}^{4}(e_{\mathbf{i}}^a\nabla_a\phi)^2+\frac{\phi^2}{2}+b\frac{\phi^4}{4}\right)e_a^1\wedge e_a^2\wedge e_a^3.$$

To calculate the restriction of the energy form to $\mathscr{I}_T^+$, the vector fields $\hat{T}^a$ and $\hat{\nabla}\phi$ are split over the basis (N^a,τ^a,v_1^a,v_2^a):

$$\hat{T}^b = \frac{||\hat{T}^b||}{2}\left(N^b+\tau^b\right) \tag{2.14}$$

$$\hat{\nabla}^b\phi = \frac{N^a\hat{\nabla}_a\phi}{2}\tau^b+\frac{\tau^a\hat{\nabla}_a\phi}{2}N^b-(v_1^a\hat{\nabla}_a\phi)v_1^b+v_2^a\hat{\nabla}_a\phi v_2^b$$

$$\hat{\nabla}_c\phi\hat{\nabla}^c\phi = N^a\hat{\nabla}_a\phi\tau^a\hat{\nabla}_a\phi-(v_1^a\hat{\nabla}_a\phi)^2-(v_2^a\hat{\nabla}_a\phi)^2. \tag{2.15}$$

The only relevant terms in the expressions of $i^\star_{\mathscr{I}^+}(\star\hat{\nabla}_b\phi)$ and $i^\star_{\mathscr{I}^+}(\hat{T}_b)$ are those which are transverse to $\mathscr{I}$, so that:

$$\begin{aligned} i^\star_{\mathscr{I}^+}(\hat{T}_b) &= \tfrac{||\hat{T}^a||}{2}i^\star_{\mathscr{I}}(\tau_a) & i^\star_{\mathscr{I}^+}(\star\hat{\nabla}_b\phi) &= \tfrac{N^a\hat{\nabla}_a\phi}{2}i^\star_{\mathscr{I}}(\tau_a)\\ &= \tfrac{||\hat{T}^a||}{2}\tau^a\lrcorner\mu[\hat{g}] & &= \tfrac{N^a\hat{\nabla}_a\phi}{2}\tau^a\lrcorner\mu[\hat{g}]\\ &= \tfrac{||\hat{T}^a||}{4}t_a^1\wedge t_a^2\wedge t_a^3 & &= \tfrac{N^a\hat{\nabla}_a\phi}{4}t_a^1\wedge t_a^2\wedge t_a^3 \end{aligned}$$

and, finally, using equations (2.14) and (2.15), since the function b vanishes at $\mathscr{I}$:

$$\begin{aligned} i^\star_{\mathscr{I}_T^+}\left(\star\hat{T}^aT_{ab}\right) &= \left(e_0^a\hat{\nabla}_a\phi N^a\hat{\nabla}_a\phi-\frac{1}{2}\hat{\nabla}_c\phi\hat{\nabla}^c\phi+\frac{\phi^2}{2}\right)\frac{||\hat{T}^a||}{4}t_a^1\wedge t_a^2\wedge t_a^3\\ &= \frac{||\hat{T}^a||}{8}\left((N^a\hat{\nabla}_a\phi)^2+(v_1^a\hat{\nabla}_a\phi)^2+(v_2^a\hat{\nabla}_a\phi)^2)+\phi^2\right)t_a^1\wedge t_a^2\wedge t_a^3 \circledast \end{aligned}$$

2.2.2 Energy estimates on U

The techniques used in this section are exactly the same as in the other section: they rely on Gronwall lemma and Stokes theorem carefully applied to the 3-form $\star\hat{T}^aT_{ab}$.

The first step of the proof consists in establishing a decay result for the energy on slices $\{t = constant\}$.

Proposition 2.16. *Let $E(\Sigma_t)$ be the energy on the slice Σ_t:*

$$E(\Sigma_t)=\int_{\Sigma_t}\left(\frac{1}{2}\sum_{i=0}^{4}(e_{\mathbf{i}}^a\nabla_a\phi)^2+\frac{\phi^2}{2}+b\frac{\phi^4}{4}\right)\mu_{\Sigma_t}$$

where $\mu_{\Sigma_t} = e_a^1 \wedge e_a^2 \wedge e_a^3$.
Then this energy satisfies:

$$E(\Sigma_t) \approx \int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star \hat{T}^a T_{ab}\right)$$

and for s *and* t *two real numbers in* $[T, T_{max}]$*, such that* $t \geq s$*:*

$$E(\Sigma_t) \lesssim E(\Sigma_s)$$

Remark 2.17. *a. In this section, the calculations are made with respect to* $E(\Sigma_T)$ *rather than* $\int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star \hat{T}^a T_{ab}\right)$.

Proof. Since $\hat{T}^a$ is a non-vanishing timelike vector field on the compact $\hat{M}$, there exists a positive constant C such that:

$$\frac{1}{C} \leq ||\hat{T}^a|| \leq C$$

and, as a consequence of lemma 2.14, the energy $E(\Sigma_t)$ is equivalent to $\int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star \hat{T}^a T_{ab}\right)$ since:

$$\frac{1}{2C} E(\Sigma_t) \leq \int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star \hat{T}^a T_{ab}\right) \leq C E(\Sigma_t).$$

Remark 2.18. *The same result holds for the energy on* $\mathscr{I}^+_T$*, that it to say that:*

$$\int_{\mathscr{I}^+_T} i^\star_{\mathscr{I}^+_T}\left(\star \hat{T}^a T_{ab}\right) \approx \int_{\mathscr{I}^+_T} \left((N^a \hat{\nabla}_a \phi)^2 + (v_1^a \hat{\nabla}_a \phi)^2 + (v_2^a \hat{\nabla}_a \phi)^2) + \frac{\phi^2}{2}\right) t_a^1 \wedge t_a^2 \wedge t_a^3$$

We denote by $E(\mathscr{I}^+_T)$ *the right-hand side of this equation. The expression of this energy is not intrisic, since it depends on the one hand, on the parametrization of* $\mathscr{I}^+$ *by the function* φ *and, on the other hand, on the choice of a basis. The energy used by Hörmander suffers from the same property that it depends on the graph and on the chosen coordinate system.*

We assume here that $t > s$ and apply Stokes theorem between the surfaces Σ_t and Σ_s. The part of $\mathscr{I}^+$ between the time t and s is denoted $\mathscr{I}^t_s$ and the part of U between Σ_t and Σ_s, U_s^t. The following equality holds:

$$\begin{aligned}&\int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star \hat{T}^a T_{ab}\right) + \int_{\mathscr{I}^t_s} i^\star_{\Sigma_t}\left(\star \hat{T}^a T_{ab}\right) - \int_{\Sigma_s} i^\star_{\Sigma_s}\left(\star \hat{T}^a T_{ab}\right)\\ &= \int_{U_s^t} \left(\hat{\nabla}^{(a}\hat{T}^{b)} T_{ab} + (1 - \frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi \hat{T}^a \hat{\nabla}_a \phi + \hat{T}^a \hat{\nabla}_a \frac{\phi^4}{4}\right) \mu[\hat{g}]\end{aligned}$$

As it was noticed in lemma 2.14, the integral over $\mathscr{I}^+$ of the energy 3-form restricted to $\mathscr{I}$ is positive. So the following inequalities holds:

$$\begin{aligned}&\left|\int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star \hat{T}^a T_{ab}\right) + \int_{\mathscr{I}^t_s} i^\star_{\Sigma_t}\left(\star \hat{T}^a T_{ab}\right) - \int_{\Sigma_s} i^\star_{\Sigma_s}\left(\star \hat{T}^a T_{ab}\right)\right|\\ &\leq \int_{U_s^t} \left|\left(\hat{\nabla}^{(a}\hat{T}^{b)} T_{ab} + (1 - \frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi \hat{T}^a \hat{\nabla}_a \phi + \hat{T}^a \hat{\nabla}_a b \frac{\phi^4}{4}\right)\right| \mu[\hat{g}]\end{aligned}$$

and, as a consequence,

$$\begin{aligned}&\int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star \hat{T}^a T_{ab}\right) + \int_{\mathscr{I}^t_s} i^\star_{\Sigma_t}\left(\star \hat{T}^a T_{ab}\right)\\ &\leq \int_{U_s^t} \left|\hat{\nabla}^{(a}\hat{T}^{b)} T_{ab}\right| + \left|(1 - \frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi \hat{T}^a \hat{\nabla}_a \phi\right| + \left|\hat{T}^a \hat{\nabla}_a b\right| \frac{\phi^4}{4} \mu[\hat{g}] + \int_{\Sigma_s} i^\star_{\Sigma_s}\left(\star \hat{T}^a T_{ab}\right).\end{aligned}$$

Since $\int_{\mathscr{I}_s^t} i^\star_{\Sigma_t}$ is non-negative (see remark 2.18) and

$$\int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^aT_{ab}\right) \approx E(\Sigma_r),$$

it remains:

$$E(\Sigma_t) \lesssim \int_{U_s^t} \left|\hat{\nabla}^{(a}\hat{T}^{b)}T_{ab}\right| + \left|(1-\frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi\hat{T}^a\hat{\nabla}_a\phi\right| + \left|\hat{T}^a\hat{\nabla}_a b\right|\frac{\phi^4}{4}\mu[\hat{g}] + E(\Sigma_s).$$

Since $\overline{U}$ is compact, there exists a contant c depending on $\hat{\nabla}^a\hat{T}^b$, $\mathrm{Scal}_{\hat{g}}$ and $\hat{T}^a\hat{\nabla}_a b$ which controls each term in the error

$$\int_{U_s^t}\left(\hat{\nabla}^{(a}\hat{T}^{b)}T_{ab} + (1-\frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi\hat{T}^a\hat{\nabla}_a\phi + \hat{T}^a\hat{\nabla}_a b\frac{\phi^4}{4}\right)\mu[\hat{g}]$$

in function of the energy on a slice at time r. To perform such an estimate, the 2-form $\hat{\nabla}^{(a}\hat{T}^{b)}$ is split over the orthonormal basis $(e^a_{\mathbf{i}})_{i=0,\dots,4}$; each of the components is bounded by c. The remaining terms, when contracting with T_{ab}, are of either products of derivatives or functions which can be estimated by ϕ^2 or $b\phi^4$ and their derivatives.

Finally, the volume form $\mu[\hat{g}]$ is decomposed over the basis $(e^a_{\mathbf{i}})_{i=0,\dots,4}$, to obtain:

$$E(\Sigma_t) \lesssim \int_s^t E(\Sigma_r)\mathrm{d}r + E(\Sigma_s)$$

where the form $\mathrm{d}r$ is e^0_a. So, applying Gronwall lemma, we obtain since we are working in finite time:

$$E(\Sigma_t) \lesssim E(\Sigma_s).\circledast$$

A straightforward consequence of this proposition is that all the energies on slices are controlled by the energy on Σ_T. This is a necessary step to establish the following proposition:

Proposition 2.19. *The energies on $\mathscr{I}_T^+$ and on Σ_T are equivalent:*

$$E(\mathscr{I}_T^+) \approx E(\Sigma_T).$$

Proof.The proof is based on the use of Stokes theorem. We denote by, for t between T and T_{max}:

- U_t the part of U for time greater than t;
- $\mathscr{I}_t^+$ the part of $\mathscr{I}^+$ for time greater than t.

Stokes theorem is used between the hypersurfaces $\mathscr{I}_t^+$ and Σ_t:

$$\int_{\mathscr{I}_t^+} i^\star_{\mathscr{I}^+}\left(\star\hat{T}^aT_{ab}\right) - \int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^aT_{ab}\right)$$
$$= \int_{U_t}\left(\hat{\nabla}^{(a}\hat{T}^{b)}T_{ab} + (1-\frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi\hat{T}^a\hat{\nabla}_a\phi + \hat{T}^a\hat{\nabla}_a\frac{\phi^4}{4}\right)\mu[\hat{g}]$$

Using exactly the same estimate as in proposition 2.16 of the error term, we obtain:

$$\left|\int_{\mathscr{I}_t^+} i^\star_{\mathscr{I}^+}\left(\star\hat{T}^aT_{ab}\right) - \int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^aT_{ab}\right)\right| \lesssim \int_t^{T_{max}} E(\Sigma_r)\mathrm{d}r.$$

As a consequence, the two following inequalities hold:

$$\int_{\mathscr{I}_t^+} i^\star_{\mathscr{I}^+}\left(\star\hat{T}^a T_{ab}\right) - \int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) \lesssim \int_t^{T_{max}} E(\Sigma_r)\mathrm{d}r. \tag{2.16}$$

and

$$\int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) - \int_{\mathscr{I}_T^+} i^\star_{\mathscr{I}^+}\left(\star\hat{T}^a T_{ab}\right) \lesssim \int_t^{T_{max}} E(\Sigma_r)\mathrm{d}r, \tag{2.17}$$

since

$$\int_{\mathscr{I}_T^+} i^\star_{\mathscr{I}^+}\left(\star\hat{T}^a T_{ab}\right) \geq \int_{\mathscr{I}_t^+} i^\star_{\mathscr{I}^+}\left(\star\hat{T}^a T_{ab}\right).$$

We first deal with inequality (2.16). Since, according to proposition 2.16, all the energies on a slice $\{t = constant\}$ are controlled by $E(\Sigma_T)$ for $t \geq T$, the integral $\int_t^{T_{max}} E(\Sigma_s)\mathrm{d}r$ satisfies:

$$\int_t^{T_{max}} E(\Sigma_r)\mathrm{d}r \lesssim E(\Sigma_t).$$

Using inequality (2.16), a straightforward consequence is:

$$\int_{\mathscr{I}_T^+} i^\star_{\mathscr{I}^+}\left(\star\hat{T}^a T_{ab}\right) \lesssim E(\Sigma_T)$$

and with remark 2.18:

$$E(\mathscr{I}_T^+) \lesssim E(\Sigma_T).$$

On the other hand, to obtain the second inequality, we use Gronwall lemma in inequality (2.17); this gives:

$$E(\Sigma_t) \lesssim \int_{\mathscr{I}^+} i^\star_{\mathscr{I}^+}\left(\star\hat{T}^a T_{ab}\right)$$

and, consequently, for $t = T$:

$$E(\Sigma_T) \lesssim E(\mathscr{I}_T^+).\circledast$$

2.3 Estimates on V

The geometric situation in this section is almost the same as in the previous one since the hypersurface S_{u_0} is known to be null, the only difference being that an additional term comes from the boundary of the future of $\Sigma_0^{u_0<}$. The energy estimates will then be obtained in exactly the same way. Nonetheless, we wish to keep the term with an energy on the hypersurface S_{u_0}, in order to compare these terms with the inequalities obtained in section 2.1.

It is clear that the time function which was defined in the previous section cannot be used here since $\hat{T}^a$ is not necessarily orthogonal to Σ_0. We now consider another time function τ, defined in $\overline{V}$ (or in a neighborhood of V, as done in remark 2.20 below) such that the hypersurface $\{\tau = 0\}$ corresponds to Σ_0 and the hypersurface $\{\tau = 1\}$ to $\Sigma = \Sigma_T$. We consider on V the orthonormal basis $(e_0^a, e_{\mathbf{i}}^a)_{i=1,2,3}$ such that:

$$e_0^a = \frac{\nabla^a\tau}{\hat{g}_{cd}\nabla^c\tau\nabla^d\tau}.$$

By construction, the vector fields $(e_{\mathbf{i}}^a)_{i=1,2,3}$ are tangent to the time slices.

We introduce the following function:

$$\alpha = 1 + \sum_{i=1,2,3} (\hat{g}_{cd} f^c e_{\mathbf{i}}^d)^2 \geq 1$$

where f^c is the normalization with respect to the metric $\hat{g}$ of the vector field $\hat{T}^a$.

Remark 2.20. • *The constant α is used to construct the so-called Lipschitz norm of the foliation Σ_t.*

- *Such a time function τ can be constructed as follows: since $\hat{M}$ is globally hyperbolic, there exists a time function on $\hat{M}$. Let $\tilde{\tau}$ be a time function. Let $\Psi_{\tilde{\tau}}$ be the flow associated with τ and let V_0 be the preimage of Σ on Σ_0 by the flow. We then obtain a diffeomorphism defined by:*

$$\begin{array}{ccc} V_0 & \longrightarrow & \Sigma \\ x & \longmapsto & \Psi_{\phi(x)}(x) \end{array}$$

where $\phi(x)$ is the time at which the curve $\tilde{\tau} \mapsto \Psi_{\tilde{\tau}}(x)$ hits Σ. The new time function τ is then defined as: let p be a point lying between V_0 and Σ, p is written $\Psi_{\tilde{\tau}}(x)$; then

$$\tau(p) = \frac{\tilde{\tau}(p)}{\phi(x)}$$

satisfies the required assumption.

- *$\hat{T}^a$ and e_0^a are both uniformly timelike. Therefore, the scalar product*

$$\beta = \hat{g}_{cd}\hat{f}^c e_0^a$$

defines a positive function over $\overline{V}$ since T^a and e_0^a are both future directed and timelike over a compact set.

The following notations will be used, in order to be coherent with section 2.1:

- the section of the initial data surface below u_0 is denoted $\Sigma_0^{u_0<}$;
- as previously introduced, the slices $\{t = contant\}$ in V are denoted Σ_t;
- the part V between time t and s (with $t < s$) is denoted by V_t^s;
- the part of S_{u_0} between time t and s (with $t < s$) is denoted by S_t^s;

The expression of the energy on S_{u_0} are the same as the one define in 2.1 for (see proposition 2.3). Since we are not working with an orthonormal basis, the expression of the energy is adapted to this surface.

Following the method already used in this paper, a geometric description of of the energy 3-form is given and an equivalence result of the integral of the 3-form with an well chosen energy is established:

Proposition 2.21. *The restriction of the energy 3-form to Σ_t is given by:*

$$i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) = \left\{\frac{(f^c\hat{\nabla}_c\phi)^2}{2(1+\sum_{i=1,2,3}(\hat{g}_{cd}f^c e^d c_i)^2)}\right.$$

$$\left. + \frac{1}{2}\left(\sum_{i=1,2,3}\left(1 - \frac{(\hat{g}_{cd}f^c e^d c_i)^2}{1+\sum_{i=1,2,3}(\hat{g}_{cd}f^c e^d c_i)^2}\right)(e_{\boldsymbol{i}}^a\hat{\nabla}_a\phi)^2\right) + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right\}\frac{\hat{g}_{cd}\hat{T}^c\hat{T}^d}{\hat{g}_{cd}\hat{T}^c e_0^d} e_1^a \wedge e_2^a \wedge e_3^a$$

and, as a consequence, the following equivalence holds, for all t in $[0,1]$:

$$\int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) \approx \int_{\Sigma_t}\left((f^c\hat{\nabla}_c\phi)^2 + \sum_{i=1,2,3}(e_{\boldsymbol{i}}^a\hat{\nabla}_a\phi)^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right) e_1^a \wedge e_2^a \wedge e_3^a.$$

We denote by $E(\Sigma_t)$ this energy:

$$E(\Sigma_t) = \int_{\Sigma_t}\left((f^c\hat{\nabla}_c\phi)^2 + \sum_{i=1,2,3}(e_{\boldsymbol{i}}^a\hat{\nabla}_a\phi)^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right) e_1^a \wedge e_2^a \wedge e_3^a.$$

Remark 2.22. *This proposition together with proposition 2.16 states that the energy on a space-like slice for two uniformly timelike (for the metric $\hat{g}$) vector fields are equivalent. This justifies that we write in the same way the energy in proposition 2.16 and in this proposition.*

Proof.the strategy of the proof is the same as usual: the geometric objects are split over the basis $(f^a, e^a_{\mathbf{i}})_{i=1,2,3}$ where f^a is transverse to Σ_t and $(e^a_i)_{i=1,2,3}$ tangent to Σ_t.

Considering the 4-form

$$f_a \wedge e^a_1 \wedge e^a_2 \wedge e^a_3$$

f^a is decomposed as follows:

$$f^a = \beta e^a_0 - \sum_{i=1,2,3} \delta^{\mathbf{i}} e^a_{\mathbf{i}}$$

with

$$\delta^{\mathbf{i}} = \hat{g}_{cd} f^c e^d_{\mathbf{i}} \text{ and } \beta^2 = 1 + \sum_{i=1,2,3} (\delta^{\mathbf{i}})^2.$$

The volume form $\mu[\hat{g}]$ satisfies:

$$\mu[\hat{g}] = \frac{1}{\beta} f_a \wedge e^a_1 \wedge e^a_2 \wedge e^a_3$$

and its contraction with the vector f^a is:

$$f^a \lrcorner \mu[\hat{g}] = \left(\beta e^a_0 - \sum_{i=1,2,3} \delta^{\mathbf{i}} e^a_{\mathbf{i}} \right) \lrcorner \mu[\hat{g}] = \beta e^a_1 \wedge e^a_2 \wedge e^a_3$$

as a consequence, we have:

$$i^\star_{\Sigma_t}(\star \hat{T}_a) = ||\hat{T}^a|| i^\star_{\Sigma_t}(\star f_a) = ||\hat{T}^a|| f^a \lrcorner \mu[\hat{g}] = ||\hat{T}^a|| \beta e^a_1 \wedge e^a_2 \wedge e^a_3.$$

We then deal with $\hat{\nabla}^c \phi$ which can be written:

$$\hat{\nabla}^c \phi = b f^c - \sum_{i=1,2,3} a^{\mathbf{i}} e^c_{\mathbf{i}}$$

where

$$b = \frac{\hat{g}_{ab} \hat{\nabla}^a \phi e^b_0}{\beta} = \frac{f^a \hat{\nabla}_a \phi + \sum_{i=1,2,3} \delta^{\mathbf{i}} e^a_{\mathbf{i}} \hat{\nabla}_a \phi}{\beta^2} \text{ and } a_{\mathbf{i}} = e^a_{\mathbf{i}} \hat{\nabla}_a \phi - b \delta^{\mathbf{i}} \text{ for } i \in \{1,2,3\}.$$

Consequently, its norm is:

$$\hat{\nabla}^c \phi \hat{\nabla}_c \phi = \frac{(f^a \hat{\nabla}_a \phi + \sum_{i=1,2,3} \delta^{\mathbf{i}} e^a_{\mathbf{i}} \hat{\nabla}_a \phi)^2}{\beta^2} - \sum_{i=1,2,3} (e^a_{\mathbf{i}} \hat{\nabla}_a \phi)^2$$

and the restriction to Σ_t of $\star \nabla_b \phi$ is:

$$\begin{aligned} i^\star_{\Sigma_t}(\star \nabla_b \phi) &= \frac{(f^a \hat{\nabla}_a \phi + \sum_{i=1,2,3} \delta^{\mathbf{i}} e^a_{\mathbf{i}} \hat{\nabla}_a \phi)}{\beta^2} i^\star_{\Sigma_t}(\star f_a) \\ &= \left(f^a \hat{\nabla}_a \phi + \sum_{i=1,2,3} \delta^{\mathbf{i}} e^a_{\mathbf{i}} \hat{\nabla}_a \phi \right) \frac{e^a_1 \wedge e^a_2 \wedge e^a_3}{\beta}. \end{aligned} \tag{2.18}$$

Using these results, the energy 3-form is given by:

$$i^\star_{\Sigma_t}(\star\hat{T}^a T_{ab}) = \hat{T}^a\hat{\nabla}_a\phi\left(\left(f^a\hat{\nabla}_a\phi + \sum_{i=1,2,3}\delta^{\mathbf{i}} e^a_{\mathbf{i}}\hat{\nabla}_a\phi\right)\frac{e^a_1\wedge e^a_2\wedge e^a_3}{\beta}\right)$$

$$+\left(-\frac{1}{2}\left(\frac{(f^a\hat{\nabla}_a\phi + \sum_{i=1,2,3}\delta^{\mathbf{i}} e^a_{\mathbf{i}}\hat{\nabla}_a\phi)^2}{\beta^2} - \sum_{i=1,2,3}(e^a_{\mathbf{i}}\hat{\nabla}_a\phi)^2\right) + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)||\hat{T}^a||\beta e^a_1\wedge e^a_2\wedge e^a_3$$

$$= \left\{\frac{1}{2}\left((f^c\hat{\nabla}_c\phi)^2 + \sum_{i=1,2,3}(e^a_i\hat{\nabla}_a\phi)^2(\beta^2 - (\delta^{\mathbf{i}})^2)\right) + \beta^2\left(\frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\right\}\frac{||\hat{T}^a||^2}{\beta} e^a_1\wedge e^a_2\wedge e^a_3$$

To get the equivalence, it is sufficient to notice that:

- $\hat{g}_{cd}\hat{T}^c\hat{T}^d$ is a positive function over the compact $\overline{V}$ and, as such, is bounded below and behind by two positive constants;
- as already noticed in remark 2.20, β is a positive function over $\overline{V}$;
- and, finally, the scalar products β, $\delta_{\mathbf{i}}$ and the difference $\beta^2 - \delta_{\mathbf{i}}$ are clearly bounded below by 1 and above by a certain constant since we are working on a compact.

The energy equivalence then holds:

$$\int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) \approx \int_{\Sigma_t}\left(\frac{(f^c\hat{\nabla}_c\phi)}{2}^2 + \frac{1}{2}\sum_{i=1,2,3}(e^a_{\mathbf{i}}\hat{\nabla}_a\phi)^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right) e^a_1\wedge e^a_2\wedge e^a_3\circledast$$

From now on, the strategy is exactly the same as in subsection 2.2.2. The fact that the energy over Σ dominates the energy on all slices Σ_t is established:

Proposition 2.23. *The following estimate holds, for all t in $[0,1]$:*

$$E(\Sigma_t) \lesssim E(\Sigma)$$

Proof. Let t be in $[0,1]$. The energy 3-form is integrated over the surfaces Σ, Σ_t and S^1_t; Stokes theorem then gives:

$$\int_\Sigma i^\star_\Sigma\left(\star\hat{T}^a T_{ab}\right) - \int_{S^1_t} i^\star_{S_{u_0}}\left(\star\hat{T}^a T_{ab}\right) - \int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right)$$
$$= \int_{V^1_t}\left(\hat{\nabla}^{(a}\hat{T}^{b)}T_{ab} + (1 - \frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi\hat{T}^a\hat{\nabla}_a\phi + \hat{T}^a\hat{\nabla}_a\frac{\phi^4}{4}\right)\mu[\hat{g}].$$

As in the proof of proposition 2.19, the error term is estimated considering that $\nabla^{(a}\hat{T}^{b)}$ has bounded coefficients in the given basis, that $\mathrm{Scal}_{\hat{g}}$ is bounded and using the behavior of b. The volume form is decomposed on the basis $(\mathrm{d}t, e^{\mathbf{i}}_a)_{1,2,3}$ as:

$$\mu[\hat{g}] = \frac{\mathrm{d}t\wedge e^1_a\wedge e^2_a\wedge e^3_a}{e^a_0\hat{\nabla}_a t},$$

where $e^a_0\hat{\nabla}_a t$ is a positive function and, as a consequence, is bounded above and below by two positives constants.

We obtain, when contracting the volume form through e^a_0 in order to integrate in time:

$$\left|\int_{V^1_t}\left(\hat{\nabla}^{(a}\hat{T}^{b)}T_{ab} + (1 - \frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi\hat{T}^a\hat{\nabla}_a\phi + \hat{T}^a\hat{\nabla}_a\frac{\phi^4}{4}\right)\mu[\hat{g}]\right| \lesssim \int_t^1 E(\Sigma_t)\mathrm{d}t.$$

Using the energy equivalence proved in proposition 2.21, the following inequality then holds:

$$E(S_t^1) + E(\Sigma_t) \quad \lesssim \quad \int_t^1 E(\Sigma_t)\mathrm{d}t + E(\Sigma). \tag{2.19}$$

Since $E(S_t^1)$ is non negative (see lemma 1.5 and proposition 2.3), (2.19) turns into the integral inequality:

$$E(\Sigma_t) \lesssim \int_t^1 E(\Sigma_t)\mathrm{d}t + E(\Sigma).$$

Using Growall's lemma, we get:

$$E(\Sigma_t) \lesssim E(\Sigma)\circledast.$$

Finally, the following proposition holds:

Proposition 2.24. *The following estimates are satisfied on V:*

$$E(\Sigma) \approx E(S_{u_0}) + E(\Sigma_0^{u_0<})$$

Proof.The energy 3-form is integrated over the surfaces Σ_0, Σ_t and S_0^t; the application of Stokes theorem gives:

$$\begin{aligned}&\int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) - \int_{S_0^t} i^\star_{S_{u_0}}\left(\star\hat{T}^a T_{ab}\right) - \int_{\Sigma_0} i^\star_{\Sigma_0}\left(\star\hat{T}^a T_{ab}\right)\\ &= \int_{V_0^t}\left(\hat{\nabla}^{(a}\hat{T}^{b)}T_{ab} + (1 - \frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi\hat{T}^a\hat{\nabla}_a\phi + \hat{T}^a\hat{\nabla}_a\frac{\phi^4}{4}\right)\mu[\hat{g}].\end{aligned}$$

The error term satisfy the same inequality as in proposition 2.23:

$$\left|\int_{V_0^t}\left(\hat{\nabla}^{(a}\hat{T}^{b)}T_{ab} + (1 - \frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi\hat{T}^a\hat{\nabla}_a\phi + \hat{T}^a\hat{\nabla}_a\frac{\phi^4}{4}\right)\mu[\hat{g}]\right| \lesssim \int_0^t E(\Sigma_s)\mathrm{d}s.$$

As a consequence, the two following inequalities hold:

$$E(S_0^t) + E(\Sigma_0) \quad \lesssim \quad \int_0^t E(\Sigma_t)\mathrm{d}t + E(\Sigma_t) \tag{2.20}$$

$$E(\Sigma_t) \quad \lesssim \quad \int_0^t E(\Sigma_t)\mathrm{d}t + E(S_0^t) + E(\Sigma_0). \tag{2.21}$$

The right-hand side of inequality (2.20) is estimated via proposition 2.23 as:

$$\int_0^t E(\Sigma_t)\mathrm{d}t \lesssim E(\Sigma)$$

and, as a consequence, for $t = 1$, the first part of the equivalence can be stated:

$$E(S_{u_0}) + E(\Sigma_0) \lesssim E(\Sigma_1).$$

Using the positivity of $E(S_0^t)$, inequality (2.21) becomes:

$$E(\Sigma_t) \lesssim \int_0^t E(\Sigma_t)\mathrm{d}t + E(S_{u_0}) + E(\Sigma_0).$$

Using Gronwall's lemma and setting $t = 1$, the second part of the equivalence is obtained:

$$E(\Sigma) \lesssim E(S_{u_0}) + E(\Sigma_0).\circledast$$

2.4 Final estimates

Finally, using the three propositions 2.8, 2.19 and 2.24, the following a priori global estimates hold:

Theorem 2.25. *Let u be a smooth solution with compactly supported data of the nonlinear wave equation:*

$$\Box u + \frac{1}{6} Scal_{\hat{g}} u + bu^3 = 0.$$

Then, the a priori estimates hold:

$$E(\Sigma_0) \approx E(\mathscr{I}^+).$$

Remark 2.26.

a. *The solution of the wave equation is assumed to be smooth in order to avoid the problem of defining trace operators for weak solutions of the equation. Nonetheless, using a usual trace theorem, as soon as u is in $H^{\frac{3}{2}}(\hat{M})$, its trace over $\mathscr{I}^+$ and Σ_0 is well defined.*

b. *Furthermore, in the framework of a (characteristic) Cauchy problem, it is know that the solution is in $H^1(\hat{M})$. Using the same theorem of existence of trace operators, u is only in $H^{\frac{1}{2}}(\Sigma_0)$ or $H^{\frac{1}{2}}(\mathscr{I}^+)$, which is clearly not sufficient to write such estimates. It will be shown in section 4.1 that these operators are well defined and with values in $H^1(\Sigma_0)$ or $H^1(\mathscr{I}^+)$.*

Proof. Let us consider the hypersurface Σ_0. This hypersurface is split in $\Sigma_0^{u_0<}$ and $\Sigma_0^{u_0>}$:

$$E(\Sigma_0) = E(\Sigma_0^{u_0<}) + E(\Sigma_0^{u_0>}).$$

Using proposition 2.8:

$$E(\Sigma_0^{u_0>}) \lesssim E(\mathscr{I}^+_{u_0}) + E(S_{u_0}),$$

proposition 2.24:

$$E(S_{u_0}) + E(\Sigma_0^{u_0<}) \lesssim E(\Sigma_T),$$

and proposition 2.19:

$$E(\Sigma_T) \lesssim E(\mathscr{I}^+_T),$$

we obtain the first part of the apriori estimate:

$$E(\Sigma_0) \lesssim E(\mathscr{I}^+_T) + E(\mathscr{I}^+_{u_0}) = E(\mathscr{I}^+).$$

Conversely, let us consider

$$E(\mathscr{I}^+) = E(\mathscr{I}^+_{u_0}) + E(\mathscr{I}^+_T).$$

Using proposition 2.19:

$$E(\mathscr{I}^+_T) \lesssim E(\Sigma_T),$$

proposition 2.24:

$$E(\Sigma_T) \lesssim E(S_{u_0}) + E(\Sigma_0^{u_0<}),$$

and proposition 2.8:

$$E(S_{u_0}) + E(\mathscr{I}^+_{u_0}) \lesssim E(\Sigma_0^{u_0>}),$$

we get the other side of the inequality:

$$E(\mathscr{I}^+) \lesssim E(\Sigma_0^{u_0>}) + E(\Sigma_0^{u_0<}) = E(\Sigma_0).\circledast$$

3 Goursat problem on $\mathscr{I}^+$

We show in this section that there exists a unique solution to the Goursat problem on $\mathscr{I}^+$ for characteristic data in $H^1(\mathscr{I}^+)$:

$$\begin{cases} \hat{\Box}\phi + \frac{1}{6}\mathrm{Scal}_{\hat{g}}\phi + b\phi^3 = 0 \\ \phi\big|_{\mathscr{I}^+} = \theta \in H^1(\mathscr{I}^+). \end{cases} \tag{3.1}$$

It is known (see [30]) that the linear Goursat problem on a smooth weakly hypersurface admits a global solution; nonetheless, due to technical problems coming from the singularity at i^0 (essentially the fact that some Sobolev embeddings are not valid), the existence of a global solution must be justified carefully.

The proof of the existence for problem (3.1) is made in three steps:

a. considering two solutions of the wave equation, estimates on the difference of these solutions are established; the main technical problem, which consists in obtaining uniform Sobolev estimates, is encountered and solved in section 3.1.

b. Let $\mathcal{S}$ be a uniformly spacelike hypersurface for the metric $\hat{g}$ in the future of Σ_0 and close enough to $\mathscr{I}^+$. The existence of solutions for problem (3.1) with characteristic data whose compact support contains neither i^0 nor i^+ is obtain through a Picard iteration in the future of $\mathcal{S}$ for small data in section 3.2.

c. Then this solution is extended down to Σ_0, by means of a Cauchy problem on $\mathcal{S}$ and density results in section 3.3.1.

d. Finally, using estimates for the propagator between $\mathscr{I}^+$ and Σ_0 obtained in section 3.1, the result is extended to $H^1(\mathscr{I}^+)$ for small data in section 3.3.2.

3.1 Continuity result

This section is devoted to the proof of a continuity result in function of the characteristic data, although an existence theorem has not yet been stated. Consequences of these estimates wil be required to obtain well-posedness of problem (3.1).

3.1.1 Technical tools

The greatest problem when dealing with foliations is the difficulty to obtain uniform estimates of the non-linearity. This requires that the constants associated with the Sobolev embeddings are controlled uniformly over the foliation. Two theorems are given to control these constants.

The first one is a result by Hébey-Vaugon which is used to obtain a uniform control over the Sobolev constant for the embedding from H^1 into L^6 on the leaves of a foliation. The result is the following (theorem 7.2 in [29], adapted to dimension 3) :

Theorem 3.1 (Hébey–Vaugon, 1995)**.** *Let (Σ, g) be a smooth complete Riemannian manifold of dimension 3. Suppose that its Riemann curvature R_{ijkl} and its injectivity radius inj_g satisfy:*

$$\exists(\lambda_1, \lambda_1, i), ||R_{ijkl}||_g \leq \lambda_1,\ ||\nabla^a R_{ijkl}||_g \leq \lambda_2 \text{ and } inj_g \geq i.$$

Then there exists a constant B depending only on λ_1, λ_2 and i such that, for all functions u in $H^1(M)$:

$$\left(\int_M |u|^6 d\mu[g]\right)^3 \leq \sqrt{\frac{4}{3\omega_3^{\frac{2}{3}}}}\int_M |\nabla u|^2 d\mu[g] + B\int_M u^2 d\mu[g],$$

where ω_3 is the volume of the unit sphere in $\mathbb{R}^4$.

The second useful result is an extension theorem. It is used in the following as an intermediary result to apply theorem 3.1. As previously, it is necessary to control the norm of the map (see chapter VI, theorems 5 and 5' in [39]):

Theorem 3.2 (Extension theorem). *Let U be a bounded open set in $\mathbb{R}^n$; we assume that its boundary ∂U is C^1 differentiable.*
Then there exists an operator $\mathscr{E}$ from $H^1(U)$ into $H^1(\mathbb{R}^n)$ such that:

a. *for all f in $H^1(U)$, $\mathscr{E}(f)\Big|_U = f$; $\mathscr{E}$ is an extension operator;*

b. *there exists a constant C depending only on the Lipschitz constant of the boundary such that, for all f in $H^1(U)$:*
$$||\mathscr{E}(f)||_{H^1(\mathbb{R}^n)} \leq C||f||_{H^1(U)}.$$

Remark 3.3. *There exist variations of this result: the constant C can de replaced by the L^∞-norm of the maps on the boundary or the curvatures and its derivatives when ∂U is more regular. More precise estimates can be found in [8].*

3.1.2 Uniform Sobolev estimates on uniformly spacelike foliations

When dealing with apriori estimates in section 2, the problem of controlling the nonlinearity was avoided by assumptions on the asymptotic behavior of the function b. Controlling the non-linearity in function of the H^1-norm is a way to remove these assumptions. This is done by obtaining uniform Sobolev embeddings over a foliation.

Proposition 3.4. *Let (M, g) be a four dimensional smooth Lorentzian manifold; let $(\Sigma_t)_{t\in I}$ be a foliation of M by uniformly spacelike hypersurfaces with smooth boundary; we assume that I is a compact interval in $\mathbb{R}$ and that the hypersurfaces Σ_t can be embedded in simply connected open sets of $\mathbb{R}^3$.*

Then there exists a constant K_{sob}, depending only on the geometry of the foliation such that:
$$\forall t \in I, \forall f \in H^1(\Sigma_t), ||f||_{L^6(\Sigma_t)} \leq K_{Sob}||f||_{H^1(\Sigma_t)}.$$

Remark 3.5. *This constant depends on:*

- *the supremum of the Lipshitz constant of the boundaries of the hypersurfaces Σ_t;*
- *the supremum of the curvatures and its derivatives of the hypersurfaces of a given extension of the Σ_t;*
- *the infimum of the injectivity radii of a given extension of the Σ_t.*

Proof. To obtain such estimates, the method is the following:

a. we notice that the boundaries of the hypersurfaces Σ_t have the same Lipschitz norm L;

b. the slices Σ_t can be considered as a family of compact hypersurfaces with trivial topology which can be extended in $\mathbb{R}^3$; the metrics $g\Big|_{\Sigma_t}$ are extended smoothly to $\mathbb{R}^3$ so that they are equal to the euclidean metric of $\mathbb{R}^3$ outside a compact set; these extensions are denoted by $\tilde{\Sigma}_t$ and their metrics $\tilde{g}_t$;

c. we then obtain a family of unbounded 3-dimensional manifolds $(\tilde{\Sigma}_t, \tilde{g}_t)$; since these manifolds are Euclidean outside a compact set, there exist three constants λ_1, λ_2, i such that, uniformly in t:
$$\exists(\lambda_1, \lambda_1, i), ||R_{ijkl}|| \leq \lambda_1, \; ||\nabla^a R_{ijkl}|| \leq \lambda_2 \text{ and } inj_{\tilde{g}_t} \geq i.$$

d. using theorem 3.1, there exists a constant such that uniformly in t the following Sobolev embeddings hold:

$$\forall t \in I, \forall f \in H^1(\tilde{\Sigma}_t), ||f||_{L^6\tilde{\Sigma}_t} \leq K_1||f||_{H^1\tilde{\Sigma}_t}$$

e. using theorem 3.2, there exists a family of extension operators $\mathscr{E}_t$ from $H^1(\Sigma_t)$ into $H^1(\tilde{\Sigma}_t)$. Since the boundaries of Σ_t have the same Lipschitz constant L, there exists a constant K_2, which depends only on L, such that

$$\forall t \in I, \forall f \in H^1(\Sigma_t),\ ||\mathscr{E}_t(f)||_{H^1(\tilde{\Sigma}_t)} \leq K_2||f||_{H^1(\tilde{\Sigma}_t)}$$

Finally using these extensions and the Sobolev imbeddings from $H^1(\tilde{\Sigma}_t)$ into $L^6(\tilde{\Sigma}_t)$, we obtain, for all n and t:

$$\begin{aligned} ||u_n||_{L^6(\Sigma_t)} &\leq ||\mathscr{E}_t(u_n)||_{L^6(\tilde{\Sigma}_t)} \\ &\leq K_1||\mathscr{E}_t(u_n)||_{H^1(\tilde{\Sigma}_t)} \\ &\leq K_2K_1||u_n||_{H^1(\Sigma_t)}. \end{aligned}$$

We denote by K_{Sob} the constant K_1K_2.⊛

3.1.3 Continuity in terms of the initial data for the Cauchy problem

The purpose of this section is to establish estimates on the difference of two solutions of the wave equation. The purpose of these estimates is to obtain continuity in terms of initial data, characteristic or not. This step is an important one to obtain the continuity of the scattering operator.

Finally, it is important to notice that the proof which is made here does not require that the function b vanishes on $\mathscr{I}^+$ or that it satisfies the decay condition:

$$|\hat{T}^a\hat{\nabla}_a b| \leq cb.$$

Using the method developed below, it is then possible to obtain the apriori estimates of section 2.

Let ϕ and ψ be two smooth functions:

$$\hat{\Box}u + \frac{1}{6}\mathrm{Scal}_{\hat{g}}u + bu^3 = 0.$$

We assume that they satisfy one of the problems:

- an initial value problem on Σ_0 with data in $H_0^1(\Sigma_0) \times L^2(\Sigma_0)$ with compact support in Σ_0;
- an characteristic initial value problem with data in $H^1(\mathscr{I}^+)$ with compact support which contains neither i^0 nor i^+.

This ensures that the support of u and v does not contain the singularity i^0.

Theorem 3.6. *Let ϕ and ψ two smooth solutions of the nonlinear problem:*

$$\Box u + \frac{1}{6}Scal_{\hat{g}}u + bu^3 = 0.$$

Then there exist two constants, depending on $Scal_{\hat{g}}$, $\hat{\nabla}^{(a}\hat{T}^{b)}$, b and the energies of ϕ and ψ on Σ_0 and $\mathscr{I}^+$ such that the following estimates hold:

$$||\phi - \psi||^2_{H^1(\mathscr{I}^+)} \leq C(E_\phi(\Sigma_0), E_\psi(\Sigma_0))E_{\phi-\psi}(\Sigma_0)$$

and

$$E_{\phi-\psi}(\Sigma_0) \leq \tilde{C}(||\phi||_{H^1(\mathscr{I}^+)}, ||\psi||_{H^1(\mathscr{I}^+)})||\phi-\psi||^2_{H^1(\mathscr{I}^+)},$$

where the energy of a function on Σ_0 is chosen to be:

$$E_\phi(\Sigma_0) \approx \int_{\Sigma_0} i^\star_{\Sigma_0}\left(\hat{T}^a T_{ab}\right) \approx \int_{\Sigma_0} (\hat{T}^a\hat{\nabla}\phi)^2 + \sum_{i=1,2,3} (e^a_{\boldsymbol{i}}\hat{\nabla}_a\phi)^2 + \phi^2 \, d\mu_{\Sigma_0},$$

where T_{ab} is the energy tensor associated with the linear wave equation and $(e^a_{\boldsymbol{i}})_{i=1,2,3}$ is an orthonormal basis of $T\Sigma_0$.

Remark 3.7. *Because of the a priori estimates, the constants can be chosen indifferently to depend on the energy on $\mathscr{I}^+$ or Σ_0.*

Proof.The proof relies on exactly the same strategy as in the first section when establishing the a priori estimates. Let δ be the difference between ϕ and ψ:

$$\delta = \phi - \psi.$$

δ satisfies the partial differential equation:

$$\Box\delta + \frac{1}{6}\mathrm{Scal}_{\hat{g}}\delta + b(\psi^2 + \psi\phi + \phi^2)\delta = 0.$$

To establish the inequality, let us consider the energy tensor associated with the linear equation:

$$T_{ab} = \hat{\nabla}_a\delta\hat{\nabla}_b\delta + \hat{g}_{ab}\left(-\frac{1}{2}\hat{\nabla}_c\delta\hat{\nabla}^c\delta + \frac{\delta^2}{2}\right).$$

The error term associated with this tensor is:

$$\hat{\nabla}^a(\hat{T}^b T_{ab}) = (\hat{\nabla}^{(a}\hat{T}^{b)}T_{ab}) + \underbrace{\delta(\hat{T}^a\hat{\nabla}_a\delta) - (\hat{T}^a\hat{\nabla}_a\delta)\left(\frac{1}{6}\mathrm{Scal}_{\hat{g}}\delta + b(\psi^2+\psi\phi+\phi^2)\delta\right)}_{A}.$$

The term A can be estimated by:

$$\begin{aligned} A &\leq \frac{1}{2}\left(\delta^2 + (\hat{T}^a\hat{\nabla}_a\delta)^2\right) + \frac{1}{2}\sup_{\hat{M}}(|\mathrm{Scal}_{\hat{g}}|)\left(\delta^2 + (\hat{T}^a\hat{\nabla}_a\delta)^2\right) + 2\sup_{\hat{M}}|b|\left((\hat{T}^a\hat{\nabla}_a\delta(\phi^2+\psi^2)\delta)\right) \\ A &\leq 2\max\left(\sup_{\hat{M}}(|\mathrm{Scal}_{\hat{g}}|), \sup_{\hat{M}}|b|, 1\right)\left((\hat{T}^a\hat{\nabla}_a\delta)^2 + \delta^2 + \delta^2\psi^4 + \delta^2\phi^4\right). \end{aligned}$$

The estimates will then be obtained in exactly the same way as in section 1, provided that we are able to use the Sobolev embeddings on the spacelike slices. The main problem then arises when working in the Schwarzschild section because of the choice of the foliation $\mathcal{H}_s$ which contains the singularity i^0.

On U, the analogue of the equation (2.17) is:

$$\begin{aligned} &E_\delta(\Sigma_t) - ||\delta||^2_{H^1(\mathscr{I}^+_T)} \\ &\leq \max\left\{1, \sup(|\mathrm{Scal}_{\hat{g}}|), \sup(|b|), \sup\left(||\nabla^{(a}\hat{T}^{b)}||\right)\right\}\left(\int_t^{T_{max}}\left(E_\delta(\Sigma_t) + \int_{\Sigma_t}\left(\delta^2\psi^4 + \delta^2\phi^4\right)\mu_{\Sigma_t}\right)\mathrm{d}t\right), \end{aligned}$$

where:

$$E_\delta(\Sigma_t) = \int_{\Sigma_t}\frac{1}{2}\left(\sum_{i=0}^4(e^a_{\mathbf{i}}\nabla_a\delta)^2 + \frac{\delta^2}{2}\right)\mu_{\Sigma_t} = \frac{1}{2}\left(||u||^2_{H^1(\Sigma_t)} + ||\hat{T}^a\hat{\nabla}_a\delta||^2_{L^2(\Sigma_t)}\right). \tag{3.2}$$

The foliation (Σ_t) satisfies the assumption of proposition 3.4: using this proposition, the integral inequality then becomes:

$$\begin{aligned}&E_\delta(\Sigma_t) - ||\delta||^2_{H^1(\mathscr{I}^+_T)}\\ &\leq \left(C_1 + ||\phi||^4_{H^1(\Sigma)} + ||\psi||^4_{H^1(\Sigma)}\right)\left(\int_t^{T_{max}} E_\delta(\Sigma_t)\mathrm{d}t\right).\end{aligned}$$

This gives, using Gronwall estimates:

$$E_\delta(\Sigma_t) \leq \exp\left(\left(C_1 + ||\phi||^4_{H^1(\Sigma)} + ||\psi||^4_{H^1(\Sigma)}\right)(T_{max} - t)\right)||\delta||^2_{H^1(\mathscr{I}^+_T)}.$$

The other estimate is obtained when noticing that the inequality also holds:

$$\begin{aligned}&||\delta||^2_{H^1(\mathscr{I}^+_T)} - E_\delta(\Sigma_t)\\ &\leq \left(C_1 + ||\phi||^4_{H^1(\Sigma)} + ||\psi||^4_{H^1(\Sigma)}\right)\left(\int_t^{T_{max}} E_\delta(\Sigma_t)\mathrm{d}t\right).\end{aligned}$$

Using proposition 2.19, the H^1-norm of ϕ and ψ on Σ is controlled by the H^1-norm of ψ and ϕ on $\mathscr{I}^+_T$ and, as a consequence, by the H^1-norm of ψ and ϕ on $\mathscr{I}^+$. We then finally obtain on U the following inequality: there exist two increasing functions c_U and C_U such that:

$$\begin{aligned}E_\delta(\Sigma) &\leq c_U\left((||\phi||^4_{H^1(\mathscr{I}^+)} + ||\psi||^4_{H^1(\mathscr{I}^+)}\right)||\delta||^2_{H^1(\mathscr{I}^+)}\\ ||\delta||^2_{H^1(\mathscr{I}^+)} &\leq C_U\left((||\phi||^4_{H^1(\mathscr{I}^+)} + ||\psi||^4_{H^1(\mathscr{I}^+)}\right)E_\delta(\Sigma) \qquad (3.3)\end{aligned}$$

On V, the principle is exactly the same: the only modification comes from the fact that a new boundary term arises, corresponding to the boundary of the Schwarzschild section. We work with the same geometric configuration. The equivalent of equation (2.19) is then:

$$E_\delta(S^1_t) + E_\delta(\Sigma_t) - E_\delta(\Sigma) \leq C_2\int_t^1 E_\delta(\Sigma_t)\mathrm{d}t + \int_{\Sigma_t}\delta^2\psi^4 + \delta^2\phi^4\mathrm{d}t$$

where E_δ has the same expression as in equation (3.2). We finally obtain the energy equivalence:

$$\begin{aligned}E_\delta(\Sigma) &\leq c_V\left((||\phi||^4_{H^1(\mathscr{I}^+)} + ||\psi||^4_{H^1(\mathscr{I}^+)}\right)\left(E_\delta(\Sigma_0^{u_0<}) + E_\delta(S_{u_0})\right)\\ E_\delta(\Sigma_0^{u_0<}) + E_\delta(S_{u_0}) &\leq C_V\left((||\phi||^4_{H^1(\mathscr{I}^+)} + ||\psi||^4_{H^1(\mathscr{I}^+)}\right)E_\delta(\Sigma) \qquad (3.4)\end{aligned}$$

Remark 3.8. *In the subset V of $\hat{M}$, the energy on a slice is controlled by the upper slice, which is denoted by Σ as said in proposition 2.23. As this energy is controlled by proposition 2.19 by the H^1-norm on $\mathscr{I}^+_T$ and, as a consequence, on $\mathscr{I}^+$, this explains why the energy on $\mathscr{I}^+$ appears in the inequality.*

Finally, on $\Omega^+_{u_0}$, it is not possible to use the same method as above to control uniformly the Sobolev constant. The strategy consists in adopting the same foliation by the hypersurfaces $\mathcal{H}_s$. The energy on this foliation is weighted Sobolev norm with a precise decay. The Sobolev embeddings must then be adaptated to that decay. The identifying vector field is used to write the integral (see formulae (2.9) and (2.10)). The error term can then be expressed as:

$$\begin{aligned}\int_0^{\tau(s)}&\left(\int_{\mathcal{H}_\tau}\left\{4mR^2(3+uR)(\partial_R\phi)^2 + (1-12mR)\phi\left(u^2\partial_u\phi - 2(1+uR)\partial_R\phi\right)\right.\right.\\ &\left.\left.-(\hat{T}^a\hat{\nabla}_a\delta)\delta(\phi^2+\phi\psi+\psi^2)\right\}(r^*R)^{\frac{3}{2}}(1-2mR)\sqrt{\frac{R}{|u|}}\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}\right)\mathrm{d}\tau\end{aligned}$$

In this subset of $\hat{M}$, the error is bounded above by, using Hölder estimates:

$$\int_0^{\tau(s)}\left\{E_\delta(\mathcal{H}_\tau)+\left(\int_{\mathcal{H}_\tau}\delta^6\sqrt{\frac{R}{|u|}}\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}\right)^{\frac{1}{3}}\left(\int_{\mathcal{H}_\tau}\left(\phi^6+\psi^6\right)\sqrt{\frac{R}{|u|}}\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}\right)^{\frac{2}{3}}\right\}\mathrm{d}\tau$$

$$\leq\int_0^{\tau(s)}\left\{E_\delta(\mathcal{H}_\tau)+\sqrt{\frac{\epsilon}{2m|u_0|}}\left(\int_{\mathcal{H}_\tau}\delta^6\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}\right)^{\frac{1}{3}}\left(\int_{\mathcal{H}_\tau}\left(\phi^6+\psi^6\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}\right)^{\frac{2}{3}}\right\}\mathrm{d}\tau.$$

The Sobolev embedding from H^1 into L^6 must then be realized uniformly in τ with regard to the volume form $\mathrm{d}u\wedge\omega_{\mathbb{S}^2}$, which is the volume form associated with the cylinder $]u_0,+\infty[\times\mathbb{S}^2$ and the metric $(\mathrm{d}u)^2+\mathrm{d}\omega^2_{\mathbb{S}^2}$. Since the Sobolev embedding from $H^1(]u_0,+\infty[\times\mathbb{S}^2)$ into $L^6(]u_0,+\infty[\times\mathbb{S}^2)$ is valid in this geometry, we obtain, in the coordinate system $(u,\omega_{\mathbb{S}^2})$ the following Sobolev inequality:

$$\int_{\mathcal{H}_s}\phi^6\mathrm{d}u\mathrm{d}\omega_{\mathbb{S}^2}=\int_{]u_0,+\infty[\times\mathbb{S}^2}\phi^6\mathrm{d}u\mathrm{d}\omega_{\mathbb{S}^2}$$

$$\leq K\int_{]u_0,+\infty[\times\mathbb{S}^2}(\partial_u(\phi\Big|_{\mathcal{H}_s=\{u=-sr^*\}}))^2+|\nabla_{\mathbb{S}^2}\phi|^2+\phi^2\mathrm{d}u\mathrm{d}\omega_{\mathbb{S}^2}$$

$$\leq\int_{]u_0,+\infty[\times\mathbb{S}^2}\left(\partial_u\phi+\frac{r^*R^2(1-2mR)}{|u|}\partial_R\phi\right)^2+|\nabla_{\mathbb{S}^2}\phi|^2+\phi^2\mathrm{d}u\mathrm{d}\omega_{\mathbb{S}^2}$$

$$\leq\int_{]u_0,+\infty[\times\mathbb{S}^2}2(\partial_u\phi)^2+2(r^*R)^2(1-2mR)^2\left(\frac{R}{|u|}\right)^2(\partial_R\phi^2)|+\nabla_{\mathbb{S}^2}\phi|^2+\phi^2\mathrm{d}u\mathrm{d}\omega_{\mathbb{S}^2}$$

$$\leq\int_{]u_0,+\infty[\times\mathbb{S}^2}2\frac{u^2}{u_0^2}(\partial_u\phi)^2+(1+\epsilon)^2\frac{\epsilon}{2m|u_0|}\frac{R}{|u|}(\partial_R\phi^2)+|\nabla_{\mathbb{S}^2}\phi|^2+\phi^2\mathrm{d}u\mathrm{d}\omega_{\mathbb{S}^2}.$$

Then there exists a constant K, depending on u_0 and ϵ such that, uniformly in s:

$$\left(\int_{\mathcal{H}_\tau}\phi^6\sqrt{\frac{R}{|u|}}\mathrm{d}u\mathrm{d}\omega_{\mathbb{S}^2}\right)^{\frac{2}{3}}\leq K||\phi||^4_{H^1(\mathcal{H}_\tau)}\text{ and }\left(\int_{\mathcal{H}_\tau}\delta^6\sqrt{\frac{R}{|u|}}\mathrm{d}u\mathrm{d}\omega_{\mathbb{S}^2}\right)^{\frac{1}{3}}\leq KE_\delta(\mathcal{H}_\tau).$$

Using the fact that (see equation (2.12)):

$$||\phi||^4_{H^1(\mathcal{H}_{\tau(s)})}\lesssim\left(E_\phi(\Sigma_0^{u_0>})\right)^2,$$

the following integral inequality holds:

$$\left|E_\delta(\mathcal{H}_{\tau(s)}+E_\delta(S_{u_0})-E_\delta(\Sigma_0^{u_0>})\right|$$

$$\leq(C+K(E_\phi(\Sigma_0^{u_0>})^2+E_\psi(\Sigma_0^{u_0>})^2))\int_0^{\tau(s)}E_\delta(\mathcal{H}_\tau)\mathrm{d}\tau$$

$$\leq(C+K(E_\phi(\Sigma_0)^2+E_\psi(\Sigma_0)^2))\int_0^{\tau(s)}E_\delta(\mathcal{H}_\tau)\mathrm{d}\tau$$

and using the a priori estimates given by theorem 2.25,

$$\left|E_\delta(\mathcal{H}_{\tau(s)}+E_\delta(S_{u_0})-E_\delta(\Sigma_0^{u_0>})\right|\leq\left(C+\tilde{K}\left((||\phi||^4_{H^1(\mathscr{I}^+)}+||\psi||^4_{H^1(\mathscr{I}^+)}\right)\right)\int_0^{\tau(s)}E_\delta(\mathcal{H}_\tau)\mathrm{d}\tau.$$

At the end, using the method as previously, there exist two increasing functions $c_{\Omega^+_{u_0}}$ and $C_{\Omega^+_{u_0}}$ such that:

$$\begin{aligned}\mathbb{E}_\delta(\mathscr{I}^+_{u_0})+E_\delta(S_{u_0})&\leq c_{\Omega^+_{u_0}}\left((||\phi||^4_{H^1(\mathscr{I}^+)}+||\psi||^4_{H^1(\mathscr{I}^+)}\right)E_\delta(\Sigma_0^{u_0>})\\E_\delta(\Sigma_0^{u_0>})&\leq C_{\Omega^+_{u_0}}\left((||\phi||^4_{H^1(\mathscr{I}^+)}+||\psi||^4_{H^1(\mathscr{I}^+)}\right)\left(E_\delta(\mathscr{I}^+_{u_0})+E_\delta(S_{u_0})\right).\quad(3.5)\end{aligned}$$

Eventually, combining the inequalities (3.3), (3.4) and (3.5) as in section 2 for the proof of theorem 2.25, we get the existence of two increasing functions c and C such that:

$$\begin{aligned} E_\delta(\Sigma_0) &\leq c\left((||\phi||^4_{H^1(\mathscr{I}^+)} + ||\psi||^4_{H^1(\mathscr{I}^+)}\right) ||\delta||^2_{H^1(\mathscr{I}^+)} \\ ||\delta||^2_{H^1(\mathscr{I}^+)} &\leq C\left((||\phi||^4_{H^1(\mathscr{I}^+)} + ||\psi||^4_{H^1(\mathscr{I}^+)}\right) E_\delta(\Sigma_0). \end{aligned}$$

Because of the a priori estimates given by theorem 2.25, the H^1-norm of ϕ and ψ on $\mathscr{I}^+$ can be replaced by the energy of ϕ and ψ on Σ_0.⊛

This result is equivalent to the result obtained by Hörmander at the beginning of his paper.

Finally, as already mentioned, a by-product of this result is the continuity result for the Cauchy problem for the nonlinear wave equation on a uniformly spacelike hypersurface $\mathcal{S}$, transverse to $\mathscr{I}^+$. The problems are exactly the same: obtaining uniform Sobolev estimates near i^0 and in the equivalent of the region V. The techniques to solve this problem are then exactly the same. We are working with functions ϕ and ψ which satisfy the same assumptions as for theorem 3.6.

Proposition 3.9. *Let $\mathcal{S}$ be a uniformly spacelike hypersurface transverse to $\mathscr{I}^+$. Let ϕ and ψ be two smooth solutions of the nonlinear problem:*

$$\Box u + \frac{1}{6} Scal_{\hat{g}} u + bu^3 = 0.$$

Then there exists two constants, depending on $Scal_{\hat{g}}$, $\hat{\nabla}^a \hat{T}^b$, b and the energy of ϕ and ψ on Σ_0 and $\mathcal{S}$ such that the following estimates hold:

$$E_{\phi-\psi}(\mathcal{S}) \leq C(E_\phi(\Sigma_0), E_\psi(\Sigma_0)) E_{\phi-\psi}(\Sigma_0)$$

and

$$E_{\phi-\psi}(\Sigma_0) \leq \tilde{C}(||\phi||_{H^1(\mathscr{I}^+)}, ||\psi||_{H^1(\mathscr{I}^+)}) E_{\phi-\psi}(\mathcal{S}),$$

where the energy of a function u on a uniformly spacelike hypersurface Σ is chosen to be:

$$E_u(\Sigma) \approx \int_{\Sigma_0} (\hat{T}^a T_{ab}) \approx \int_{\Sigma_0} (\hat{T}^a \hat{\nabla} u)^2 + \sum_{i=1,2,3} (e^a_{\boldsymbol{i}} \hat{\nabla}_a u)^2 + u^2 d\mu_{\Sigma_0},$$

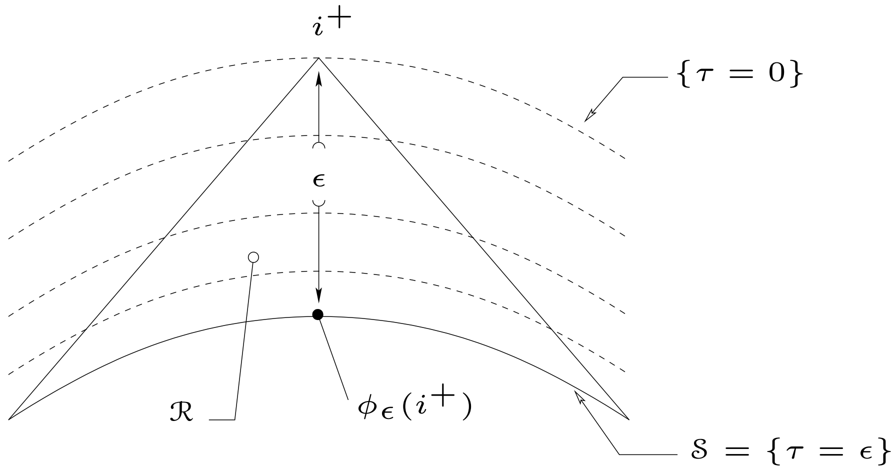
where T_{ab} is the energy tensor associated with the linear wave equation and $(e^a_{\boldsymbol{i}})_{i=1,2,3}$ is an orthonormal basis of $T\Sigma$.

3.2 Solution of the Goursat problem near $\mathscr{I}^+$ with small initial data.

The existence of solutions to the nonlinear problem is established from the linear problem through a Picard iteration in the future of a spacelike hypersurface $\mathcal{S}$, close enough to $\mathscr{I}^+$. This hypersurface is constructed as follows.

We choose to work here with a function in $H^1(\mathscr{I}^+)$ with compact support which contains neither i^0 nor i^+

Let τ be a "reverse" time function on $\hat{M}$, in the sense that its gradient is past directed with respect to $\hat{T}^a$. We assume that $\tau(i^+) = 0$. $\hat{M}$ is endowed with an orthonormal basis $(e^a_{\mathbf{i}})_{i=0,1,2,3}$ such that e^a_0 is colinear to $\hat{\nabla}\tau$ and $(e^a_{\mathbf{i}})_{i=1,2,3}$ is tangent to the time slices $\{\tau = \textit{constant}\}$. The integral flow associated with ∂_τ is denoted by Φ_τ.



Let ϵ be a positive constant smaller than 1 and consider $\Phi_\epsilon(i^+)$. Let $\mathcal{S}$ be a uniformly spacelike hypersurface for the metric $\hat{g}$ between $\mathscr{I}^+$ and $\{\Phi_\epsilon(p)|p \in \mathscr{I}^+\}$. We assume that $\mathcal{S}$ is uniformly spacelike, transverse to $\mathscr{I}^+$ in the past of the support of the characteristic data, θ, and contains $\Phi_\epsilon(i^+)$.

Remark 3.10. *The geometric framework is then exactly the same as in section 2.3: the timelike vector field which will be used for the energy is not colinear to the gradient defining the foliation. Nonetheless, since the hypersurface $\mathcal{S}$ is uniformly spacelike, the same estimates as in section 2.3 hold without the nonlinearity.*

Finally, the future of S in $\hat{M}$ is foliated by the surfaces $\mathcal{S}_\tau = \{\Phi_{\epsilon-\tau}(p)|p \in \mathcal{S}\}$ for τ in $[0, \epsilon]$ so that $\mathcal{S} = \mathcal{S}_\epsilon$ and $\mathcal{S}_0 = \{i^+\}$. The future of $\mathcal{S}$ is denoted by $\mathcal{R}$ and the subset of $\hat{M}$ between $\mathcal{S}_0$ and $\mathcal{S}_\tau$, $\mathcal{R}_\tau$.

The solution of the nonlinear problem is approximated via solutions of the linear problem on $\mathscr{I}^+$. Hormander solved this problem in [30]:

Proposition 3.11 (Hormander)**.** *Let us consider the linear inhomogeneous characteristic Cauchy problem on $\mathscr{I}^+$:*

$$\left\{ \begin{array}{l} \hat{\Box}\phi + \frac{1}{6}Scal_{\hat{g}}\phi = f \\ \phi\big|_{\mathscr{I}^+} = \theta \in H^1(\mathscr{I}^+). \end{array} \right.$$

where θ is a function whose compact support does not contain i^+ or i^0. Then, this problem admits a unique global solution in the future of Σ_0 in $C^0([0, \epsilon], H^1(\mathcal{S}_\tau))$.

Using this proposition and estimates for the linear problem, the following theorem holds:

Theorem 3.12. *Let us consider the nonlinear characteristic Cauchy problem on $\mathscr{I}^+$:*

$$\left\{ \begin{array}{l} \hat{\Box}u + \frac{1}{6}Scal_{\hat{g}}u + bu^3 = 0 \\ u\big|_{\mathscr{I}^+} = \theta \in H^1(\mathscr{I}^+). \end{array} \right.$$

where θ is a function whose compact support does not contain i^+ or i^0. Then, for $||\theta||_{H^1(\mathscr{I}^+)}$ small enough, there exists a uniformly spacelike hypersurface $\mathcal{S}$ close enough to $\mathscr{I}^+$ such that this problem admits a smooth global solution on $\mathcal{R}$ in $C^0([0, \epsilon], H^1(\mathcal{S}_\tau))$.

Remark 3.13. *The proof of the well-posedness in $C^0([0, \epsilon], H^1(\mathcal{S}_\tau))$ is given in section 3.1.3 were the geometric estimates required to obtain it are established (see theorem 3.6 which remains true in that context).*

Proof. Let u_0 be a solution on $\mathcal{R}$ of the problem:

$$\left\{ \begin{array}{l} \hat{\Box}\phi + \frac{1}{6}\mathrm{Scal}_{\hat{g}}\phi = 0 \\ \phi\big|_{\mathscr{I}^+} = \theta \in H^1(\mathscr{I}^+). \end{array} \right.$$

and let $(u_n)_{n\in\mathbb{N}}$ be the sequence of smooth functions on $\mathcal{R}$ defined by the recursion:

$$\left\{ \begin{array}{l} \hat{\Box}u_{n+1} + \frac{1}{6}\mathrm{Scal}_{\hat{g}}u_{n+1} + bu_n^3 = 0 \\ u_{n+1}\big|_{\mathscr{I}^+} = \theta \in H^1(\mathscr{I}^+). \end{array} \right.$$

The sequence defined by the difference of two consecutive terms of this sequence is denoted by $(\delta_n = u_{n+1} - u_n)_{n\in\mathbb{N}}$. For $n \in \mathbb{N}$, the smooth function δ_n satisfies the Cauchy problem:

$$\left\{ \begin{array}{l} \hat{\Box}\delta_n + \frac{1}{6}\mathrm{Scal}_{\hat{g}}\delta_n = -b(u_n^2 + u_n u_{n-1} + u_{n-1}^2)\delta_{n-1} \\ \delta_n = 0 \text{ on } \mathscr{I}^+. \end{array} \right.$$

The proof of the convergence is made in two steps: the first one consists in proving that, for initial data which are small enough, the sequence (u_n) is bounded; the second part proves the convergence of (u_n) by showing that the sequence (δ_n) is summable.

Proposition 3.14. *For $||\theta||_{H^1(\mathscr{I}^+)}$ small enough, the sequence $\sup_{\tau\in[0,\epsilon]}||u_n||_{H^1(\mathcal{S}_\tau)}$ is bounded.*

Proof. Let n be a integer greater than 1. Let us finally consider the energy tensor associated with the linear wave equation:

$$T_{ab} = \hat{\nabla}_a u_{n+1}\hat{\nabla}_b u_{n+1} + \hat{g}_{ab}\left(-\frac{1}{2}\nabla_c u_{n+1}\nabla^c u_{n+1} + \frac{u_{n+1}^2}{2}\right).$$

The energy associated with a time slice $\mathcal{S}_\tau$ is written as:

$$E_{u_n}(\mathcal{S}_\tau) = \int_{\mathcal{S}_\tau}\left(\sum_{i=0,1,2,3}(e_{\mathbf{i}}^a\hat{\nabla}u_n)^2 + \frac{u_n^2}{2}\right)\mathrm{d}\mu_{\mathcal{S}_\tau}$$

and it is equivalent to $\int_{\mathcal{S}_\tau} i^\star(\star e_0^a T_{ab})$ (with constants which only depend on the geometric data, the L^∞-norm of b and the Killing form of e_0^a).

The error term associated to this energy tensor is:

$$\hat{\nabla}^a\left(e_0^b T_{ab}\right) = \hat{\nabla}^{(a}e_0^{b)}T_{ab} + u_{n+1}e_0^a\nabla_a u_{n+1} - \frac{1}{6}\mathrm{Scal}_{\hat{g}}u_{n+1}e_0^a\nabla_a u_{n+1} - (e_0^a\nabla_a u_{n+1})bu_n^3,$$

which is smaller than, in absolute value:

$$|\hat{\nabla}^a e_0^b T_{ab}| \leq C\left(\sum_{i=0}^{3}(e_{\mathbf{i}}^a\hat{\nabla}_a u_{n+1})^2 + u_{n+1}^2\right) + Cu_n^6,$$

where C is a positive constant depending on $\sup(|\mathrm{Scal}_{\hat{g}}|)$, $\sup\left(||\hat{\nabla}^{(a}e_0^{b)}||\right)$, $\sup(|b|)$ and the foliation $\mathcal{S}_\tau$.

The next step consists in using the Sobolev embedding of $H^1(\mathcal{S}_\tau)$ in $L^6(\mathcal{S}_\tau)$. There exist two obstacles to the use of this embedding:

a. the first is the fact that the estimates must be uniform over the foliation in the sense that it must not depend on the parameter τ of the foliation (or the Sobolev constant must be the same all along the foliation);

b. the second comes the fact that we must deal with the singularity in i^+.

To deal with the second problem, the manifold $\hat{M}$ is extended beyond $\mathscr{I}^+$ by pulling backwards the hypersurface $\mathcal{S}$ through the flow associated with the vector field ∂_τ of the time function τ. Since the regularity of the metric is arbitrarily smooth at i^+ (say, at least C^2, in order to insure the existence of the different curvatures), this gives an extension as a smooth Lorentzian manifold of the manifold $(\hat{M}, \hat{g})$ in the neighborhood of i^+. $\mathscr{I}^+$ is then the past light cone from i^+ obtained from a C^2-extension of the metric $\hat{g}$ behind i^+.

To obtain a Sobolev embedding from $H^1(\mathcal{S}_\tau)$ into $L^6(\mathcal{S}_\tau)$ uniformly in τ, it is necessary to have a uniform bound for the Sobolev constant on the hypersurface $\mathcal{S}_\tau$. This is achieved by using proposition 3.4 for the foliation $\mathcal{S}_\tau$: there exists a constant K_{Sob} depending on the foliation $\mathcal{S}_\tau$ such that, uniformly in τ:

$$\forall \tau \in [0, \epsilon], \forall u \in H^1(\mathcal{S}_\tau), ||u||_{L^6(\mathcal{S}_\tau)} \leq K_{Sob} ||u||_{H^1(\mathcal{S}_\tau)}.$$

Remark 3.15. *a. The constant K_{Sob} depends on the foliation and, as such, of the parameter ϵ.*

b. The hypersurfaces $\mathcal{S}_\tau$ shrink as τ tends to zero. This does not affect the fact that the Lipschitz constants for the $\mathcal{S}_\tau$, for $\tau > 0$ remain bounded. Furthermore, the initial data are taken to be with compact support away from i^0. As a consequence, the functions (u_n) vanish in a neighborhood of i^+.

A direct consequence of the uniform Sobolev embeddings of $H^1(\mathcal{S}_\tau)$ in $L^6(S_\tau)$ in dimension 3 is the following inequality:

$$\begin{aligned} \int_{\mathcal{S}_\tau} |\hat{\nabla}^a e_0^b T_{ab}| \mu_{\mathcal{S}_\tau} &\leq C E_{u_{n+1}}(\mathcal{S}_\tau) + C K_{sob} ||u_n||^4_{H^1(\mathcal{S}_\tau)} \\ &\leq C E_{u_{n+1}}(\mathcal{S}_\tau) + C K_{sob} \left(\sup_{\tau \in [0,\epsilon]} E_{u_n}(\mathcal{S}_\tau) \right)^2. \end{aligned}$$

As a consequence, there exists a constant $\tilde{C}$ which depends only on the scalar curvature, the Killing form of e_0^a, the supremum of b and Sobolev constants such that:

$$E_{u_{n+1}}(\mathcal{S}_\tau) \leq \tilde{C} \left(\int_0^\tau E_{u_{n+1}}(\mathcal{S}_r) \mathrm{d}r + ||\theta||^2_{H^1(\mathscr{I}^+)} + \left(\sup_{r \in [0,\epsilon]} \left(E_{u_n}(\mathcal{S}_r) \right) \right)^2 \right).$$

Remark 3.16. *The constant $\tilde{C}$ can be chosen arbitrarily high. As a consequence, it is rescaled later without consequence for the proof (see remark 3.18 in the proof of proposition 3.17).*

Using Stokes theorem and Gronwall lemma, as in section 2.2.2, the energy of u_{n+1} satisfies:

$$E_{u_{n+1}}(\mathcal{S}_\tau) \leq \tilde{C} \exp(\tilde{C}\epsilon) \left(||\theta||^2_{H^1(\mathscr{I}^+)} + \left(\sup_{\tau \in [0,\epsilon]} E_{u_n}(\mathcal{S}_\tau) \right)^2 \right).$$

For $n = 0$, we have:

$$E_{u_0}(\mathcal{S}_\tau) \leq \tilde{C} \exp(\tilde{C}\epsilon) ||\theta||^2_{H^1(\mathscr{I}^+)}.$$

We denote by (C_n) the sequence defined by:

$$C_n = \sup_{\tau \in [0,\epsilon]} \left\{ E_{u_n}(\mathcal{S}_\tau) \right\}.$$

This sequence satisfies the inequality:

$$\forall n \in \mathbb{N}, C_{n+1} \leq \underbrace{\tilde{C} \exp(\tilde{C}\epsilon)}_{\alpha} \left(C_n^2 + \underbrace{||\theta||^2_{H^1(\mathscr{I}^+)}}_{\beta} \right) \text{ with } C_0 \leq \tilde{C} \exp(\tilde{C}\epsilon) ||\theta||^2_{H^1(\mathscr{I}^+)}.$$

Let us then consider the sequence $(c_n)_n$ defined by:

$$\left\{\begin{array}{lcl} c_0 & = & \alpha\beta \\ c_{n+1} & = & \alpha\left(c_n^2+\beta\right). \end{array}\right.$$

The purpose is to choose correctly $||\theta||_{H^1(\mathscr{I}^+)}$ such that the sequence is bounded. The function $x \mapsto \alpha\left(x^2+\beta\right)$ has two fixed points provided that:

$$1-4\beta\alpha^2>0,\ \text{ie } 1>2\tilde{C}\exp(\tilde{C}\epsilon)||\theta||^2_{H^1(\mathscr{I}^+)}. \tag{3.6}$$

If the initial condition c_0 is below the repulsive fixed point (the greater fixed point) of the function $x \mapsto \alpha\left(x^2+\beta\right)$, that is to say if

$$\frac{1+\sqrt{1-4\beta\alpha^2}}{2\alpha} \geq \alpha\beta. \tag{3.7}$$

This inequality is always satisfied as soon as $1-4\beta\alpha^2>0$. As a consequence, assuming that

$$\frac{1}{2\tilde{C}\exp(\tilde{C}\epsilon)} > ||\theta||^2_{H^1(\mathscr{I}^+)},$$

the sequence (c_n) converges to the remaining attractive fixed point; (c_n) is bounded and so is (C_n), which is the expected result.

Another useful consequence of the convergence of the sequence (c_n) is the following. The limit of this sequence satisfies:

$$\begin{array}{rcl} \dfrac{1-\sqrt{1-4\beta\alpha^2}}{2\alpha} & = & \dfrac{2\beta\alpha}{1+\sqrt{1-4\beta\alpha^2}} \\ & \leq & \dfrac{1}{2\alpha}. \end{array}$$

As a consequence, there exits a integer n_0 such that:

$$\forall n \geq n_0,\ \sup_{\tau\in[0,\epsilon]}\{E_{u_n}(\mathcal{S}_\tau)\} \leq c_n \leq \frac{1}{\alpha} = \frac{1}{\tilde{C}\exp(\tilde{C}\epsilon)}.\circledast \tag{3.8}$$

Proposition 3.17. *The sequence (u_n) converges on $\mathcal{R}$ in $C^0([0,\epsilon],H^1(\mathcal{S}_\tau))$, that is to say in the norm $\left(\sup_{\tau\in[0,\epsilon]}||u||^2_{H^1(\mathcal{S}_\tau)}\right)$.*

Proof. The method is exactly the same as in the previous proposition. Let n be a positive integer and consider the energy tensor associated with the linear wave equation for the function δ_n

$$T_{ab} = \hat{\nabla}_a\delta_n\hat{\nabla}_b\delta_n + \hat{g}_{ab}\left(-\frac{1}{2}\nabla_c\delta_n\nabla^c\delta_n+\frac{\delta_n^2}{2}\right).$$

The energy associated with a time slice $\mathcal{S}_\tau$ is written as in the previous proposition:

$$E_{\delta_n}(\mathcal{S}_\tau) = \int_{\mathcal{S}_\tau}\left(\frac{1}{2}\sum_{i=0,1,2,3}(e_{\mathbf{i}}^a\hat{\nabla}\delta_n)^2+\frac{\delta_n^2}{2}\right)\mathrm{d}\mu_{\mathcal{S}_\tau}$$

and it is equivalent to $\int_{\mathcal{S}_\tau} i^\star_{\mathcal{S}_\tau}(\star e_0^a T_{ab})$ (with constant which only depends on the geometric data, b and the Killing form of e_0^a).

Finally, the error term is:

$$\hat{\nabla}^a\left(e_0^b T_{ab}\right) = \hat{\nabla}^{(a}e_0^{b)}T_{ab} + \delta_n e_0^a\nabla_a\delta_n - \frac{1}{6}\mathrm{Scal}_{\hat{g}}\delta_n e_0^a\nabla_a\delta_n - b(e_0^a\nabla_a\delta_n)\delta_{n-1}(u_n^2+u_nu_{n-1}+u_{n-1}^2).$$

and can be estimated in absolute value by:

$$\int_{\mathcal{S}_\tau} |\hat{\nabla}^a e_0^b T_{ab}|\mu_{\mathcal{S}_\tau} \leq C E_{\delta_n}(\mathcal{S}_\tau) + 2\int_{\mathcal{S}_\tau} \delta_{n-1}^2 (u_n^4 + u_{n-1}^4)\mu_{\mathcal{S}_\tau}.$$

where C is a positive constant depending on $\sup(|\mathrm{Scal}_{\hat{g}}|)$, $\sup\left(||\hat{\nabla}^{(a} e_0^{b)}||\right)$, $\sup(|b|)$ and the foliation $\mathcal{S}_\tau$.

Using Hölder inequality and proposition 3.4 for the foliation $\mathcal{S}_\tau$, the non-linearity in the error term is estimated by:

$$\begin{aligned}\int_{\mathcal{S}_\tau} \delta_{n-1}^2 u_n^4 \mu_{\mathcal{S}_\tau} &\leq \left(\int_{\mathcal{S}_\tau} \delta_{n-1}^6 \mu_{\mathcal{S}_\tau}\right)^{\frac{1}{3}} \left(\int_{\mathcal{S}_\tau} u_n^6 \mu_{\mathcal{S}_\tau}\right)^{\frac{2}{3}} \\ &\leq K_{Sob^6} ||\delta_n||^2_{H^1(\mathcal{S}_\tau)} ||u_n||^4_{H^1(\mathcal{S}_\tau)}.\end{aligned}$$

The same inequality holds for $\delta_{n-1}^2 u_{n-1}^4$.

Finally, there exists a constant K, such that:

$$\int_{\mathcal{S}_\tau} |\hat{\nabla}^a e_0^b T_{ab}|\mu_{\mathcal{S}_\tau}$$

$$\leq K\left(\int_0^t E_{\delta_n}(\mathcal{S}_r)\mathrm{d}r + \epsilon^2 \left(\sup_{r\in[0,\epsilon]} \left(||\delta_{n-1}||_{H^1(\mathcal{S}_r)}\right)^2\right) \sup_{k\geq n-1} \left(\left(\sup_{r\in[0,\epsilon]} \left(||u_k||_{H^1(\mathcal{S}_r)}\right)^4\right)\right)\right).$$

Stokes theorem is then applied beween $\mathcal{S}_\tau$ and $\mathscr{I}^+$: since the characteristic data for δ_n are zero, the only remaining term is the energy on the surface $\mathcal{S}_\tau$. Modulo a constant which only depends on the same data as the constant $\tilde{C}$, the integral inequality holds:

$$E_{\delta_n}(\mathcal{S}_\tau) \leq \tilde{K}\left(\int_0^\tau E_{\delta_n}(\mathcal{S}_r)\mathrm{d}r + \epsilon^2 \sup_{k\geq n-1}\left(\sup_{r\in[0,\epsilon]}\left(||u_n||_{H^1(\mathcal{S}_r)}\right)^4\right)\left(\sup_{r\in[0,\epsilon]}\left(||\delta_{n-1}||_{H^1(\mathcal{S}_r)}\right)^2\right)\right),$$

for some contant $\tilde{K}$ and, using Gronwall's lemma, we get:

$$E_{\delta_n}(\mathcal{S}_\tau) \leq \tilde{K}\exp(\tilde{K}\epsilon)\epsilon^2 \sup_{k\geq n-1}\left(\sup_{r\in[0,\epsilon]}\left(||u_n||_{H^1(\mathcal{S}_r)}\right)^4\right)\left(\sup_{r\in[0,\epsilon]}\left(||\delta_{n-1}||_{H^1(\mathcal{S}_r)}\right)^2\right).$$

Remark 3.18. *The constant $\tilde{K}$, as the constant $\tilde{C}$ depends only the foliation $\mathcal{S}_\tau$, its scalar curvature, the Killing form of e_0^a and the supremum of b. As a consequence, up to a rescaling of $\tilde{C}$ or $\tilde{K}$, these constants can be chosen to be equal.*

Finally, the sequence (δ_n) satisfies:

$$\sup_{r\in[0,\epsilon]}\left(||\delta_n||_{H^1(\mathcal{S}_r)}\right)^2 \leq \tilde{C}\exp(\tilde{C}\epsilon)\epsilon^2 \sup_{k\geq n-1}\left(\sup_{r\in[0,\epsilon]}\left(||u_n||_{H^1(\mathcal{S}_r)}\right)^4\right)\left(\sup_{r\in[0,\epsilon]}\left(||\delta_{n-1}||_{H^1(\mathcal{S}_r)}\right)^2\right).$$

Since the sequence (u_n) is bounded by $\frac{1}{\alpha}$ for $n > n_0$, we have:

$$\tilde{C}\exp(\tilde{C}\epsilon)\epsilon^2 \sup_{n\geq n_0}\left(\sup_{r\in[0,\epsilon]}\left(||u_n||_{H^1(\mathcal{S}_r)}\right)^4\right) \leq \epsilon^2 \leq 1.$$

The sequence (δ_n) is then eventually contracting. The series of $(\sup_{r\in[0,\epsilon]}\left(||\delta_n||_{H^1(\mathcal{S}_r)}\right)^2)_n$ converges in the norm $\left(\sup_{\tau\in[0,\epsilon]}||u||^2_{H^1(\mathcal{S}_\tau)}\right)$, that is to say in $C^0([0,\epsilon], H^1(\mathcal{S}_\tau))$, and so does the sequence (u_n) .

End of the proof of theorem 3.12. The proof of the local existence is a direct consequence of the fact the sequence $(u_n)_{n\in\mathbb{N}}$ converges strongly on $\mathcal{R}$ for the norm

$$\sup_{r\in[0,\epsilon]}\left(||u||_{H^1(\mathcal{S}_r)}\right).$$

Let u be the limit of the sequence $(u_n)_n$. The only remaining thing to show is that the limit solves the problem of theorem 3.12. It is clear that u satisfies the initial conditions since all the functions u_n are identically equal to θ on $\mathscr{I}^+$. Finally, when noticing that u is in $H^1(\mathcal{R})$ which is continuously embedded in $L^3(\mathcal{R})$ (since $\mathcal{R}$ is four dimensional), the sequence $(u_n^3)_n$ converges in $L^1(\mathcal{R})$ and, as a consequence, in the distribution sense. u then satisfied the equation

$$\hat{\Box}u+\frac{1}{6}\mathrm{Scal}_{\hat{g}}u+bu^3=0$$

in the distribution sense.⊛

3.3 Global characteristic Cauchy problem

A global Cauchy problem is finally derived in two steps:

a. a preliminary result about the Cauchy problem for a hypersurface in the future of Σ_0 whose past contains i^0;

b. the characteristic Cauchy problem is then solved for small data with compact support which contains neither i^0 nor i^+ and then extended to functions in $H^1(\mathscr{I}^+)$.

3.3.1 Global Cauchy problem for compactly supported data

Starting from the same data θ in $H^1(\mathscr{I}^+)$ whose support contains neither i^+ nor i^0, the solution obtained in theorem 3.1 is extended to the future of Σ_0 by using density results and continuity of the propagator. The purpose of this section is to show that is it possible, starting from the hypersurface $\mathcal{S}$ in the future of Σ_0, to obtain a solution down to Σ_0 despite the singularity in i^0.

Proposition 3.19. *Let V^a be an orthogonal and normalized vector field to the uniformly spacelike hypersurface $\mathcal{S}$.*
The non-linear problem on $\mathcal{S}$:

$$\begin{cases}\hat{\Box}v+\frac{1}{6}Scal_{\hat{g}}v+bv^3=0\\ v\big|_{\Sigma_0}=\xi\in H_0^1(\mathcal{S})\\ V^a\hat{\nabla}_a v\big|_{\Sigma_0}=\zeta\in L^2(\mathcal{S})\end{cases}$$

admits a global unique solution down to Σ_0 in $C^0(\mathbb{R},H_0^1(\mathcal{S}))$.

Proof.The method consists in approximating the solution by solutions of the same problem with truncated data since the existence result for the Cauchy problem given by theorem 1.17 cannot be applied here directly because of the singularity in i^0. The uniqueness directly comes from theorem 3.6 and its corollary.

Let $(\chi_n)_{n\in\mathbb{N}}$ be a sequence of smooth functions with compact support in the interior of $\mathcal{S}$ such that:

$$\forall n\in\mathbb{N},\mathrm{supp}\,(\chi_n)\subset\mathrm{supp}\,(\chi_{n+1})\ \text{ and }\ \bigcup_{n\in\mathbb{N}}\mathrm{supp}\,(\chi_n)=\mathcal{S}\backslash\partial\mathcal{S}.$$

Let $(v_n)_{n\in\mathbb{N}}$ be the sequence defined by:

$$\begin{cases}\hat{\Box}v_n+\frac{1}{6}\mathrm{Scal}_{\hat{g}}v_n+bv_n^3=0\\ v_n\big|_{\mathcal{S}}=\chi_n\xi\in H^1(\mathcal{S}).\\ V^a\hat{\nabla}_a v_n\big|_{\mathcal{S}}=\chi_n\zeta\in L^2(\mathcal{S}).\end{cases}$$

Since the data are with compact support in the interior of $\mathcal{S}$, their pasts do not intersect $\mathscr{I}^+$ and, as a consequence, i^0.

Using proposition 3.9, this sequence converges towards a function v in the past of $\mathcal{S}$ down to Σ_0 for the $L^\infty H^1$ norm. This function clearly satisfies the initial conditions:

$$v\big|_{\mathcal{S}} = \xi \text{ on } \mathcal{S} \text{ and } V^a \hat{\nabla}_a v\big|_{\mathcal{S}} = \zeta.$$

Furthermore, proposition 3.9 also gives convergence in $H^1(J^-(\mathcal{S}))$ and, using Sobolev embeddings, as a consequence, in $L^6(J^-(\mathcal{S}))$. v then satisfies the nonlinear wave equation in the distribution sense. ⊛

Remark 3.20. *A direct consequence of this construction is that the trace of the solution of the Cauchy problem on Σ_0 is in $H_0^1(\Sigma_0)$*

3.3.2 Global characteristic Cauchy problem for small initial data in $H^1(\mathscr{I}^+)$

A global solution to the Goursat problem with compact which contains neither i^+ nor i^0 is then obtained by gluing solution of the local characteristic Cauchy problem and the solution of a well-chosen Cauchy problem on $\mathcal{S}$:

Proposition 3.21. *Let u be a solution to the Goursat problem for data θ with compact support which contains neither i^+ nor i^0.*
Then u can be extended from $\mathcal{S}$ down to Σ_0 in $C^0(\mathbb{R}, H^1(\mathcal{S}))$.

Proof.Consider the Cauchy problem on $\mathcal{S}$:

$$\begin{cases} \hat{\Box} v + \frac{1}{6}\mathrm{Scal}_{\hat{g}} v + bv^3 = 0 \\ v\big|_{\mathcal{S}} = u \in H^1(\mathcal{S}). \\ V^a \hat{\nabla}_a v\big|_{\mathcal{S}} = V^a \hat{\nabla}_a u \in L^2(\mathcal{S}). \end{cases}$$

According to proposition 3.19, this problem admits a global solution v down to Σ_0 in $C^0(\mathbb{R}, H^1(\mathcal{S}))$. Finally, the function w defined piecewise by:

$$w = u \text{ on } J^+(S) \text{ and } w = v \text{ on } J^+(\Sigma_0) \cap J^-(S).$$

satisfies the Goursat problem:

$$\begin{cases} \hat{\Box} u + \frac{1}{6}\mathrm{Scal}_{\hat{g}} u + bu^3 = 0 \\ \phi\big|_{\mathscr{I}^+} = \theta \in H^1(\mathscr{I}^+) \end{cases} \quad ⊛$$

Using proposition 3.21 and the continuity result, we can state the theorem of existence of the Goursat problem for small initial data:

Theorem 3.22. *Let us consider the nonlinear characteristic Cauchy problem on $\mathscr{I}^+$:*

$$\begin{cases} \hat{\Box} u + \frac{1}{6} Scal_{\hat{g}} u + bu^3 \quad = \quad 0 \\ u\big|_{\mathscr{I}^+} = \theta \in H^1(\mathscr{I}^+). \end{cases}$$

Then, for $||\theta||_{H^1(\mathscr{I}^+)}$ small enough, this problem admits a global unique solution down to the future of Σ_0 in $C^0(\mathbb{R}, H^1(\Sigma_0))$.

Remark 3.23. *As previously said in section 1.2.2 (see proposition 1.13), the singularity in i^+ is removable for Sobolev space in the sense that it is a regular point of a bigger manifold.*

Proof.The proof relies on the density of data with compact support in $H^1(\mathscr{I}^+)$ which does not neither i^0 nor i^+ in $H^1(\mathscr{I}^+)$ (proposition 1.13) and proposition 3.21.⊛

Remark 3.24. *As noticed above, the trace of the solution of the Goursat problem on Σ_0 is in $H_0^1(\Sigma)$.*

4 Construction of the scattering operator

The construction of the scattering operator can now be done by the mean of Cauchy problem on $\mathscr{I}^+$, $\mathscr{I}^-$ and Σ_0 via the composition of trace operators.

4.1 Existence and continuity of trace operators

The purpose of this section is to define trace operators for the solution of the wave equation on the hypersurfaces Σ_0 and $\mathscr{I}^+$. A symmetric construction can of course be realized on the past null infinity $\mathscr{I}^-$.

These trace operators are obtained using the following theorem ([40], p. 287):

Theorem 4.1. *Let M be a smooth compact manifold with piecewise C^1 boundary and consider the application T defined by:*

$$T : \left\{ \begin{array}{ccc} C^0(M) & \longrightarrow & C^0(\partial M) \\ f & \longmapsto & f\big|_{\partial M}. \end{array} \right.$$

Then, for all $s > \frac{1}{2}$, the operator T extends uniquely to a continuous map from $H^s(M)$ into $H^{s-\frac{1}{2}}(\partial M)$.

Existence theorems 1.17 and 3.22 give solutions to the initial (characteristic) problem in $H^1(\mathcal{J}^+(\Sigma_0))$. As a consequence, their traces on Σ_0 and $\mathscr{I}^+$ are respectively in $H^{\frac{1}{2}}(\Sigma_0)$ and $H^{\frac{1}{2}}(\mathscr{I}^+)$. Nonetheless, using the a priori estimates, they are in fact $H^1(\mathscr{I}^+)$ and $H^1(\Sigma_0)$.

Remark 4.2. *The singularity in i^+ is not a threat to the existence of a trace since the manifold and the metric can be extended with arbitrary regularity in a neighborhood of i^+. The problem with the singularity i^0 is avoided since the function spaces $H^1(\mathscr{I}^+)$ and $H^1_0(\Sigma_0)$ are the completions of smooth functions whose compact support does not contain i^0.*

Let us consider the trace operators:

$$T_0^+ := \left\{ \begin{array}{ccc} C_0^\infty(\Sigma_0) \times C_0^\infty(\Sigma_0) & \longrightarrow & H^1(\mathscr{I}^+) \\ (\theta, \tilde{\theta}) & \longmapsto & \phi\big|_{\mathscr{I}^+} \end{array} \right. \tag{4.1}$$

where ϕ is the unique solution of the problem:

$$\left\{ \begin{array}{l} \hat{\Box}\phi + \frac{1}{6}\mathrm{Scal}_{\hat{g}}\phi + b\phi^3 = 0 \\ \phi\big|_{\Sigma_0} = \theta \in C_0^\infty(\Sigma_0) \\ \hat{T}^a\hat{\nabla}_a\phi\big|_{\Sigma_0} = \tilde{\theta} \in C_0^\infty(\Sigma_0) \end{array} \right.$$

obtained by theorem 1.17 and

$$T_+^0 := \left\{ \begin{array}{ccc} \mathscr{E} & \longrightarrow & H^1_0(\Sigma_0) \times L^2(\Sigma_0) \\ \theta & \longmapsto & (\phi\big|_{\Sigma_0}, (\hat{T}^a\hat{\nabla}_a\phi)\big|_{\Sigma_0}) \end{array} \right. \tag{4.2}$$

where $\mathscr{E}$ is the set of smooth functions with compact support which contains neither i^+ nor i^0 and ϕ is the unique solution of the problem:

$$\left\{ \begin{array}{lcl} \hat{\Box}\phi + \frac{1}{6}\mathrm{Scal}_{\hat{g}}\phi + b\phi^3 & = & 0 \\ \phi\big|_{\mathscr{I}^+} = \theta \in C_0^\infty(\mathscr{I}^+) & & \end{array} \right.$$

obtained by theorem 3.22.

Remark 4.3. *The operator T_+^0 is not globally defined on $\mathscr{E}$ since theorem 3.22 only gives existence for small data. We denote by $\mathcal{B}^\infty_{\mathscr{I}^+}$ the trace of the open ball $\mathcal{B}_{\mathscr{I}^+}$ centered in zero in $H^1(\mathscr{I}^+) \cap \mathscr{E}$ on which T_+^0 is defined.*

These operators can be extended to $H^1(\Sigma_0)$ and $H^1(\mathscr{I}^+)$:

Proposition 4.4. *The operator T_0^+ can be extended to a locally Lipschitz operator from $H_0^1(\Sigma_0)\times L^2(\Sigma_0)$ to $H^1(\mathscr{I}^+)$.*
The operator T_+^0 can be extended to an Lipschitz operator from $\mathcal{B}_{\mathscr{I}^+}\subset H^1(\mathscr{I}^+)$ to $H_0^1(\Sigma_0)\times L^2(\Sigma_0)$.

Proof.The proof is done for the operator T_0^+ (it is exactly the same on the other side).

Let R be a positive constant. We denote by B_R the ball centered in 0 with radius R in $H^1(\Sigma_0)\times L^2(\Sigma_0)$ for the norm:

$$||\star||^2_{H^1(\Sigma_0)}+||\star||^2_{H^1(\Sigma_0)}$$

Using theorem 3.6, this operator satisfies:

$$\forall(\theta,\tilde{\theta},\xi,\tilde{\xi})\in(C_0^\infty(\Sigma_0)\cap B_R))^4,$$
$$||T_0^+(\theta,\tilde{\theta})-T_0^+(\xi,\tilde{\xi})||_{H^1(\mathscr{I}^+)}\leq C(R)\left(||\theta-\xi||^2_{H^1(\Sigma_0)}+||\tilde{\theta}-\tilde{\xi}||^2_{L^2(\Sigma_0)}\right)^{\frac{1}{2}}.$$

As a consequence, since the smooth functions wit compact support in Σ_0 are dense in $H_0^1(\Sigma_0)$, it admits a unique locally-Lipschitz extension from $H_0^1(\Sigma_0)\times L^2(\Sigma_0)$ into $H^1(\mathscr{I}^+)$.

The same proof holds for the operator T_+^0. The only difference is the that, due to theorem 3.22 which only provides us with solutions for small data, this operator is defined on an open all of $H^1(\mathscr{I}^+)$ and, as a consequence, is globally Lipschitz. ⊛

As already noted at the beginning of this section, a similar construction can be achieved on the past null infinity: there exist two trace operators T_-^0 and T_0^-, respectively locally Lipschitz and Lipschitz defined by:

$$T_0^-:=\left\{\begin{array}{ccc} H_0^1(\Sigma_0)\times L^2(\Sigma_0) & \longrightarrow & H^1(\mathscr{I}^-)\\ (\theta,\tilde{\theta}) & \longmapsto & \phi\big|_{\mathscr{I}^+}\end{array}\right. \tag{4.3}$$

where ϕ is the unique solution of the problem:

$$\left\{\begin{array}{l}\hat{\Box}\phi+\frac{1}{6}\mathrm{Scal}_{\hat{g}}\phi+b\phi^3=0\\ \phi\big|_{\Sigma_0}=\theta\in H_0^1(\Sigma_0)\\ \hat{T}^a\hat{\nabla}_a\phi\big|_{\Sigma_0}=\tilde{\theta}\in L^2(\Sigma_0)\end{array}\right.$$

obtained by theorem 1.17 and

$$T_-^0:=\left\{\begin{array}{ccc}\mathcal{B}_{\mathscr{I}^-} & \longrightarrow & H_0^1(\Sigma_0)\times L^2(\Sigma_0)\\ \theta & \longmapsto & (\phi\big|_{\Sigma_0},(\hat{T}^a\hat{\nabla}_a\phi)\big|_{\Sigma_0})\end{array}\right. \tag{4.4}$$

where $\mathcal{B}_{\mathscr{I}^-}$ is an open ball in $H^1(\mathscr{I}^-)$ and ϕ is the unique solution of the problem:

$$\left\{\begin{array}{lcl}\hat{\Box}\phi+\frac{1}{6}\mathrm{Scal}_{\hat{g}}\phi+b\phi^3 & = & 0\\ \phi\big|_{\mathscr{I}^-}=\theta\in H^1(\mathscr{I}^-) & & \end{array}\right.$$

obtained by theorem 3.22.

4.2 Conformal scattering operator

Finally, the conformal scattering operator is obtained as the composition of two trace operators. Following the idea of of Friedlander in [22] and applied by Mason-Nicolas in [32] for the Dirac and wave equations, the conformal scattering operator S is defined by the composition of the operators T_-^0 and T_0^+:

$$S=T_0^+\circ T_-^0:H^1(\mathscr{I}^-)\longrightarrow H^1(\mathscr{I}^-) \tag{4.5}$$

and its inverse is given by

$$S^{-1} = T_0^- \circ T_+^0 : H^1(\mathscr{I}^+) \longrightarrow H^1(\mathscr{I}^-) \tag{4.6}$$

These operators are not defined globally on $H^1(\mathscr{I}^+)$ or $H^1(\mathscr{I}^+)$ due to the restrictions imposed by theorem 3.22. Its domain of definition in our context is obtained from the domains of definition of T_-^0 and T_+^0 as follows: let $\mathcal{B}$ be the open set defined by:

$$\mathcal{B} = T_-^0(\mathcal{B}_{\mathscr{I}^-}) \cap T_+^0(\mathcal{B}_{\mathscr{I}^+}).$$

The images of $\mathcal{B}$ under T_0^- and T_0^+ give the domains of definition of S and S^{-1}, respectively. Finally, the following existence result for the conformal scattering operator can be stated:

Theorem 4.5 (Scattering operator)**.** *The operator S is an inversible, Lipschitz operator from $T_0^-(\mathcal{B})$ in $H^1(\mathscr{I}^-)$ into $T_0^+(\mathcal{B})$ in $H^1(\mathscr{I}^+)$. This operator is called conformal scattering operator.*

Proof.The proof is an immediate consequence of proposition 4.4.⊛

Remark 4.6. *The conformal scattering operator was introduced to avoid the use of the spectral theory which requires the metric to be static. It is nonetheless possible to talk about geometric scattering at least in the Schwarzschild part of the manifold and wonder wether it is possible to establish an equivalence in this region. Some answers to this question can be found in [32] (section 4.2) for the Dirac and Maxwell equations.*

Concluding remarks

There exist several possible extensions to this work:

- the case where the metric in the neighborhood of i^0 is the Kerr-Newman metric;
- the nonlinearity could be modified and the equation could, for instance, be quasilinear, or satisfy the null condition;
- following [33], these results could be extended to peeling results for the same cubic defocusing wave equation.

One of the main problems of general relativity is the construction of solutions to the Einstein equations. One intermediate step is to establish the same kind of result for the Yang-Mills equations.

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