Recurrence for random dynamical systems

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RECURRENCE FOR RANDOM DYNAMICAL SYSTEMS

PHILIPPE MARIE AND JEROME ROUSSEAU

Abstract. This paper is a first step in the study of the recurrence behavior in random dynamical systems and randomly perturbed dynamical systems. In particular we define a concept of quenched and annealed return times for systems generated by the composition of random maps. We moreover prove that for super-polynomially mixing systems, the random recurrence rate is equal to the local dimension of the stationary measure.

1. Introduction

Two important properties of discrete dynamical systems that have been deeply studied in the last years are recurrence (e.g. [4, 5, 12]) and stability under small random perturbations, or more generally random dynamical systems (e.g. [7, 8, 9, 10]). As far as we know and despite of the success of these both surveys, there is no result about the recurrence of random dynamical systems. The purpose of this paper is to define the main needed objects to start this study and to give first results to describe the recurrence behavior in the random case.

The evolution of a random dynamical system generated by random transformations will be as described in what follows: we consider an indexed family of maps $\{T_\omega\}_{\omega\in\Omega}$ and ϑ a map preserving a probability measure $\mathbb{P}$ on Ω, let $x \in X$ be an initial state, an ω is chosen according to the probability $\mathbb{P}$ and the system moves to the state $x_1 = T_\omega x$. Then the system moves to $T_{\vartheta\omega}(T_\omega x)$ and so on such that a random orbit will be a subset $\{x_n^\omega\}_{n\in\mathbb{N}^*}$ of the phase space, where for every n: $x_n^\omega = T_{\vartheta^n\omega} \circ \ldots \circ T_\omega x$. We will consider systems where there exists a stationary measure, i.e. a probability measure μ on X such that the measure $\mathbb{P} \otimes \mu$ is invariant for the skew-product associated to the random dynamical system (see for example [7, 8, 13] for an introduction to this theory).

In section 2 we firstly introduce some new definitions that are the counterpart of usual recurrence objects in the random framework. More precisely, for random dynamical systems generated by the composition of random transformations we define a *quenched* and an *annealed* version of return times. The quenched one is the return time of a random orbit for a special realisation ω, which is not the best object to study the global behavior of the system, while the annealed one defined by

$$\mathbf{T}_B(x) := \int_\Omega \inf\{n > 0 : T_{\vartheta^n\omega} \circ \ldots \circ T_\omega x \in B\} d\mathbb{P}(\omega)$$

appears to us as the best quantity to consider for this study. We will justify the choice of these definitions applying classical results of recurrence in the deterministic case (Poincaré's theorem, Kac's lemma) to an usual representation of random maps, namely the skew-product transformation, and a good borelian subset.

Then, in section 3, using a recent result from Rousseau and Saussol [11] about the recurrence for observations applied to the skew product representation, we establish a link

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between random return times and local dimension of stationary measures in both cases: quenched and annealed. Moreover, in our main theorem (Theorem 12),we prove that, for systems whose decay of correlations hold super-polynomially fast for Lipschitz observables, the random recurrence rates are equal to the local dimension of the stationary measure. This is a first step in the description of the recurrence for non-deterministic dynamical systems.

In the fourth section we will give two simple examples of random toral automorphisms for which our results hold and in the fifth one we will consider the particular case where random dynamical systems are used as a perturbation of an initial dynamics. The last part is devoted to the proofs of our main theorems.

2. Random dynamical systems and return time

Let us first recall some generalities about random dynamical systems. Let Ω be a metric space with $\mathcal{B}(\Omega)$ its borelian σ-algebra and $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$ a probability space which represent the space of the randomness. Let ϑ be a $\mathbb{P}$-preserving map which represents the time evolution of the randomness. Ω indexes a family of maps $\{T_\omega\}_{\omega\in\Omega}$ from a compact riemannian manifold X into itself. A random dynamical system $\mathcal{T}$ on X over $(\Omega, \mathcal{B}(\Omega), \mathbb{P}, \vartheta)$ is generated by mappings T_ω so that the map $(\omega, x) \to T_\omega x$ is measurable. The random orbit generated by a realisation $\omega \in \Omega$ starting from x is the set $\{x_n^\omega\}_{n\in\mathbb{N}^*} \subset X$ where for every n:

$$x_n^\omega := T_{\vartheta^n\omega} \circ \ldots \circ T_\omega x := T_\omega^n x.$$

One useful representation of this system is given by the following skew product transformation:

$$\begin{aligned} S:\ \Omega \times X &\longrightarrow \Omega \times X \\ (\omega, x) &\longmapsto (\vartheta\omega, T_\omega x). \end{aligned}$$

We focus on random dynamical systems such that the skew-product S has an invariant measure ν with marginal $\mathbb{P}$ on Ω. Moreover we only deal with random dynamical systems for which the skew product invariant measure is a product measure $\nu = \mathbb{P}\otimes\mu$, i.e. systems where the sample measures are identical (for more details on sample measures see e.g. [9, 8]).

Definition 1. *We will say that μ is a stationary measure for the random dynamical system.*

We remark that for i.i.d. random dynamical systems under weak assumptions such measures always exist (for example if T_ω is continuous for all $\omega \in \Omega$ [7]).

We can also define an ergodic property for random dynamical system [10]:

Definition 2. *We say that the random dynamical system $\mathcal{T}$ on X over $(\Omega, \mathcal{B}(\Omega), \mathbb{P}, \vartheta)$ is ergodic with respect to a stationary measure μ if the deterministic system $(\Omega\times X, S, \mathbb{P}\otimes\mu)$ is ergodic.*

We now introduce a return time concept for random orbits. Since $(\Omega \times X, S, \mathbb{P}\otimes\mu)$ is a deterministic dynamical system, we can define the first return time into a borelian subset $A \times B \subset \Omega \times X$ as usually for $(\omega, x) \in A \times B$:

$$\tau^S_{A\times B}(\omega, x) = \inf\{k > 0\ :\ S^k(\omega, x) \in A \times B\}.$$

Choosing $A = \Omega$ we remark that $S^n(\omega, x) \in \Omega \times B$ if and only if $T^n_\omega x \in B$. This naturally drives to the following definition:

Definition 3. *For a fixed $\omega \in \Omega$ the first quenched random return time in a measurable subset $B \subset X$ of the random orbit starting from a point $x \in B$ is:*

$$\begin{aligned} \tau_B^\omega(x) &= \tau^S_{\Omega\times B}(\omega, x) \\ &= \inf\{k > 0 \,:\, T_{\vartheta^k\omega} \circ \cdots \circ T_\omega x \in B\}. \end{aligned}$$

Remark that the Poincaré recurrence theorem applied to the skew product ensures that this quantity is almost everywhere finite.

From now on we assume that X is a metric space with metric d. For $\omega \in \Omega$, we are interested in the behavior as $r \to 0$ of the quenched random return time of a point $x \in X$ into the open ball $B(x, r)$ defined by

$$\tau_r^\omega(x) := \inf\{k > 0 : T_{\vartheta^k\omega} \circ ... \circ T_\omega x \in B(x,r)\} = \inf\{k > 0 : d(T_{\vartheta^k\omega} \circ ... \circ T_\omega x, x) < r\}.$$

Definition 4. *The random dynamical system $\mathcal{T}$ on X over $(\Omega, \mathcal{B}(\Omega), \mathbb{P}, \vartheta)$ with a stationary measure μ is called random-aperiodic if*

$$\mathbb{P} \otimes \mu\{(\omega, x) \in \Omega \times X : \exists n \in \mathbb{N},\, T_{\vartheta^n\omega} \circ \ldots \circ T_\omega x = x\} = 0.$$

To avoid some problem with the first return time for non-random-aperiodic system (as the one presented on the Section 2.3 of [11]) we need to introduce the non-instantaneous return times.

Definition 5. *Let $r > 0$. For $x \in X$, $\omega \in \Omega$ and $p \in \mathbb{N}$ we define the p-non-instantaneous quenched random return time:*

$$\tau_{r,p}^\omega(x) := \inf\{k > p \ :\ d(T_{\vartheta^k\omega} \circ ... \circ T_\omega x, x) < r\}.$$

Then we define the non-instantaneous quenched random lower and upper recurrence rates:

$$\underline{R}^\omega(x) := \lim_{p\to\infty} \liminf_{r\to 0} \frac{\log \tau_{r,p}^\omega(x)}{-\log r} \qquad \text{and} \qquad \overline{R}^\omega(x) := \lim_{p\to\infty} \limsup_{r\to 0} \frac{\log \tau_{r,p}^\omega(x)}{-\log r}.$$

Finally we recall that *the lower and upper pointwise or local dimension* of a Borel probability measure μ on X at a point $x \in X$ are defined by

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \qquad \text{and} \qquad \overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

3. Quenched recurrence rate, annealed recurrence rate and pointwise dimension

We now state our results about the recurrence behavior for the random maps. We first focus on the quenched case (random orbits) in section 3.1 and then on the annealed one (random recurrence) in section 3.2.

3.1. **Recurrence for random orbits.**

Theorem 6. *Let $\mathcal{T}$ be a random dynamical system on X over $(\Omega, \mathcal{B}(\Omega), \mathbb{P}, \vartheta)$ with a stationary measure μ. For $\mathbb{P} \otimes \mu$-almost every $(\omega, x) \in \Omega \times X$*

$$\underline{R}^\omega(x) \le \underline{d}_\mu(x) \qquad \text{and} \qquad \overline{R}^\omega(x) \le \overline{d}_\mu(x).$$

Even if these inequalities can be strict, with more assumptions on the random dynamical system one can prove that the equalities hold. This drives us to introduce the decay of correlations for a random dynamical system:

Definition 7. *A random dynamical system with a stationary measure μ has a super-polynomial decay of correlations if for all $n \in \mathbb{N}^*$ and ψ , φ Lipschitz observables from X to $\mathbb{R}$:*

$$|\int_X \int_\Omega \psi(T_{\vartheta^n \omega} \circ \ldots \circ T_\omega x)\varphi(x) d\mathbb{P}(\omega) d\mu(x) - \int_X \psi d\mu \int_X \varphi d\mu| \leq \|\psi\| \|\varphi\| \theta_n$$

with $\lim_{n\to\infty} \theta_n n^p = 0$ for any $p > 0$ and where $\|.\|$ is the Lipschitz norm.

Theorem 8. *Let $\mathcal{T}$ be a random dynamical system on X over $(\Omega, \mathcal{B}(\Omega), \mathbb{P}, \vartheta)$ with a stationary measure μ. If the random dynamical system has a super-polynomial decay of correlations then*

$$\underline{R}^\omega(x) = \underline{d}_\mu(x) \qquad \text{and} \qquad \overline{R}^\omega(x) = \overline{d}_\mu(x)$$

for $\mathbb{P} \otimes \mu$-almost every $(\omega, x) \in \Omega \times X$ such that $\underline{d}_\mu(x) > 0$.

Remark. *If the system is random-aperiodic, one can remark that for $\mathbb{P} \otimes \mu$-almost every (ω, x) and for every p, $\tau^\omega_{r,p} = \tau^\omega_r(x)$ for r small enough. So, for such systems, the non-instantaneous return time are not necessary and these two theorems are proved for the instantaneous quenched random lower and upper recurrence rate (defined using the first return time instead of the p-non-instantaneous return times).*

Proof. These two theorems are proved using Theorem 2 and Theorem 5 of [11] applied to $(\Omega \times X, \mathcal{B}(\Omega \times X), \mathbb{P} \otimes \mu, S)$ with the observation f defined by

$$\begin{aligned} f \ : \Omega \times X &\longrightarrow X \\ (\omega, x) &\longrightarrow x. \end{aligned}$$

We emphasize that in Theorem 5 of [11] the condition of the super-polynomial decay of correlations is applied to the skew-product, in fact in our case this is not necessary. A weaker assumption, the super-polynomial decay of correlations for the random dynamical system, allows us to prove Lemma 15 of [11] for the observation f defined above. □

3.2. **Random recurrence.** For B a Borelian subset of X, Poincaré's recurrence theorem applied to S ensures that the first return time is finite for $\mathbb{P} \otimes \mu$-almost all $(\omega, x) \in \Omega \times B$. Let us now suppose that the random dynamical system is ergodic then we can apply Kac's lemma and we get:

$$\begin{aligned} \int_{\Omega \times B} \tau^S_{\Omega \times B}(\omega, x) \, d\mathbb{P} \otimes \mu(\omega, x) &= \int_{\Omega \times B} \tau^\omega_B(x) \, d\mathbb{P} \otimes \mu(\omega, x) \\ &= \int_B \int_\Omega \tau^\omega_B(x) \, d\mathbb{P}(\omega) d\mu(x) \\ &= 1. \end{aligned} \tag{1}$$

Which is an argument to set:

Definition 9. *The first annealed random return time for $x \in B$ in the set B is:*

$$\mathbf{T}_B(x) = \int_\Omega \tau^\omega_B(x) \, d\mathbb{P}(\omega).$$

With this definition and (1), we have Kac's lemma for the first annealed random return time:

$$\int_B \mathbf{T}_B(x) d\mu(x) = 1.$$

As previously, we are interested in the behaviour of the return time of $x \in X$ into $B(x, r)$ when $r \to 0$, which drives to the following definitions:

Definition 10. *Let $r > 0$ and $p \in \mathbb{N}$. We define the p-non-instantaneous annealed random return time*

$$\mathbf{T}_{r,p}(x) := \int_{\Omega} \tau^{\omega}_{r,p}(x) d\mathbb{P}(\omega).$$

When $p = 0$, we denote this return time $\mathbf{T}_r(x)$. Then we define the non-instantaneous annealed random lower and upper recurrence rates

$$\underline{\mathbf{R}}(x) := \lim_{p\to\infty} \liminf_{r\to 0} \frac{\log \mathbf{T}_{r,p}(x)}{-\log r} \qquad \overline{\mathbf{R}}(x) := \lim_{p\to\infty} \limsup_{r\to 0} \frac{\log \mathbf{T}_{r,p}(x)}{-\log r}.$$

The main results are the two following theorems:

Theorem 11. *Let $\mathcal{T}$ be a random dynamical system on X over $(\Omega, \mathcal{B}(\Omega), \mathbb{P}, \vartheta)$ with a stationary measure μ. For μ-almost every $x \in X$*

$$\underline{\mathbf{R}}(x) \le \underline{d}_\mu(x) \qquad \text{and} \qquad \overline{\mathbf{R}}(x) \le \overline{d}_\mu(x).$$

Theorem 12. *Let $\mathcal{T}$ be a random dynamical system on X over $(\Omega, \mathcal{B}(\Omega), \mathbb{P}, \vartheta)$ with a stationary measure μ. If the random dynamical system has a super-polynomial decay of correlations then*

$$\underline{\mathbf{R}}(x) = \underline{d}_\mu(x) \qquad \text{and} \qquad \overline{\mathbf{R}}(x) = \overline{d}_\mu(x)$$

for μ-almost every x such that $\underline{d}_\mu(x) > 0$.

Remark. *As in the previous theorems, we emphasize that if the system is random-aperiodic, the first annealed return time is equal to the p-non-instantaneous one for r small enough almost everywhere and these theorems hold for the instantaneous annealed random lower and upper recurrence rate.*

4. Random toral automorphisms

We now give two examples of random dynamical systems for which our theorems hold. We emphasize that the first example is a non-i.i.d. random dynamical system where there exists a stationary measure.

4.1. **Non-i.i.d. random linear maps.** Let $X = \mathbb{T}^1$ be the one-dimensional torus. Consider the two linear maps which preserve Lebesgue measure *Leb* on X

$$\begin{array}{rcl} T_1 : X & \longrightarrow & X \\ x & \longrightarrow & 2x \end{array} \quad \text{and} \quad \begin{array}{rcl} T_2 : X & \longrightarrow & X \\ x & \longrightarrow & 3x. \end{array}$$

The random orbit is construct by choosing one of these two maps following a Markov process with the stochastic matrix

$$A = \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}.$$

In fact, this random dynamical system is represented by the following skew-product

$$\begin{array}{rcl} S : \Omega \times X & \longrightarrow & \Omega \times X \\ (\omega, x) & \longleftarrow & (\vartheta(\omega), T_\omega x) \end{array}$$

with $\Omega = [0,1]$, $T_\omega = T_1$ if $\omega \in [0, 2/5)$, $T_\omega = T_2$ if $\omega \in [2/5, 1]$ and where ϑ is the following piecewise linear map

$$\vartheta(\omega) = \begin{cases} 2\omega & \text{if } \omega \in [0, 1/5) \\ 3\omega - 1/5 & \text{if } \omega \in [1/5, 2/5) \\ 2\omega - 4/5 & \text{if } \omega \in [2/5, 3/5) \\ 3\omega/2 - 1/2 & \text{if } \omega \in [3/5, 1]. \end{cases}$$

Since ϑ preserves Lebesgue measure $\mathbb{P} = Leb$, the skew-product S is $Leb \otimes Leb$-invariant and so Leb is a stationary measure for this random dynamical system. Since S has an exponential decay of correlations [2], Theorem 8 and Theorem 12 hold. Moreover, one can easily see that this system is random-aperiodic, so for $Leb \otimes Leb$-almost every $(\omega, x) \in [0,1] \times \mathbb{T}^1$

$$\lim_{r\to 0} \frac{\log \tau_r^{\omega}(x)}{-\log r} = 1$$

and for Leb-almost every $x \in \mathbb{T}^1$

$$\lim_{r\to 0} \frac{\log [\mathbf{T}_r(x)]}{-\log r} = 1.$$

4.2. **Random hyperbolic toral automorphisms.** Let $X = \mathbb{T}^2$, we recall that a hyperbolic toral automorphism is a map $A : \mathbb{T}^2 \to \mathbb{T}^2$ acting through the matrix $x \mapsto Ax$ (mod 1), such that the matrix A has integer entries, eigenvalues with modulus different from 1 and $\det A = \pm 1$. We will restrict our example to the case where the matrix has positive entries, it is possible to consider more general automorphisms under an invariant cone assumption, see [1]. Let $\Omega = \{0,1\}^{\mathbb{N}}$ and $\vartheta = \sigma$ be the left shift on Ω. Let A_0, A_1 two hyperbolic automorphisms with positive entries. Let A_0 be chosen with a probability q and A_1 with a probability $1-q$, i.e. $\mathbb{P} = P^{\mathbb{N}}$ with $P(0) = q$ and $P(1) = 1-q$. Then the Lebesgue measure is stationary and the decay of correlations is exponentially fast for Lipschitz observables (indeed for strong Hölder observables (see [1])) f and g which satisfy $\int_{\mathbb{T}^2} f(x)dx = \int_{\mathbb{T}^2} g(x)dx = 0$.

Proposition 13. *This system is random-aperiodic.*

Proof. For any fixed $\omega \in \Omega$

$$\mu(\{x \in \mathbb{T}^2 \,:\, \exists n \in \mathbb{N}^*\,,\, A_\omega^n x = x\}) \le \mu\Bigg(\bigcup_{\substack{n\in\mathbb{N}^* \\ 1 \text{ eigenvalue of } A_\omega^n}} E_1(A_\omega^n) \Bigg)$$

where $A_\omega^n = A_{\omega_n} \ldots A_{\omega_2} A_{\omega_1}$ and $E_1(A_\omega^n)$ is the eigenspace of A_ω^n associated to the eigenvalue 1. Since A_ω^n has determinant 1 or -1, $\dim E_1(A_\omega^n) = 2$ implies $A_\omega^n = Id$ which is a contradiction with A_0 and A_1 have positive entries:

$$\underbrace{A_{\omega_n} \ldots A_{\omega_2}}_{=A_{\omega_1}^{-1}} A_{\omega_1} = Id$$

and $A_{\omega_1}^{-1}$ must have some negative entries. So the set A_ω^n has dimension one and then

$$\begin{aligned}
\mu(\{x \in \mathbb{T}^2 \,:\, \exists n \in \mathbb{N}^*\,,\, A_\omega^n x = x\}) &\le \sum_{\substack{n\in\mathbb{N}^* \\ 1 \text{ eigenvalue of } A_\omega^n}} \underbrace{\mu(E_1(A_\omega^n))}_{0} \\
&\le 0.
\end{aligned}$$

Then, this system satisfies

$$\mathbb{P} \otimes \mu(\{(\omega, x) \in \{0,1\}^{\mathbb{N}} \times \mathbb{T}^2 \,:\, \exists n \in \mathbb{N}^*\,,\, A_\omega^n x = x\}) = 0.$$

□

So Proposition 13 and Theorem 8 give for $\mathbb{P} \otimes Leb$-almost every $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times \mathbb{T}^2$

$$\lim_{r\to 0} \frac{\log \tau_r^{\omega}(x)}{-\log r} = 2$$

and Theorem 12 gives for *Leb*-almost every $x \in \mathbb{T}^2$

$$\lim_{r\to 0} \frac{\log [\mathbf{T}_r(x)]}{-\log r} = 2.$$

5. Small random perturbations

A particular case of random dynamical systems is the one where all the maps are chosen arbitraly close to a fixed initial map, namely the random dynamical system is a small random perturbation of a map T. More precisely we consider a measurable map $T : X \to X$ which plays the role of an initial dynamics, and we add a small amount of noise during its evolution, modelled by a random dynamical system. More precisely, for any small $\varepsilon > 0$ (which is the noise level) we consider a probability space $(\Lambda_\varepsilon, \mathcal{B}(\Lambda_\varepsilon), P_\varepsilon)$ where Λ_ε is a metric space and $\mathcal{B}(\Lambda_\varepsilon)$ the borelian σ-algebra. Then let us consider a parametrized family of maps $\{T_\lambda\}_{\lambda\in\Lambda_\varepsilon}$ which are ε-close to T in some $\mathcal{C}^0$ sense. The framework is the following $\Omega = \Lambda_\varepsilon^{\mathbb{N}}$, $\mathbb{P} = P_\varepsilon^{\mathbb{N}}$, $\vartheta = \sigma$ the left shift on Ω and $T_\omega = T_{\lambda_1}$ for all $\omega = (\lambda_1, \lambda_2, \dots) \in \Omega$. Let us consider the following map:

$$\begin{aligned} \Phi_\varepsilon :\ \Lambda_\varepsilon \times X &\longrightarrow X \\ (\lambda, x) &\longmapsto T_\lambda(x) \end{aligned}$$

such that for all $x \in X$, $\Phi_\varepsilon(., x)$ is measurable and there is a $\lambda^* \in \Lambda_\varepsilon$ which satisfies $\Phi_\varepsilon(\lambda^*, x) = T(x)$ for all $x \in X$. We assume that the probability measure $(\Phi_x^\varepsilon)_* P_\varepsilon$ satisfies the the following assumptions:

(RT1) For all $x \in X$, $(\Phi_x^\varepsilon)_* P_\varepsilon$ is absolutely continuous with respect to the Lebesgue measure.

(RT2) For all $x \in X$, $\operatorname{supp}(\Phi_x^\varepsilon)_* P_\varepsilon \subset B_\varepsilon(Tx)$.

Remark. *Our assumptions (RT2) is a kind of $\mathcal{C}^0$ closeness since: $(\Phi_x^\varepsilon)_* P_\varepsilon(B_\varepsilon(Tx)) = 1$, i.e for all $x \in M$ and for P_ε almost all λ, $d(Tx, T_\lambda x) \le \varepsilon$.*

Again, we do not need non-instantaneous return time:

Proposition 14. *Under the hypothesis (RT1), the system is random-aperiodic.*

Proof. We will prove that for all $x \in X$

$$P_\varepsilon^{\mathbb{N}}(\{\omega = (\lambda_1, \lambda_2, ...) \in \Lambda_\varepsilon^{\mathbb{N}} \,:\, \exists n \in \mathbb{N}^*\,,\, T_\omega^n x = x\}) = 0$$

where $T_\omega^n = T_{\lambda_n} \circ \cdots \circ T_{\lambda_2} \circ T_{\lambda_1}$.

Let $x \in X$. Let $A \subset X$ with $Leb(A) = 0$. Since the measure $(\Phi_x^\varepsilon)_* P_\varepsilon$ is absolutely continuous by (RT1) we have $(\Phi_x^\varepsilon)_* P_\varepsilon(A) = P_\varepsilon(\{\lambda_1 \in \Lambda_\varepsilon \,:\, T_{\lambda_1} x \in A\}) = 0$. Moreover, we remark that

$$(\Phi_x^\varepsilon)^2{}_* P_\varepsilon^2(A) := P_\varepsilon^2(\{(\lambda_1, \lambda_2) \in \Lambda_\varepsilon^2 \,:\, T_{\lambda_2} T_{\lambda_1} x \in A\}) = 0.$$

Indeed

$$(\Phi_x^\varepsilon)^2{}_* P_\varepsilon^2(A) = \int_{\Lambda_\varepsilon} (\Phi_{T_{\lambda_1} x}^\varepsilon)_* P_\varepsilon(A) dP_\varepsilon(\lambda_1)$$

but since $T_{\lambda_1} x \in X$ for all $\lambda_1 \in \Lambda_\varepsilon$, (RT1) gives $(\Phi_{T_{\lambda_1} x}^\varepsilon)_* P_\varepsilon(A) = 0$ for all $\lambda_1 \in \Lambda_\varepsilon$. Using this idea, with an easy recurrence argument, we can prove that for every $n \in \mathbb{N}^*$

$$(\Phi_x^\varepsilon)^n{}_* P_\varepsilon^n(A) := P_\varepsilon^n(\{(\lambda_1, \dots, \lambda_n) \in \Lambda_\varepsilon^n \,:\, T_{\lambda_n} \dots T_{\lambda_1} x \in A\}) = 0. \tag{2}$$

Finally, we get

$$\begin{aligned}
P_\varepsilon^{\mathbb{N}}(\{\omega\in\Lambda_\varepsilon^{\mathbb{N}}:\exists n\in\mathbb{N}^*\,,\,T_\omega^n x=x\}) &= P_\varepsilon^{\mathbb{N}}(\bigcup_{n\in\mathbb{N}^*}\{\omega\in\Lambda_\varepsilon^{\mathbb{N}}\,:\,T_\omega^n x=x\})\\
&\leq \sum_{n\in\mathbb{N}^*} P_\varepsilon^{\mathbb{N}}(\{\omega\in\Lambda_\varepsilon^{\mathbb{N}}\,:\,T_\omega^n x=x\})\\
&\leq \sum_{n\in\mathbb{N}^*} P_\varepsilon^{n}(\{(\lambda_1,\ldots,\lambda_n)\in\Lambda_\varepsilon^{n}\,:\,T_{\lambda_n}\ldots T_{\lambda_1}x=x\})\\
&\leq \sum_{n\in\mathbb{N}^*} \underbrace{P_\varepsilon^{n}(\{(\lambda_1,\ldots,\lambda_n)\in\Lambda_\varepsilon^{n}:T_{\lambda_n}\ldots T_{\lambda_1}x\in\{x\}\})}_{=0 \text{ by } (2)}\\
&\leq 0.
\end{aligned}$$

□

5.1. **Small random perturbations of expanding maps of the circle.** Let $X=\mathcal{S}^1$ and T be an expanding $\mathcal{C}^r$ $(2\leq r<\infty)$ transformation of $\mathcal{S}^1$. We put an additive noise to this system, namely $T_\lambda x=Tx+\lambda$ where λ is a random variable distributed according to a density supported on $(-\varepsilon,+\varepsilon)$. Then it is well known (see for example [3]) that the random dynamical system admits an absolutely continuous stationary measure μ whose density is $\mathcal{C}^{r-1}$ and is exponentially mixing for $\mathcal{C}^{r-1}$ observables. Even if the decay of correlations is exponential for not Lipschitz observables, it is possible to go from C^{r-1} observables to Lipschitz observables with some simple approximations arguments. Then by Theorem 8 and Proposition 14 for $\mathbb{P}\otimes\mu$-almost every $(\omega,x)\in(-\varepsilon,\varepsilon)^{\mathbb{N}}\times\mathcal{S}^1$ we have

$$\lim_{r\to0}\frac{\log\tau_r^\omega(x)}{-\log r}=1$$

and by Theorem 12 we get for μ-almost every $x\in\mathcal{S}^1$

$$\lim_{r\to0}\frac{\log\left[\mathbf{T}_r(x)\right]}{-\log r}=1.$$

5.2. **Small random perturbations of piecewise expanding maps of the interval.** Let $X=[0,1]$ and T be a $\mathcal{C}^2$-piecewise expanding map without periodic turning point (see [3]) [1], then for the same additive perturbation than in the previous example, Baladi and Young [3] have proved the existence (and the stability) of an absolutely continuous stationary measure μ and the exponential decay of correlations for observables that are of bounded variations, and therefore for Lipschitz ones. We obtain by Theorem 8 and Proposition 14 that for $\mathbb{P}\otimes\mu$-almost every $(\omega,x)\in(-\varepsilon,\varepsilon)^{\mathbb{N}}\times[0,1]$

$$\lim_{r\to0}\frac{\log\tau_r^\omega(x)}{-\log r}=1$$

and we obtain by Theorem 12 that for μ-almost every $x\in[0,1]$

$$\lim_{r\to0}\frac{\log\left[\mathbf{T}_r(x)\right]}{-\log r}=1.$$

[1] Remark that similar results can be obtained for maps whose derivative is uniformly larger than 2, see [8].

6. Proofs

We recall the definition of a weakly diametrically regular measure:

Definition 15. *A measure μ is weakly diametrically regular (wdr) on the set $Z \subset X$ if for any $\eta > 1$, for μ-almost every $x \in Z$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that if $r < \delta$ then $\mu(B(x, \eta r)) \leq \mu(B(x,r))\, r^{-\varepsilon}$.*

Proof of Theorem 11. By [5] we know that for all $d \in \mathbb{N}^*$ any probability measure is weakly diametrically regular on $\mathbb{R}^d$ and then the measure μ is weakly diametrically regular on X. In the definition of weakly diametrically regular we emphasize that the function $\delta(\cdot, \varepsilon, \eta)$ can be made measurable for every fixed ε and η. Let us fix $\varepsilon > 0$ and $\eta = 4$. We choose $\delta > 0$ sufficiently small to have $\mu(X_\delta) > \mu(X) - \varepsilon = 1 - \varepsilon$ where

$$X_\delta := \{x \in X : \delta(x, \varepsilon, \eta) > \delta\}.$$

In order to use Borel-Cantelli lemma, we prove that for every $p \in \mathbb{N}$, $\sum_{n \in \mathbb{N}} \mu(A_\varepsilon(e^{-n})) < \infty$ where for $r > 0$

$$A_\varepsilon(r) := \left\{ y \in X_\delta : \mathbf{T}_{6r,p}(y) \mu(B(y, 2r)) \geq r^{-2\varepsilon} \right\}.$$

Definition 16. *Given $r > 0$, a countable set $E \subset F$ is a maximal r-separated set for F if*

(1) $B(x, \frac{r}{2}) \cap B(y, \frac{r}{2}) = \emptyset$ for any two distinct $x, y \in E$.
(2) $\mu(F \setminus \bigcup_{x \in E} B(x, r)) = 0$.

Let $p \in \mathbb{N}$ and $r > 0$. Let $C \subset X_\delta$ a maximal $2r$-separated set for X_δ.

$$\begin{aligned} \mu(A_\varepsilon(r)) &= \mu\left(\left\{ y \in X_\delta : \mathbf{T}_{6r,p}(y) \mu(B(y, 2r)) \geq r^{-2\varepsilon} \right\} \right) \\ &\leq \sum_{x \in C} \mu\left(\left\{ y \in B(x, 2r) : \mathbf{T}_{6r,p}(y) \mu(B(y, 2r)) \geq r^{-2\varepsilon} \right\} \right). \end{aligned} \tag{3}$$

For $y \in B(x, 4r)$, we define

$$\tau^\omega_{4r,p}(y, x) := \inf \{ k > p : d(T_{\vartheta^k \omega} \circ \cdots \circ T_\omega y, x) < 4r \}$$

and

$$\mathbf{T}_{4r,p}(y, x) := \int_\Omega \tau^\omega_{4r,p}(y, x) d\mathbb{P}(\omega).$$

If $d(x, y) < 2r$, for all $\omega \in \Omega$ we have

$$\tau^\omega_{4r,p}(y, x) \geq \tau^\omega_{6r,p}(y)$$

and then

$$\mathbf{T}_{4r,p}(y, x) \mu(B(x, 4r)) \geq \mathbf{T}_{6r,p}(y) \mu(B(y, 2r)). \tag{4}$$

So we have for every $x \in C$

$$\mu\left(\left\{ y \in B(x, 2r) : \mathbf{T}_{6r,p}(y) \mu(B(y, 2r)) \geq r^{-2\varepsilon} \right\} \right) \leq \mu(D_{r,x}) \tag{5}$$

where

$$D_{r,x} := \left\{ y \in B(x, 4r) : \mathbf{T}_{4r,p}(y, x) \mu(B(x, 4r)) \geq r^{-2\varepsilon} \right\}.$$

Markov's inequality gives:

$$\mu(D_{r,x}) \leq r^{2\varepsilon} \mu(B(x, 4r)) \int_{B(x,4r)} \mathbf{T}_{4r,p}(y, x)\, d\mu(y) \tag{6}$$

$$= r^{2\varepsilon} \mu(B(x, 4r)) \int_{B(x,4r) \times \Omega} \tau^\omega_{4r,p}(y, x)\, d\mu(y) d\mathbb{P}(\omega). \tag{7}$$

Since $\tau^{\omega}_{4r,p}(y,x)$ is bounded by the p^{th} return time of (y,ω) in the set $B(x,4r)\times\Omega$, by Kac's lemma we have:

$$\int_{B(x,4r)\times\Omega}\tau^{\omega}_{4r,p}(y,x)\,d\mu(y)d\mathbb{P}(\omega)\le p. \tag{8}$$

Using (6) and (8), we have:

$$\mu(D_{r,x})\le pr^{2\varepsilon}\mu\left(B\left(x,4r\right)\right). \tag{9}$$

Then

$$\begin{aligned}
\mu(A_\varepsilon(r)) &\le \sum_{x\in C}\mu(D_{r,x}) && \text{by (3) and (5)}\\
&\le p\,r^{2\varepsilon}\sum_{x\in C}\mu\left(B(x,4r)\right) && \text{by (9)}\\
&\le p\,r^{\varepsilon}\sum_{x\in C}\mu\left(B(x,r)\right) && \text{since } \mu \text{ is wdr and with } \eta=4\\
&\le p\,r^{\varepsilon} && \text{according to the definition of } C.
\end{aligned}$$

Finally:

$$\sum_{n,e^{-n}<\delta}\mu(A_\varepsilon(e^{-n}))=\sum_{n>-\log\delta}\mu(A_\varepsilon(e^{-n}))\le p\sum_n e^{-\varepsilon n}<\infty.$$

Then, thanks to the Borel-Cantelli lemma, for μ-almost every $x\in X_\delta$

$$\mathbf{T}_{6e^{-n},p}(x)\mu\left(B(x,2e^{-n})\right)\le e^{2\varepsilon n}$$

for any n sufficiently large. Then

$$\frac{\log\mathbf{T}_{6e^{-n},p}(x)}{n}\le 2\varepsilon+\frac{\log\mu(B(x,2e^{-n}))}{-n}. \tag{10}$$

One can easily prove that for all $a>0$ we have:

$$\underline{d}_\mu(x)=\liminf_{n\to\infty}\frac{\log\mu\left(B\left(x,ae^{-n}\right)\right)}{-n}\quad\text{and}\quad\overline{d}_\mu(x)=\limsup_{n\to\infty}\frac{\log\mu\left(B\left(x,ae^{-n}\right)\right)}{-n}$$

$$\underline{\mathbf{R}}(x)=\lim_{p\to\infty}\liminf_{n\to\infty}\frac{\log\mathbf{T}_{ae^{-n},p}(x)}{n}\quad\text{and}\quad\overline{\mathbf{R}}(x)=\lim_{p\to\infty}\limsup_{n\to\infty}\frac{\log\mathbf{T}_{ae^{-n},p}(x)}{n}$$

and taking the limit inferior or the limit superior and then the limit over p in (10) prove the results since ε can be chosen arbitrarily small. □

Proof of Theorem 12. Let $x\in X$ be such that $\underline{R}^\omega(x)\ge\underline{d}_\mu(x)>0$ for $\mathbb{P}$-almost every $\omega\in\Omega$, the existence of such a x is ensured by Theorem 6. Let $0<\varepsilon<1$, by the proof of Theorem 5 in [11], we know that for $\mathbb{P}$-almost every $\omega\in\Omega$ it exists $N\in\mathbb{N}$ such that for every $p>N$, $\underline{\lim}_{r\to 0}\mu(B(x,r))^{1-\varepsilon}\tau^{\omega}_{r,p}(x)=+\infty$, that is to say that:

$$\mathbb{P}(\left\{\omega\in\Omega:\exists N\in\mathbb{N},\,\forall p>N,\,\liminf_{r\to0}\mu(B(x,r))^{1-\varepsilon}\tau^{\omega}_{r,p}(x)=+\infty\right\})=\mathbb{P}(\Omega)=1. \tag{11}$$

Let us denote by

$$\tilde{\Omega}(N):=\left\{\omega\in\Omega\,:\,\forall p>N,\,\liminf_{r\to0}\mu(B(x,r))^{1-\varepsilon}\tau^{\omega}_{r,p}(x)=+\infty\right\}.$$

It exists $N_1\in\mathbb{N}$ such that $\mathbb{P}(\tilde{\Omega}(N_1))>0$, otherwise $\mathbb{P}(\cup_{N\in\mathbb{N}}\tilde{\Omega}(N))\le\sum_{N\in\mathbb{N}}\mathbb{P}(\tilde{\Omega}(N))=0$ and it would contradict the fact that:

$$1=\mathbb{P}(\bigcup_{N\in\mathbb{N}}\tilde{\Omega}(N))=\mathbb{P}(\left\{\omega\in\Omega:\exists N\in\mathbb{N},\,\forall p>N,\,\liminf_{r\to0}\mu(B(x,r))^{1-\varepsilon}\tau^{\omega}_{r,p}(x)=+\infty\right\}).$$

Let $p > N_1$, we therefore have:

$$\int_{\tilde{\Omega}} \liminf_{r\to 0} \mu(B(x,r))^{1-\varepsilon}\tau^{\omega}_{r,p}(x)d\mathbb{P}(\omega) = +\infty. \tag{12}$$

Moreover, Fatou's lemma gives

$$\begin{aligned} +\infty = \int_{\tilde{\Omega}} \liminf_{r\to 0} \mu(B(x,r))^{1-\varepsilon}\tau^{\omega}_{r,p}(x)d\mathbb{P}(\omega) &\leq \int_{\Omega} \liminf_{r\to 0} \mu(B(x,r))^{1-\varepsilon}\tau^{\omega}_{r,p}(x)d\mathbb{P}(\omega) \\ &\leq \liminf_{r\to 0} \int_{\Omega} \mu(B(x,r))^{1-\varepsilon}\tau^{\omega}_{r,p}(x)d\mathbb{P}(\omega) \\ &= \liminf_{r\to 0} \mu(B(x,r))^{1-\varepsilon}\mathbf{T}_{r,p}(x). \end{aligned}$$

We thus get that:

$$\liminf_{r\to 0} \mu(B(x,r))^{1-\varepsilon}\mathbf{T}_{r,p}(x) = \limsup_{r\to 0} \mu(B(x,r))^{1-\varepsilon}\mathbf{T}_{r,p}(x) = +\infty.$$

Finally we have that for all $p > N_1$ and $M > 0$, it exists $R > 0$ such that for all $r < R$, $\mu(B(x,r))^{1-\varepsilon}\tau^{\Omega}_{r,p}(x) \geq M$ and then:

$$(1-\varepsilon)\frac{\log \mu(B(x,r))}{\log r} \leq \frac{M}{\log r} - \frac{\log \tau^{\Omega}_{r,p}(x)}{\log r}$$

which drives to:

$$\underline{\mathbf{R}}(x) \geq (1-\varepsilon)\underline{d}_{\mu}(x) \qquad \text{and} \qquad \overline{\mathbf{R}}(x) \geq (1-\varepsilon)\overline{d}_{\mu}(x).$$

Since these inequalities hold for every $0 < \varepsilon < 1$, we get:

$$\underline{\mathbf{R}}(x) \geq \underline{d}_{\mu}(x) \qquad \text{and} \qquad \overline{\mathbf{R}}(x) \geq \overline{d}_{\mu}(x). \tag{13}$$

By Theorem 8, the equation (13) is satisfied for μ-almost every x such that $\underline{d}_{\mu}(x) > 0$ and then the theorem is proved using Theorem 11. □

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